REAL AND COMPLEX ANALYSIS

Third Edition

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McGraw-Hill Book Company

New York St. Louis San Francisco Auckland Bogotá Hamburg
London Madrid Mexico Milan Montreal New Delhi
Panama Paris São Paulo Singapore Sydney Tokyo Toronto

REAL AND COMPLEX ANALYSIS
INTERNATIONAL EDITION

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2nd Printing 1987

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This book was set in Times Roman.
The editor was Peter R. Devine.
The production supervisor was Diane Renda.
Project supervision was done by Santype International Limited.

Library of Congress Cataloging-in-Publication Data

Rudin, Walter, 1921 —
Real and complex analysis.

Bibliography: p.
Includes index.
1. Mathematical analysis. I. Title.

When ordering this title use ISBN 0-07-100276-6

Printed and bound in Singapore by B & Jo Enterprise Pte Ltd.

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His research has dealt mainly with harmonic analysis and with complex variables. He has written three research monographs on these topics, *Fourier Analysis on Groups*, *Function Theory in Polydiscs*, and *Function Theory in the Unit Ball of* $\mathbf{C}^n$.

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PREFACE

This book contains a first-year graduate course in which the basic techniques and theorems of analysis are presented in such a way that the intimate connections between its various branches are strongly emphasized. The traditionally separate subjects of "real analysis" and "complex analysis" are thus united; some of the basic ideas from functional analysis are also included.

Here are some examples of the way in which these connections are demonstrated and exploited. The Riesz representation theorem and the Hahn-Banach theorem allow one to "guess" the Poisson integral formula. They team up in the proof of Runge's theorem. They combine with Blaschke's theorem on the zeros of bounded holomorphic functions to give a proof of the Müntz-Szasz theorem, which concerns approximation on an interval. The fact that L^2 is a Hilbert space is used in the proof of the Radon-Nikodym theorem, which leads to the theorem about differentiation of indefinite integrals, which in turn yields the existence of radial limits of bounded harmonic functions. The theorems of Plancherel and Cauchy combined give a theorem of Paley and Wiener which, in turn, is used in the Denjoy-Carleman theorem about infinitely differentiable functions on the real line. The maximum modulus theorem gives information about linear transformations on L^p-spaces.

Since most of the results presented here are quite classical (the novelty lies in the arrangement, and some of the proofs are new), I have not attempted to document the source of every item. References are gathered at the end, in Notes and Comments. They are not always to the original sources, but more often to more recent works where further references can be found. In no case does the absence of a reference imply any claim to originality on my part.

The prerequisite for this book is a good course in advanced calculus (settheoretic manipulations, metric spaces, uniform continuity, and uniform convergence). The first seven chapters of my earlier book "Principles of Mathematical Analysis" furnish sufficient preparation.

Experience with the first edition shows that first-year graduate students can study the first 15 chapters in two semesters, plus some topics from 1 or 2 of the remaining 5. These latter are quite independent of each other. The first 15 should be taken up in the order in which they are presented, except for Chapter 9, which can be postponed.

The most important difference between this third edition and the previous ones is the entirely new chapter on differentiation. The basic facts about differentiation are now derived from the existence of Lebesgue points, which in turn is an easy consequence of the so-called "weak type" inequality that is satisfied by the maximal functions of measures on euclidean spaces. This approach yields strong theorems with minimal effort. Even more important is that it familiarizes students with maximal functions, since these have become increasingly useful in several areas of analysis.

One of these is the study of the boundary behavior of Poisson integrals. A related one concerns H^p-spaces. Accordingly, large parts of Chapters 11 and 17 were rewritten and, I hope, simplified in the process.

I have also made several smaller changes in order to improve certain details: For example, parts of Chapter 4 have been simplified; the notions of equicontinuity and weak convergence are presented in more detail; the boundary behavior of conformal maps is studied by means of Lindelöf's theorem about asymptotic valued of bounded holomorphic functions in a disc.

Over the last 20 years, numerous students and colleagues have offered comments and criticisms concerning the content of this book. I sincerely appreciated all of these, and have tried to follow some of them. As regards the present edition, my thanks go to Richard Rochberg for some useful last-minute suggestions, and I especially thank Robert Burckel for the meticulous care with which he examined the entire manuscript.

Walter Rudin

PROLOGUE
THE EXPONENTIAL FUNCTION

This is the most important function in mathematics. It is defined, for every complex number z, by the formula

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}. \tag{1}$$

The series (1) converges absolutely for every z and converges uniformly on every bounded subset of the complex plane. Thus exp is a continuous function. The absolute convergence of (1) shows that the computation

$$\sum_{k=0}^{\infty} \frac{a^k}{k!} \sum_{m=0}^{\infty} \frac{b^m}{m!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} a^k b^{n-k} = \sum_{n=0}^{\infty} \frac{(a+b)^n}{n!}$$

is correct. It gives the important addition formula

$$\exp(a) \exp(b) = \exp(a+b), \tag{2}$$

valid for all complex numbers a and b.

We define the number e to be $\exp(1)$, and shall usually replace $\exp(z)$ by the customary shorter expression e^z. Note that $e^0 = \exp(0) = 1$, by (1).

Theorem

(*a*) *For every complex z we have $e^z \neq 0$.*
(*b*) exp *is its own derivative:* $\exp'(z) = \exp(z)$.

(*c*) *The restriction of* exp *to the real axis is a monotonically increasing positi function, and*

$$e^x \to \infty \text{ as } x \to \infty, \qquad e^x \to 0 \text{ as } x \to -\infty.$$

(*d*) *There exists a positive number* π *such that* $e^{\pi i/2} = i$ *and such that* $e^z = 1$ *and ônly if* $z/(2\pi i)$ *is an integer.*

(*e*) exp *is a periodic function, with period* $2\pi i$.

(*f*) *The mapping* $t \to e^{it}$ *maps the real axis onto the unit circle.*

(*g*) *If w is a complex number and* $w \neq 0$, *then* $w = e^z$ *for some z.*

PROOF By (2), $e^z \cdot e^{-z} = e^{z-z} = e^0 = 1$. This implies (*a*). Next,

$$\exp'(z) = \lim_{h \to 0} \frac{\exp(z+h) - \exp(z)}{h} = \exp(z) \lim_{h \to 0} \frac{\exp(h) - 1}{h} = \exp(z).$$

The first of the above equalities is a matter of definition, the second follows from (2), and the third from (1), and (*b*) is proved.

That exp is monotonically increasing on the positive real axis, and that $e^x \to \infty$ as $x \to \infty$, is clear from (1). The other assertions of (*c*) are consequences of $e^x \cdot e^{-x} = 1$.

For any real number t, (1) shows that e^{-it} is the complex conjugate of e^{it}. Thus

$$|e^{it}|^2 = e^{it} \cdot \overline{e^{it}} = e^{it} \cdot e^{-it} = e^{it-it} = e^0 = 1,$$

or

$$|e^{it}| = 1 \qquad (t \text{ real}). \tag{3}$$

In other words, if t is real, e^{it} lies on the unit circle. We define cos t, sin t to be the real and imaginary parts of e^{it}:

$$\cos t = \text{Re}\,[e^{it}], \qquad \sin t = \text{Im}\,[e^{it}] \qquad (t \text{ real}). \tag{4}$$

If we differentiate both sides of Euler's identity

$$e^{it} = \cos t + i \sin t, \tag{5}$$

which is equivalent to (4), and if we apply (*b*), we obtain

$$\cos' t + i \sin' t = ie^{it} = -\sin t + i \cos t,$$

so that

$$\cos' = -\sin, \qquad \sin' = \cos. \tag{6}$$

The power series (1) yields the representation

$$\cos t = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots. \tag{7}$$

Take $t = 2$. The terms of the series (7) then decrease in absolute value (except for the first one) and their signs alternate. Hence cos 2 is less than the sum of the first three terms of (7), with $t = 2$; thus $\cos 2 < -\frac{1}{3}$. Since $\cos 0 = 1$ and cos is a continuous real function on the real axis, we conclude that there is a smallest positive number t_0 for which $\cos t_0 = 0$. We define

$$\pi = 2t_0. \tag{8}$$

It follows from (3) and (5) that $\sin t_0 = \pm 1$. Since

$$\sin'(t) = \cos t > 0$$

on the segment $(0, t_0)$ and since $\sin 0 = 0$, we have $\sin t_0 > 0$, hence $\sin t_0 = 1$, and therefore

$$e^{\pi i/2} = i. \tag{9}$$

It follows that $e^{\pi i} = i^2 = -1$, $e^{2\pi i} = (-1)^2 = 1$, and then $e^{2\pi in} = 1$ for every integer n. Also, (e) follows immediately:

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z. \tag{10}$$

If $z = x + iy$, x and y real, then $e^z = e^x e^{iy}$; hence $|e^z| = e^x$. If $e^z = 1$, we therefore must have $e^x = 1$, so that $x = 0$; to prove that $y/2\pi$ must be an integer, it is enough to show that $e^{iy} \neq 1$ if $0 < y < 2\pi$, by (10).

Suppose $0 < y < 2\pi$, and

$$e^{iy/4} = u + iv \qquad (u \text{ and } v \text{ real}). \tag{11}$$

Since $0 < y/4 < \pi/2$, we have $u > 0$ and $v > 0$. Also

$$e^{iy} = (u + iv)^4 = u^4 - 6u^2v^2 + v^4 + 4iuv(u^2 - v^2). \tag{12}$$

The right side of (12) is real only if $u^2 = v^2$; since $u^2 + v^2 = 1$, this happens only when $u^2 = v^2 = \frac{1}{2}$, and then (12) shows that

$$e^{iy} = -1 \neq 1.$$

This completes the proof of (d).

We already know that $t \to e^{it}$ maps the real axis *into* the unit circle. To prove (f), fix w so that $|w| = 1$; we shall show that $w = e^{it}$ for some real t. Write $w = u + iv$, u and v real, and suppose first that $u \geq 0$ and $v \geq 0$. Since $u \leq 1$, the definition of π shows that there exists a t, $0 \leq t \leq \pi/2$, such that $\cos t = u$; then $\sin^2 t = 1 - u^2 = v^2$, and since $\sin t \geq 0$ if $0 \leq t \leq \pi/2$, we have $\sin t = v$. Thus $w = e^{it}$.

If $u < 0$ and $v \geq 0$, the preceding conditions are satisfied by $-iw$. Hence $-iw = e^{it}$ for some real t, and $w = e^{i(t+\pi/2)}$. Finally, if $v < 0$, the preceding two cases show that $-w = e^{it}$ for some real t, hence $w = e^{i(t+\pi)}$. This completes the proof of (f).

If $w \neq 0$, put $\alpha = w/|w|$. Then $w = |w|\alpha$. By (c), there is a real x such that $|w| = e^x$. Since $|\alpha| = 1$, (f) shows that $\alpha = e^{iy}$ for some real y. Hence $w = e^{x+iy}$. This proves (g) and completes the theorem. ////

We shall encounter the integral of $(1 + x^2)^{-1}$ over the real line. To eva it, put $\varphi(t) = \sin t/\cos t$ in $(-\pi/2, \pi/2)$. By (6), $\varphi' = 1 + \varphi^2$. Hence φ is a m tonically increasing mapping of $(-\pi/2, \pi/2)$ onto $(-\infty, \infty)$, and we obtain

$$\int_{-\infty}^{\infty} \frac{dx}{1 + x^2} = \int_{-\pi/2}^{\pi/2} \frac{\varphi'(t)\, dt}{1 + \varphi^2(t)} = \int_{-\pi/2}^{\pi/2} dt = \pi.$$

CHAPTER

ONE

ABSTRACT INTEGRATION

Toward the end of the nineteenth century it became clear to many mathematicians that the Riemann integral (about which one learns in calculus courses) should be replaced by some other type of integral, more general and more flexible, better suited for dealing with limit processes. Among the attempts made in this direction, the most notable ones were due to Jordan, Borel, W. H. Young, and Lebesgue. It was Lebesgue's construction which turned out to be the most successful.

In brief outline, here is the main idea: The Riemann integral of a function f over an interval $[a, b]$ can be approximated by sums of the form

$$\sum_{i=1}^{n} f(t_i)m(E_i)$$

where $E_1, \ldots, E_n$ are disjoint intervals whose union is $[a, b]$, $m(E_i)$ denotes the length of E_i, and $t_i \in E_i$ for $i = 1, \ldots, n$. Lebesgue discovered that a completely satisfactory theory of integration results if the sets E_i in the above sum are allowed to belong to a larger class of subsets of the line, the so-called "measurable sets," and if the class of functions under consideration is enlarged to what he called "measurable functions." The crucial set-theoretic properties involved are the following: The union and the intersection of any countable family of measurable sets are measurable; so is the complement of every measurable set; and, most important, the notion of "length" (now called "measure") can be extended to them in such a way that

$$m(E_1 \cup E_2 \cup E_3 \cup \cdots) = m(E_1) + m(E_2) + m(E_3) + \cdots$$

for every countable collection $\{E_i\}$ of pairwise disjoint measurable sets. This property of m is called *countable additivity.*

The passage from Riemann's theory of integration to that of Lebesque is a process of completion (in a sense which will appear more precisely later). It is of the same fundamental importance in analysis as is the construction of the real number system from the rationals.

The above-mentioned measure m is of course intimately related to the geometry of the real line. In this chapter we shall present an abstract (axiomatic) version of the Lebesgue integral, relative to *any* countably additive measure on *any* set. (The precise definitions follow.) This abstract theory is not in any way more difficult than the special case of the real line; it shows that a large part of integration theory is independent of any geometry (or topology) of the underlying space; and, of course, it gives us a tool of much wider applicability. The existence of a large class of measures, among them that of Lebesgue, will be established in Chap. 2.

Set-Theoretic Notations and Terminology

1.1 Some sets can be described by listing their members. Thus $\{x_1, \ldots, x_n\}$ is the set whose members are $x_1, \ldots, x_n$; and $\{x\}$ is the set whose only member is x. More often, sets are described by properties. We write

$$\{x: P\}$$

for the set of all elements x which have the property P. The symbol $\varnothing$ denotes the empty set. The words *collection, family,* and *class* will be used synonymously with *set.*

We write $x \in A$ if x is a member of the set A; otherwise $x \notin A$. If B is a subset of A, i.e., if $x \in B$ implies $x \in A$, we write $B \subset A$. If $B \subset A$ and $A \subset B$, then $A = B$. If $B \subset A$ and $A \neq B$, B is a *proper* subset of A. Note that $\varnothing \subset A$ for every set A.

$A \cup B$ and $A \cap B$ are the union and intersection of A and B, respectively. If $\{A_\alpha\}$ is a collection of sets, where α runs through some index set I, we write

$$\bigcup_{\alpha \in I} A_\alpha \quad \text{and} \quad \bigcap_{\alpha \in I} A_\alpha$$

for the union and intersection of $\{A_\alpha\}$:

$$\bigcup_{\alpha \in I} A_\alpha = \{x: x \in A_\alpha \text{ for at least one } \alpha \in I\}$$

$$\bigcap_{\alpha \in I} A_\alpha = \{x: x \in A_\alpha \text{ for every } \alpha \in I\}.$$

If I is the set of all positive integers, the customary notations are

$$\bigcup_{n=1}^{\infty} A_n \quad \text{and} \quad \bigcap_{n=1}^{\infty} A_n.$$

If no two members of $\{A_\alpha\}$ have an element in common, then $\{A_\alpha\}$ is a *disjoint collection* of sets.

We write $A - B = \{x: x \in A, x \notin B\}$, and denote the complement of A by A^c whenever it is clear from the context with respect to which larger set the complement is taken.

The *cartesian product* $A_1 \times \cdots \times A_n$ of the sets $A_1, \ldots, A_n$ is the set of all ordered n-tuples $(a_1, \ldots, a_n)$ where $a_i \in A_i$ for $i = 1, \ldots, n$.

The *real line* (or real number system) is R^1, and

$$R^k = R^1 \times \cdots \times R^1 \qquad (k \text{ factors}).$$

The *extended real number system* is R^1 with two symbols, ∞ and $-\infty$, adjoined, and with the obvious ordering. If $-\infty \leq a \leq b \leq \infty$, the *interval* $[a, b]$ and the *segment* (a, b) are defined to be

$$[a, b] = \{x: a \leq x \leq b\}, \qquad (a, b) = \{x: a < x < b\}.$$

We also write

$$[a, b) = \{x: a \leq x < b\}, \qquad (a, b] = \{x: a < x \leq b\}.$$

If $E \subset [-\infty, \infty]$ and $E \neq \emptyset$, the least upper bound (supremum) and greatest lower bound (infimum) of E exist in $[-\infty, \infty]$ and are denoted by $\sup E$ and $\inf E$.

Sometimes (but only when $\sup E \in E$) we write $\max E$ for $\sup E$.

The symbol

$$f: X \to Y$$

means that f is a *function* (or *mapping* or *transformation*) of the set X into the set Y; i.e., f assigns to each $x \in X$ an element $f(x) \in Y$. If $A \subset X$ and $B \subset Y$, the *image* of A and the *inverse image* (or pre-image) of B are

$$f(A) = \{y: y = f(x) \text{ for some } x \in A\},$$

$$f^{-1}(B) = \{x: f(x) \in B\}.$$

Note that $f^{-1}(B)$ may be empty even when $B \neq \emptyset$.

The *domain* of f is X. The *range* of f is $f(X)$.

If $f(X) = Y$, f is said to map X *onto* Y.

We write $f^{-1}(y)$, instead of $f^{-1}(\{y\})$, for every $y \in Y$. If $f^{-1}(y)$ consists of at most one point, for each $y \in Y$, f is said to be *one-to-one*. If f is one-to-one, then f^{-1} is a function with domain $f(X)$ and range X.

If $f: X \to [-\infty, \infty]$ and $E \subset X$, it is customary to write $\sup_{x \in E} f(x)$ rather than $\sup f(E)$.

If $f: X \to Y$ and $g: Y \to Z$, the *composite function* $g \circ f: X \to Z$ is defined by the formula

$$(g \circ f)(x) = g(f(x)) \qquad (x \in X).$$

If the range of f lies in the real line (or in the complex plane), then f is said to be a *real function* (or a *complex function*). For a complex function f, the statement "$f \geq 0$" means that all values $f(x)$ of f are nonnegative real numbers.

The Concept of Measurability

The class of measurable functions plays a fundamental role in integration theory. It has some basic properties in common with another most important class of functions, namely, the continuous ones. It is helpful to keep these similarities in mind. Our presentation is therefore organized in such a way that the analogies between the concepts *topological space*, *open set*, and *continuous function*, on the one hand, and *measurable space*, *measurable set*, and *measurable function*, on the other, are strongly emphasized. It seems that the relations between these concepts emerge most clearly when the setting is quite abstract, and this (rather than a desire for mere generality) motivates our approach to the subject.

1.2 Definition

(*a*) A collection τ of subsets of a set X is said to be a *topology in* X if τ has the following three properties:

(i) $\varnothing \in \tau$ and $X \in \tau$.
(ii) If $V_i \in \tau$ for $i = 1, \ldots, n$, then $V_1 \cap V_2 \cap \cdots \cap V_n \in \tau$.
(iii) If $\{V_\alpha\}$ is an arbitrary collection of members of τ (finite, countable, or uncountable), then $\bigcup_\alpha V_\alpha \in \tau$.

(*b*) If τ is a topology in X, then X is called a *topological space*, and the members of τ are called the *open sets* in X.

(*c*) If X and Y are topological spaces and if f is a mapping of X into Y, then f is said to be *continuous* provided that $f^{-1}(V)$ is an open set in X for every open set V in Y.

1.3 Definition

(*a*) A collection $\mathfrak{M}$ of subsets of a set X is said to be a *σ-algebra in* X if $\mathfrak{M}$ has the following properties:

(i) $X \in \mathfrak{M}$.
(ii) If $A \in \mathfrak{M}$, then $A^c \in \mathfrak{M}$, where A^c is the complement of A relative to X.
(iii) If $A = \bigcup_{n=1}^{\infty} A_n$ and if $A_n \in \mathfrak{M}$ for $n = 1, 2, 3, \ldots$, then $A \in \mathfrak{M}$.

(*b*) If $\mathfrak{M}$ is a σ-algebra in X, then X is called a *measurable space*, and the members of $\mathfrak{M}$ are called the *measurable sets* in X.

(*c*) If X is a measurable space, Y is a topological space, and f is a mapping of X into Y, then f is said to be *measurable* provided that $f^{-1}(V)$ is a measurable set in X for every open set V in Y.

It would perhaps be more satisfactory to apply the term "measurable space" to the ordered pair $(X, \mathfrak{M})$, rather than to X. After all, X is a set, and X has not been changed in any way by the fact that we now also have a σ-algebra of its subsets in mind. Similarly, a topological space is an ordered pair (X, τ). But if this sort of thing were systematically done in all mathematics, the terminology would become awfully cumbersome. We shall discuss this again at somewhat greater length in Sec. 1.21.

1.4 Comments on Definition 1.2 The most familiar topological spaces are the *metric spaces*. We shall assume some familiarity with metric spaces but shall give the basic definitions, for the sake of completeness.

A *metric space* is a set X in which a *distance function* (or *metric*) ρ is defined, with the following properties:

(*a*) $0 \leq \rho(x, y) < \infty$ for all x and $y \in X$.
(*b*) $\rho(x, y) = 0$ if and only if $x = y$.
(*c*) $\rho(x, y) = \rho(y, x)$ for all x and $y \in X$.
(*d*) $\rho(x, y) \leq \rho(x, z) + \rho(z, y)$ for all x, y, and $z \in X$.

Property (*d*) is called the *triangle inequality*.

If $x \in X$ and $r \geq 0$, the *open ball* with center at x and radius r is the set $\{y \in X: \rho(x, y) < r\}$.

If X is a metric space and if τ is the collection of all sets $E \subset X$ which are arbitrary unions of open balls, then τ is a topology in X. This is not hard to verify; the intersection property depends on the fact that if $x \in B_1 \cap B_2$, where B_1 and B_2 are open balls, then x is the center of an open ball $B \subset B_1 \cap B_2$. We leave this as an exercise.

For instance, in the real line R^1 a set is open if and only if it is a union of open segments (a, b). In the plane R^2, the open sets are those which are unions of open circular discs.

Another topological space, which we shall encounter frequently, is the extended real line $[-\infty, \infty]$; its topology is defined by declaring the following sets to be open: (a, b), $[-\infty, a)$, $(a, \infty]$, and any union of segments of this type.

The definition of continuity given in Sec. 1.2(*c*) is a global one. Frequently it is desirable to define continuity locally: A mapping f of X into Y is said to be *continuous at the point* $x_0 \in X$ if to every neighborhood V of $f(x_0)$ there corresponds a neighborhood W of x_0 such that $f(W) \subset V$.

(A *neighborhood* of a point x is, by definition, an open set which contains x.)

When X and Y are metric spaces, this local definition is of course the same as the usual epsilon-delta definition, and is equivalent to the requirement that $\lim f(x_n) = f(x_0)$ in Y whenever $\lim x_n = x_0$ in X.

The following easy proposition relates the local and global definitions of continuity in the expected manner:

1.5 Proposition *Let X and Y be topological spaces. A mapping f of X into Y is continuous if and only if f is continuous at every point of X.*

PROOF If f is continuous and $x_0 \in X$, then $f^{-1}(V)$ is a neighborhood of x_0, for every neighborhood V of $f(x_0)$. Since $f(f^{-1}(V)) \subset V$, it follows that f is continuous at x_0.

If f is continuous at every point of X and if V is open in Y, every point $x \in f^{-1}(V)$ has a neighborhood W_x such that $f(W_x) \subset V$. Therefore $W_x \subset f^{-1}(V)$. It follows that $f^{-1}(V)$ is the union of the open sets W_x, so $f^{-1}(V)$ is itself open. Thus f is continuous. ////

1.6 Comments on Definition 1.3 Let $\mathfrak{M}$ be a σ-algebra in a set X. Referring to Properties (i) to (iii) of Definition 1.3(a), we immediately derive the following facts.

(a) Since $\varnothing = X^c$, (i) and (ii) imply that $\varnothing \in \mathfrak{M}$.
(b) Taking $A_{n+1} = A_{n+2} = \cdots = \varnothing$ in (iii), we see that $A_1 \cup A_2 \cup \cdots \cup A_n \in \mathfrak{M}$ if $A_i \in \mathfrak{M}$ for $i = 1, \ldots, n$.
(c) Since

$$\bigcap_{n=1}^{\infty} A_n = \left(\bigcup_{n=1}^{\infty} A_n^c \right)^c,$$

$\mathfrak{M}$ is closed under the formation of countable (and also finite) intersections.
(d) Since $A - B = B^c \cap A$, we have $A - B \in \mathfrak{M}$ if $A \in \mathfrak{M}$ and $B \in \mathfrak{M}$.

The prefix σ refers to the fact that (iii) is required to hold for all *countable* unions of members of $\mathfrak{M}$. If (iii) is required for finite unions only, then $\mathfrak{M}$ is called an *algebra* of sets.

1.7 Theorem *Let Y and Z be topological spaces, and let $g: Y \to Z$ be continuous.*

(a) *If X is a topological space, if $f: X \to Y$ is continuous, and if $h = g \circ f$, then $h: X \to Z$ is continuous.*
(b) *If X is a measurable space, if $f: X \to Y$ is measurable, and if $h = g \circ f$, then $h: X \to Z$ is measurable.*

Stated informally, continuous functions of continuous functions are continuous; continuous functions of measurable functions are measurable.

PROOF If V is open in Z, then $g^{-1}(V)$ is open in Y, and

$$h^{-1}(V) = f^{-1}(g^{-1}(V)).$$

If f is continuous, it follows that $h^{-1}(V)$ is open, proving (a).
If f is measurable, it follows that $h^{-1}(V)$ is measurable, proving (b). ////

1.8 Theorem *Let u and v be real measurable functions on a measurable space X, let Φ be a continuous mapping of the plane into a topological space Y, and define*

$$h(x) = \Phi(u(x), v(x))$$

for $x \in X$. Then $h: X \to Y$ is measurable.

PROOF Put $f(x) = (u(x), v(x))$. Then f maps X into the plane. Since $h = \Phi \circ f$, Theorem 1.7 shows that it is enough to prove the measurability of f.

If R is any open rectangle in the plane, with sides parallel to the axes, then R is the cartesian product of two segments I_1 and I_2, and

$$f^{-1}(R) = u^{-1}(I_1) \cap v^{-1}(I_2),$$

which is measurable, by our assumption on u and v. Every open set V in the plane is a countable union of such rectangles R_i, and since

$$f^{-1}(V) = f^{-1}\left(\bigcup_{i=1}^{\infty} R_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(R_i),$$

$f^{-1}(V)$ is measurable. ////

1.9 Let X be a measurable space. The following propositions are corollaries of Theorems 1.7 and 1.8:

(*a*) *If $f = u + iv$, where u and v are real measurable functions on X, then f is a complex measurable function on X.*

This follows from Theorem 1.8, with $\Phi(z) = z$.

(*b*) *If $f = u + iv$ is a complex measurable function on X, then u, v, and $|f|$ are real measurable functions on X.*

This follows from Theorem 1.7, with $g(z) = \text{Re}\,(z)$, $\text{Im}\,(z)$, and $|z|$.

(*c*) *If f and g are complex measurable functions on X, then so are $f + g$ and fg.*

For real f and g this follows from Theorem 1.8, with

$$\Phi(s, t) = s + t$$

and $\Phi(s, t) = st$. The complex case then follows from (*a*) and (*b*).

(*d*) *If E is a measurable set in X and if*

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$$

then χ_E is a measurable function.

This is obvious. We call χ_E the *characteristic function* of the set E. The letter χ will be reserved for characteristic functions throughout this book.

(*e*) *If f is a complex measurable function on X, there is a complex measurable function α on X such that $|\alpha| = 1$ and $f = \alpha|f|$.*

PROOF Let $E = \{x: f(x) = 0\}$, let Y be the complex plane with the origin removed, define $\varphi(z) = z/|z|$ for $z \in Y$, and put

$$\alpha(x) = \varphi(f(x) + \chi_E(x)) \qquad (x \in X).$$

If $x \in E$, $\alpha(x) = 1$; if $x \notin E$, $\alpha(x) = f(x)/|f(x)|$. Since φ is continuous on Y and since E is measurable (why?), the measurability of α follows from (*c*), (*d*), and Theorem 1.7. ////

We now show that σ-algebras exist in great profusion.

1.10 Theorem *If $\mathscr{F}$ is any collection of subsets of X, there exists a smallest σ-algebra $\mathfrak{M}^*$ in X such that $\mathscr{F} \subset \mathfrak{M}^*$.*

This $\mathfrak{M}^*$ is sometimes called the σ-algebra *generated* by $\mathscr{F}$.

PROOF Let Ω be the family of all σ-algebras $\mathfrak{M}$ in X which contain $\mathscr{F}$. Since the collection of all subsets of X is such a σ-algebra, Ω is not empty. Let $\mathfrak{M}^*$ be the intersection of all $\mathfrak{M} \in \Omega$. It is clear that $\mathscr{F} \subset \mathfrak{M}^*$ and that $\mathfrak{M}^*$ lies in every σ-algebra in X which contains $\mathscr{F}$. To complete the proof, we have to show that $\mathfrak{M}^*$ is itself a σ-algebra.

If $A_n \in \mathfrak{M}^*$ for $n = 1, 2, 3, \ldots$, and if $\mathfrak{M} \in \Omega$, then $A_n \in \mathfrak{M}$, so $\bigcup A_n \in \mathfrak{M}$, since $\mathfrak{M}$ is a σ-algebra. Since $\bigcup A_n \in \mathfrak{M}$ for *every* $\mathfrak{M} \in \Omega$, we conclude that $\bigcup A_n \in \mathfrak{M}^*$. The other two defining properties of a σ-algebra are verified in the same manner. ////

1.11 Borel Sets Let X be a topological space. By Theorem 1.10, there exists a smallest σ-algebra $\mathscr{B}$ in X such that every open set in X belongs to $\mathscr{B}$. The members of $\mathscr{B}$ are called the *Borel sets* of X.

In particular, closed sets are Borel sets (being, by definition, the complements of open sets), and so are all countable unions of closed sets and all countable intersections of open sets. These last two are called F_σ's and G_δ's, respectively, and play a considerable role. The notation is due to Hausdorff. The letters F and G were used for closed and open sets, respectively, and σ refers to union (*Summe*), δ to intersection (*Durchschnitt*). For example, every half-open interval $[a, b)$ is a G_δ and an F_σ in R^1.

Since $\mathscr{B}$ is a σ-algebra, we may now regard X as a measurable space, with the Borel sets playing the role of the measurable sets; more concisely, we consider the measurable space $(X, \mathscr{B})$. If $f: X \to Y$ is a continuous mapping of X, where Y is any topological space, then it is evident from the definitions that $f^{-1}(V) \in \mathscr{B}$ for every open set V in Y. In other words, *every continuous mapping of X is Borel measurable.*

Borel measurable mappings are often called *Borel mappings*, or *Borel functions.*

1.12 Theorem *Suppose $\mathfrak{M}$ is a σ-algebra in X, and Y is a topological space. Let f map X into Y.*

(*a*) *If Ω is the collection of all sets $E \subset Y$ such that $f^{-1}(E) \in \mathfrak{M}$, then Ω is a σ-algebra in Y.*
(*b*) *If f is measurable and E is a Borel set in Y, then $f^{-1}(E) \in \mathfrak{M}$.*
(*c*) *If $Y = [-\infty, \infty]$ and $f^{-1}((\alpha, \infty]) \in \mathfrak{M}$ for every real α, then f is measurable.*
(*d*) *If f is measurable, if Z is a topological space, if $g: Y \to Z$ is a Borel mapping, and if $h = g \circ f$, then $h: X \to Z$ is measurable.*

Part (*c*) is a frequently used criterion for the measurability of real-valued functions. (See also Exercise 3.) Note that (*d*) generalizes Theorem 1.7(*b*).

PROOF (*a*) follows from the relations

$$f^{-1}(Y) = X,$$

$$f^{-1}(Y - A) = X - f^{-1}(A),$$

and

$$f^{-1}(A_1 \cup A_2 \cup \cdots) = f^{-1}(A_1) \cup f^{-1}(A_2) \cup \cdots.$$

To prove (*b*), let Ω be as in (*a*); the measurability of f implies that Ω contains all open sets in Y, and since Ω is a σ-algebra, Ω contains all Borel sets in Y.

To prove (*c*), let Ω be the collection of all $E \subset [-\infty, \infty]$ such that $f^{-1}(E) \in \mathfrak{M}$. Choose a real number α, and choose $\alpha_n < \alpha$ so that $\alpha_n \to \alpha$ as $n \to \infty$. Since $(\alpha_n, \infty] \in \Omega$ for each n, since

$$[-\infty, \alpha) = \bigcup_{n=1}^{\infty} [-\infty, \alpha_n] = \bigcup_{n=1}^{\infty} (\alpha_n, \infty]^c,$$

and since (*a*) shows that Ω is a σ-algebra, we see that $[-\infty, \alpha) \in \Omega$. The same is then true of

$$(\alpha, \beta) = [-\infty, \beta) \cap (\alpha, \infty].$$

Since every open set in $[-\infty, \infty]$ is a countable union of segments of the above types, Ω contains every open set. Thus f is measurable.

To prove (*d*), let $V \subset Z$ be open. Then $g^{-1}(V)$ is a Borel set of Y, and since

$$h^{-1}(V) = f^{-1}(g^{-1}(V)),$$

(*b*) shows that $h^{-1}(V) \in \mathfrak{M}$. ////

1.13 Definition Let $\{a_n\}$ be a sequence in $[-\infty, \infty]$, and put

$$b_k = \sup \{a_k, a_{k+1}, a_{k+2}, \ldots\} \qquad (k = 1, 2, 3, \ldots) \tag{1}$$

and

$$\beta = \inf \{b_1, b_2, b_3, \ldots\}. \tag{2}$$

We call β the *upper limit* of $\{a_n\}$, and write

$$\beta = \limsup_{n\to\infty} a_n. \tag{3}$$

The following properties are easily verified: First, $b_1 \geq b_2 \geq b_3 \geq \cdots$, so that $b_k \to \beta$ as $k \to \infty$; secondly, there is a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that $a_{n_i} \to \beta$ as $i \to \infty$, and β is the largest number with this property.

The *lower limit* is defined analogously: simply interchange sup and inf in (1) and (2). Note that

$$\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty} (-a_n). \tag{4}$$

If $\{a_n\}$ converges, then evidently

$$\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n. \tag{5}$$

Suppose $\{f_n\}$ is a sequence of extended-real functions on a set X. Then $\sup_n f_n$ and $\limsup_{n\to\infty} f_n$ are the functions defined on X by

$$\left(\sup_n f_n\right)(x) = \sup_n (f_n(x)), \tag{6}$$

$$\left(\limsup_{n\to\infty} f_n\right)(x) = \limsup_{n\to\infty} (f_n(x)). \tag{7}$$

If

$$f(x) = \lim_{n\to\infty} f_n(x), \tag{8}$$

the limit being assumed to exist at every $x \in X$, then we call f the *pointwise limit* of the sequence $\{f_n\}$.

1.14 Theorem *If $f_n: X \to [-\infty, \infty]$ is measurable, for $n = 1, 2, 3, \ldots$, and*

$$g = \sup_{n\geq 1} f_n, \qquad h = \limsup_{n\to\infty} f_n,$$

then g and h are measurable.

PROOF $g^{-1}((\alpha, \infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha, \infty])$. Hence Theorem 1.12(*c*) implies that g is measurable. The same result holds of course with inf in place of sup, and since

$$h = \inf_{k\geq 1} \left\{\sup_{i\geq k} f_i\right\},$$

it follows that h is measurable. ////

Corollaries

(a) *The limit of every pointwise convergent sequence of complex measurable functions is measurable.*

(b) *If f and g are measurable (with range in $[-\infty, \infty]$), then so are* $\max\{f, g\}$ *and* $\min\{f, g\}$. *In particular, this is true of the functions*

$$f^+ = \max\{f, 0\} \quad \text{and} \quad f^- = -\min\{f, 0\}.$$

1.15 The above functions f^+ and f^- are called the *positive and negative parts* of f. We have $|f| = f^+ + f^-$ and $f = f^+ - f^-$, a standard representation of f as a difference of two nonnegative functions, with a certain minimality property:

Proposition *If $f = g - h$, $g \geq 0$, and $h \geq 0$, then $f^+ \leq g$ and $f^- \leq h$.*

PROOF $f \leq g$ and $0 \leq g$ clearly implies $\max\{f, 0\} \leq g$. ////

Simple Functions

1.16 Definition A complex function s on a measurable space X whose range consists of only finitely many points will be called a *simple function.* Among these are the nonnegative simple functions, whose range is a finite subset of $[0, \infty)$. Note that we explicitly exclude ∞ from the values of a simple function.

If $\alpha_1, \ldots, \alpha_n$ are the distinct values of a simple function s, and if we set $A_i = \{x: s(x) = \alpha_i\}$, then clearly

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i},$$

where χ_{A_i} is the characteristic function of A_i, as defined in Sec. 1.9(*d*).

It is also clear that s is measurable if and only if each of the sets A_i is measurable.

1.17 Theorem *Let $f: X \to [0, \infty]$ be measurable. There exist simple measurable functions s_n on X such that*

(a) $0 \leq s_1 \leq s_2 \leq \cdots \leq f$.

(b) $s_n(x) \to f(x)$ *as* $n \to \infty$, *for every* $x \in X$.

PROOF Put $\delta_n = 2^{-n}$. To each positive integer n and each real number t corresponds a unique integer $k = k_n(t)$ that satisfies $k\delta_n \leq t < (k+1)\delta_n$. Define

$$\varphi_n(t) = \begin{cases} k_n(t)\delta_n & \text{if } 0 \leq t < n \\ n & \text{if } n \leq t \leq \infty. \end{cases} \tag{1}$$

Each φ_n is then a Borel function on $[0, \infty]$,

$$t - \delta_n < \varphi_n(t) \leq t \qquad \text{if } 0 \leq t \leq n, \tag{2}$$

$0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq t$, and $\varphi_n(t) \to t$ as $n \to \infty$, for every $t \in [0, \infty]$. It follows that the functions

$$s_n = \varphi_n \circ f \tag{3}$$

satisfy (*a*) and (*b*); they are measurable, by Theorem 1.12(*d*). ////

Elementary Properties of Measures

1.18 Definition

(*a*) A *positive measure* is a function μ, defined on a σ-algebra $\mathfrak{M}$, whose range is in $[0, \infty]$ and which is *countably additive*. This means that if $\{A_i\}$ is a *disjoint* countable collection of members of $\mathfrak{M}$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mu(A_i). \tag{1}$$

To avoid trivialities, we shall also assume that $\mu(A) < \infty$ for at least one $A \in \mathfrak{M}$.

(*b*) A *measure space* is a measurable space which has a positive measure defined on the σ-algebra of its measurable sets.

(*c*) A *complex measure* is a complex-valued countably additive function defined on a σ-algebra.

Note: What we have called a *positive measure* is frequently just called a *measure*; we add the word "positive" for emphasis. If $\mu(E) = 0$ for every $E \in \mathfrak{M}$, then μ is a positive measure, by our definition. The value ∞ is admissible for a positive measure; but when we talk of a *complex* measure μ, it is understood that $\mu(E)$ is a complex number, for every $E \in \mathfrak{M}$. The *real measures* form a subclass of the complex ones, of course.

1.19 Theorem *Let μ be a positive measure on a σ-algebra $\mathfrak{M}$. Then*

(*a*) $\mu(\varnothing) = 0$.

(*b*) $\mu(A_1 \cup \cdots \cup A_n) = \mu(A_1) + \cdots + \mu(A_n)$ *if* $A_1, \ldots, A_n$ *are pairwise disjoint members of* $\mathfrak{M}$.

(*c*) $A \subset B$ *implies* $\mu(A) \leq \mu(B)$ *if* $A \in \mathfrak{M}$, $B \in \mathfrak{M}$.

(*d*) $\mu(A_n) \to \mu(A)$ *as* $n \to \infty$ *if* $A = \bigcup_{n=1}^{\infty} A_n$, $A_n \in \mathfrak{M}$, *and*

$$A_1 \subset A_2 \subset A_3 \subset \cdots.$$

(*e*) $\mu(A_n) \to \mu(A)$ *as* $n \to \infty$ *if* $A = \bigcap_{n=1}^{\infty} A_n$, $A_n \in \mathfrak{M}$,

$$A_1 \supset A_2 \supset A_3 \supset \cdots,$$

and $\mu(A_1)$ *is finite.*

As the proof will show, these properties, with the exception of (*c*), also hold for complex measures; (*b*) is called *finite additivity*; (*c*) is called *monotonicity*.

PROOF

(*a*) Take $A \in \mathfrak{M}$ so that $\mu(A) < \infty$, and take $A_1 = A$ and $A_2 = A_3 = \cdots = \varnothing$ in 1.18(1).

(*b*) Take $A_{n+1} = A_{n+2} = \cdots = \varnothing$ in 1.18(1).

(*c*) Since $B = A \cup (B - A)$ and $A \cap (B - A) = \varnothing$, we see that (*b*) implies $\mu(B) = \mu(A) + \mu(B - A) \geq \mu(A)$.

(*d*) Put $B_1 = A_1$, and put $B_n = A_n - A_{n-1}$ for $n = 2, 3, 4, \ldots$. Then $B_n \in \mathfrak{M}$, $B_i \cap B_j = \varnothing$ if $i \neq j$, $A_n = B_1 \cup \cdots \cup B_n$, and $A = \bigcup_{i=1}^{\infty} B_i$. Hence

$$\mu(A_n) = \sum_{i=1}^{n} \mu(B_i) \quad \text{and} \quad \mu(A) = \sum_{i=1}^{\infty} \mu(B_i).$$

Now (*d*) follows, by the definition of the sum of an infinite series.

(*e*) Put $C_n = A_1 - A_n$. Then $C_1 \subset C_2 \subset C_3 \subset \cdots$,

$$\mu(C_n) = \mu(A_1) - \mu(A_n),$$

$A_1 - A = \bigcup C_n$, and so (*d*) shows that

$$\mu(A_1) - \mu(A) = \mu(A_1 - A) = \lim_{n \to \infty} \mu(C_n) = \mu(A_1) - \lim_{n \to \infty} \mu(A_n).$$

This implies (*e*). ////

1.20 Examples The construction of interesting measure spaces requires some labor, as we shall see. However, a few simple-minded examples can be given immediately:

(*a*) For any $E \subset X$, where X is any set, define $\mu(E) = \infty$ if E is an infinite set, and let $\mu(E)$ be the number of points in E if E is finite. This μ is called the *counting measure* on X.

(*b*) Fix $x_0 \in X$, define $\mu(E) = 1$ if $x_0 \in E$ and $\mu(E) = 0$ if $x_0 \notin E$, for any $E \subset X$. This μ may be called the *unit mass* concentrated at x_0.

(*c*) Let μ be the counting measure on the set $\{1, 2, 3, \ldots\}$, let $A_n = \{n, n + 1, n + 2, \ldots\}$. Then $\bigcap A_n = \varnothing$ but $\mu(A_n) = \infty$ for $n = 1, 2, 3, \ldots$. This shows that the hypothesis

$$\mu(A_1) < \infty$$

is not superfluous in Theorem 1.19(*e*).

1.21 A Comment on Terminology One frequently sees measure spaces referred to as "ordered triples" $(X, \mathfrak{M}, \mu)$ where X is a set, $\mathfrak{M}$ is a σ-algebra in X, and μ is a measure defined on $\mathfrak{M}$. Similarly, measurable spaces are "ordered pairs" $(X, \mathfrak{M})$.

This is logically all right, and often convenient, though somewhat redundant. For instance, in $(X, \mathfrak{M})$ the set X is merely the largest member of $\mathfrak{M}$, so if we know $\mathfrak{M}$ we also know X. Similarly, every measure has a σ-algebra for its domain, by definition, so if we know a measure μ we also know the σ-algebra $\mathfrak{M}$ on which μ is defined and we know the set X in which $\mathfrak{M}$ is a σ-algebra.

It is therefore perfectly legitimate to use expressions like "Let μ be a measure" or, if we wish to emphasize the σ-algebra or the set in question, to say "Let μ be a measure on $\mathfrak{M}$" or "Let μ be a measure on X."

What is logically rather meaningless but customary (and we shall often follow mathematical custom rather than logic) is to say "Let X be a measure space"; the emphasis should not be on the set, but on the measure. Of course, when this wording is used, it is tacitly understood that there is a measure defined on some σ-algebra in X and that it is this measure which is really under discussion.

Similarly, a topological space is an ordered pair (X, τ), where τ is a topology in the set X, and the significant data are contained in τ, not in X, but "the topological space X" is what one talks about.

This sort of tacit convention is used throughout mathematics. Most mathematical systems are sets with some class of distinguished subsets or some binary operations or some relations (which are required to have certain properties), and one can list these and then describe the system as an ordered pair, triple, etc., depending on what is needed. For instance, the real line may be described as a quadruple $(R^1, +, \cdot, <)$, where $+$, $\cdot$, and $<$ satisfy the axioms of a complete archimedean ordered field. But it is a safe bet that very few mathematicians think of the real field as an ordered quadruple.

Arithmetic in $[0, \infty]$

1.22 Throughout integration theory, one inevitably encounters ∞. One reason is that one wants to be able to integrate over sets of infinite measure; after all, the real line has infinite length. Another reason is that even if one is primarily interested in real-valued functions, the lim sup of a sequence of positive real functions or the sum of a sequence of positive real functions may well be ∞ at some points, and much of the elegance of theorems like 1.26 and 1.27 would be lost if one had to make some special provisions whenever this occurs.

Let us define $a + \infty = \infty + a = \infty$ if $0 \leq a \leq \infty$, and

$$a \cdot \infty = \infty \cdot a = \begin{cases} \infty & \text{if } 0 < a \leq \infty \\ 0 & \text{if } a = 0; \end{cases}$$

sums and products of real numbers are of course defined in the usual way.

It may seem strange to define $0 \cdot \infty = 0$. However, one verifies without difficulty that with this definition the *commutative, associative, and distributive laws hold in* $[0, \infty]$ *without any restriction.*

The cancellation laws have to be treated with some care: $a + b = a + c$ implies $b = c$ only when $a < \infty$, and $ab = ac$ implies $b = c$ only when $0 < a < \infty$.

Observe that the following useful proposition holds:

If $0 \leq a_1 \leq a_2 \leq \cdots$, $0 \leq b_1 \leq b_2 \leq \cdots$, $a_n \to a$, *and* $b_n \to b$, *then* $a_n b_n \to ab$.

If we combine this with Theorems 1.17 and 1.14, we see that *sums and products of measurable functions into* $[0, \infty]$ *are measurable.*

Integration of Positive Functions

In this section, $\mathfrak{M}$ will be a σ-algebra in a set X and μ will be a positive measure on $\mathfrak{M}$.

1.23 Definition If $s: X \to [0, \infty)$ is a measurable simple function, of the form

$$s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}, \tag{1}$$

where $\alpha_1, \ldots, \alpha_n$ are the distinct values of s (compare Definition 1.16), and if $E \in \mathfrak{M}$, we define

$$\int_E s \, d\mu = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E). \tag{2}$$

The convention $0 \cdot \infty = 0$ is used here; it may happen that $\alpha_i = 0$ for some i and that $\mu(A_i \cap E) = \infty$.

If $f: X \to [0, \infty]$ is measurable, and $E \in \mathfrak{M}$, we define

$$\int_E f \, d\mu = \sup \int_E s \, d\mu, \tag{3}$$

the supremum being taken over all simple measurable functions s such that $0 \leq s \leq f$.

The left member of (3) is called the *Lebesgue integral* of f over E, with respect to the measure μ. It is a number in $[0, \infty]$.

Observe that we apparently have two definitions for $\int_E f \, d\mu$ if f is simple, namely, (2) and (3). However, these assign the same value to the integral, since f is, in this case, the largest of the functions s which occur on the right of (3).

1.24 The following propositions are immediate consequences of the definitions. The functions and sets occurring in them are assumed to be measurable:

(*a*) *If* $0 \leq f \leq g$, *then* $\int_E f \, d\mu \leq \int_E g \, d\mu$.
(*b*) *If* $A \subset B$ *and* $f \geq 0$, *then* $\int_A f \, d\mu \leq \int_B f \, d\mu$.

(*c*) *If* $f \geq 0$ *and* c *is a constant,* $0 \leq c < \infty$, *then*

$$\int_E cf\, d\mu = c \int_E f\, d\mu.$$

(*d*) *If* $f(x) = 0$ *for all* $x \in E$, *then* $\int_E f\, d\mu = 0$, *even if* $\mu(E) = \infty$.
(*e*) *If* $\mu(E) = 0$, *then* $\int_E f\, d\mu = 0$, *even if* $f(x) = \infty$ *for every* $x \in E$.
(*f*) *If* $f \geq 0$, *then* $\int_E f\, d\mu = \int_X \chi_E f\, d\mu$.

This last result shows that we could have restricted our definition of integration to integrals over all of X, without losing any generality. If we wanted to integrate over subsets, we could then use (*f*) as the definition. It is purely a matter of taste which definition is preferred.

One may also remark here that every measurable subset E of a measure space X is again a measure space, in a perfectly natural way: The new measurable sets are simply those measurable subsets of X which lie in E, and the measure is unchanged, except that its domain is restricted. This shows again that as soon as we have integration defined over every measure space, we automatically have it defined over every measurable subset of every measure space.

1.25 Proposition *Let* s *and* t *be nonnegative measurable simple functions on* X. *For* $E \in \mathfrak{M}$, *define*

$$\varphi(E) = \int_E s\, d\mu. \tag{1}$$

Then φ *is a measure on* $\mathfrak{M}$. *Also*

$$\int_X (s + t)\, d\mu = \int_X s\, d\mu + \int_X t\, d\mu. \tag{2}$$

(This proposition contains provisional forms of Theorems 1.27 and 1.29.)

PROOF If s is as in Definition 1.23, and if $E_1, E_2, \ldots$ are disjoint members of $\mathfrak{M}$ whose union is E, the countable additivity of μ shows that

$$\varphi(E) = \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E) = \sum_{i=1}^{n} \alpha_i \sum_{r=1}^{\infty} \mu(A_i \cap E_r)$$

$$= \sum_{r=1}^{\infty} \sum_{i=1}^{n} \alpha_i \mu(A_i \cap E_r) = \sum_{r=1}^{\infty} \varphi(E_r).$$

Also, $\varphi(\varnothing) = 0$, so that φ is not identically ∞.

Next, let s be as before, let $\beta_1, \ldots, \beta_m$ be the distinct values of t, and let $B_j = \{x: t(x) = \beta_j\}$. If $E_{ij} = A_i \cap B_j$, then

$$\int_{E_{ij}} (s + t)\, d\mu = (\alpha_i + \beta_j)\mu(E_{ij})$$

and

$$\int_{E_{ij}} s\, d\mu + \int_{E_{ij}} t\, d\mu = \alpha_i \mu(E_{ij}) + \beta_j \mu(E_{ij}).$$

Thus (2) holds with E_{ij} in place of X. Since X is the disjoint union of the sets E_{ij} $(1 \leq i \leq n,\ 1 \leq j \leq m)$, the first half of our proposition implies that (2) holds. ////

We now come to the interesting part of the theory. One of its most remarkable features is the ease with which it handles limit operations.

1.26 Lebesgue's Monotone Convergence Theorem *Let $\{f_n\}$ be a sequence of measurable functions on X, and suppose that*

(a) $0 \leq f_1(x) \leq f_2(x) \leq \cdots \leq \infty$ *for every* $x \in X$,
(b) $f_n(x) \to f(x)$ *as* $n \to \infty$, *for every* $x \in X$.

Then f is measurable, and

$$\int_X f_n\, d\mu \to \int_X f\, d\mu \qquad \text{as } n \to \infty.$$

PROOF Since $\int f_n \leq \int f_{n+1}$, there exists an $\alpha \in [0, \infty]$ such that

$$\int_X f_n\, d\mu \to \alpha \qquad \text{as } n \to \infty. \tag{1}$$

By Theorem 1.14, f is measurable. Since $f_n \leq f$, we have $\int f_n \leq \int f$ for every n, so (1) implies

$$\alpha \leq \int_X f\, d\mu. \tag{2}$$

Let s be any simple measurable function such that $0 \leq s \leq f$, let c be a constant, $0 < c < 1$, and define

$$E_n = \{x: f_n(x) \geq cs(x)\} \qquad (n = 1, 2, 3, \ldots). \tag{3}$$

Each E_n is measurable, $E_1 \subset E_2 \subset E_3 \subset \cdots$, and $X = \bigcup E_n$. To see this equality, consider some $x \in X$. If $f(x) = 0$, then $x \in E_1$; if $f(x) > 0$, then $cs(x) < f(x)$, since $c < 1$; hence $x \in E_n$ for some n. Also

$$\int_X f_n\, d\mu \geq \int_{E_n} f_n\, d\mu \geq c \int_{E_n} s\, d\mu \qquad (n = 1, 2, 3, \ldots). \tag{4}$$

Let $n \to \infty$, applying Proposition 1.25 and Theorem 1.19(*d*) to the last integral in (4). The result is

$$\alpha \geq c \int_X s \, d\mu. \tag{5}$$

Since (5) holds for every $c < 1$, we have

$$\alpha \geq \int_X s \, d\mu \tag{6}$$

for every simple measurable s satisfying $0 \leq s \leq f$, so that

$$\alpha \geq \int_X f \, d\mu. \tag{7}$$

The theorem follows from (1), (2), and (7). ////

1.27 Theorem *If* $f_n: X \to [0, \infty]$ *is measurable, for* $n = 1, 2, 3, \ldots$, *and*

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \qquad (x \in X), \tag{1}$$

then

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu. \tag{2}$$

PROOF First, there are sequences $\{s_i'\}$, $\{s_i''\}$ of simple measurable functions such that $s_i' \to f_1$ and $s_i'' \to f_2$, as in Theorem 1.17. If $s_i = s_i' + s_i''$, then $s_i \to f_1 + f_2$, and the monotone convergence theorem, combined with Proposition 1.25, shows that

$$\int_X (f_1 + f_2) \, d\mu = \int_X f_1 \, d\mu + \int_X f_2 \, d\mu. \tag{3}$$

Next, put $g_N = f_1 + \cdots + f_N$. The sequence $\{g_N\}$ converges monotonically to f, and if we apply induction to (3) we see that

$$\int_X g_N \, d\mu = \sum_{n=1}^{N} \int_X f_n \, d\mu. \tag{4}$$

Applying the monotone convergence theorem once more, we obtain (2), and the proof is complete. ////

If we let μ be the counting measure on a countable set, Theorem 1.27 is a statement about double series of nonnegative real numbers (which can of course be proved by more elementary means):

Corollary *If* $a_{ij} \geq 0$ *for* i *and* $j = 1, 2, 3, \ldots$, *then*

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{ij} = \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} a_{ij}.$$

1.28 Fatou's Lemma *If* $f_n: X \to [0, \infty]$ *is measurable, for each positive integer* n, *then*

$$\int_X \left(\liminf_{n \to \infty} f_n \right) d\mu \leq \liminf_{n \to \infty} \int_X f_n \, d\mu. \tag{1}$$

Strict inequality can occur in (1); see Exercise 8.

PROOF Put

$$g_k(x) = \inf_{i \geq k} f_i(x) \qquad (k = 1, 2, 3, \ldots; x \in X). \tag{2}$$

Then $g_k \leq f_k$, so that

$$\int_X g_k \, d\mu \leq \int_X f_k \, d\mu \qquad (k = 1, 2, 3, \ldots). \tag{3}$$

Also, $0 \leq g_1 \leq g_2 \leq \cdots$, each g_k is measurable, by Theorem 1.14, and $g_k(x) \to \liminf f_n(x)$ as $k \to \infty$, by Definition 1.13. The monotone convergence theorem shows therefore that the left side of (3) tends to the left side of (1), as $k \to \infty$. Hence (1) follows from (3). ////

1.29 Theorem *Suppose* $f: X \to [0, \infty]$ *is measurable, and*

$$\varphi(E) = \int_E f \, d\mu \qquad (E \in \mathfrak{M}). \tag{1}$$

Then φ *is a measure on* $\mathfrak{M}$, *and*

$$\int_X g \, d\varphi = \int_X gf \, d\mu \tag{2}$$

for every measurable g *on* X *with range in* $[0, \infty]$.

PROOF Let $E_1, E_2, E_3, \ldots$ be disjoint members of $\mathfrak{M}$ whose union is E. Observe that

$$\chi_E f = \sum_{j=1}^{\infty} \chi_{E_j} f \tag{3}$$

and that

$$\varphi(E) = \int_X \chi_E f \, d\mu, \qquad \varphi(E_j) = \int_X \chi_{E_j} f \, d\mu. \tag{4}$$

It now follows from Theorem 1.27 that

$$\varphi(E) = \sum_{j=1}^{\infty} \varphi(E_j). \tag{5}$$

Since $\varphi(\varnothing) = 0$, (5) proves that φ is a measure.

Next, (1) shows that (2) holds whenever $g = \chi_E$ for some $E \in \mathfrak{M}$. Hence (2) holds for every simple measurable function g, and the general case follows from the monotone convergence theorem. ////

Remark The second assertion of Theorem 1.29 is sometimes written in the form

$$d\varphi = f\, d\mu. \tag{6}$$

We assign no independent meaning to the symbols $d\varphi$ and $d\mu$; (6) merely means that (2) holds for every measurable $g \geq 0$.

Theorem 1.29 has a very important converse, the Radon-Nikodym theorem, which will be proved in Chap. 6.

Integration of Complex Functions

As before, μ will in this section be a positive measure on an arbitrary measurable space X.

1.30 Definition We define $L^1(\mu)$ to be the collection of all complex measurable functions f on X for which

$$\int_X |f|\, d\mu < \infty.$$

Note that the measurability of f implies that of $|f|$, as we saw in Proposition 1.9(*b*); hence the above integral is defined.

The members of $L^1(\mu)$ are called *Lebesgue integrable* functions (with respect to μ) or *summable functions*. The significance of the exponent 1 will become clear in Chap. 3.

1.31 Definition If $f = u + iv$, where u and v are real measurable functions on X, and if $f \in L^1(\mu)$, we define

$$\int_E f\, d\mu = \int_E u^+\, d\mu - \int_E u^-\, d\mu + i \int_E v^+\, d\mu - i \int_E v^-\, d\mu \tag{1}$$

for every measurable set E.

Here u^+ and u^- are the positive and negative parts of u, as defined in Sec. 1.15; v^+ and v^- are similarly obtained from v. These four functions are measurable, real, and nonnegative; hence the four integrals on the right of (1) exist, by Definition 1.23. Furthermore, we have $u^+ \leq |u| < |f|$, etc., so that

each of these four integrals is finite. Thus (1) defines the integral on the left as a complex number.

Occasionally it is desirable to define the integral of a measurable function f with range in $[-\infty, \infty]$ to be

$$\int_E f\,d\mu = \int_E f^+\,d\mu - \int_E f^-\,d\mu, \tag{2}$$

provided that at least one of the integrals on the right of (2) is finite. The left side of (2) is then a number in $[-\infty, \infty]$.

1.32 Theorem *Suppose f and $g \in L^1(\mu)$ and α and β are complex numbers. Then $\alpha f + \beta g \in L^1(\mu)$, and*

$$\int_X (\alpha f + \beta g)\,d\mu = \alpha \int_X f\,d\mu + \beta \int_X g\,d\mu. \tag{1}$$

PROOF The measurability of $\alpha f + \beta g$ follows from Proposition 1.9(*c*). By Sec. 1.24 and Theorem 1.27,

$$\begin{aligned}\int_X |\alpha f + \beta g|\,d\mu &\le \int_X (|\alpha|\,|f| + |\beta|\,|g|)\,d\mu \\ &= |\alpha| \int_X |f|\,d\mu + |\beta| \int_X |g|\,d\mu < \infty.\end{aligned}$$

Thus $\alpha f + \beta g \in L^1(\mu)$.

To prove (1), it is clearly sufficient to prove

$$\int_X (f + g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu \tag{2}$$

and

$$\int_X (\alpha f)\,d\mu = \alpha \int_X f\,d\mu, \tag{3}$$

and the general case of (2) will follow if we prove (2) for real f and g in $L^1(\mu)$.

Assuming this, and setting $h = f + g$, we have

$$h^+ - h^- = f^+ - f^- + g^+ - g^-$$

or

$$h^+ + f^- + g^- = f^+ + g^+ + h^-. \tag{4}$$

By Theorem 1.27,

$$\int h^+ + \int f^- + \int g^- = \int f^+ + \int g^+ + \int h^-, \tag{5}$$

and since each of these integrals is finite, we may transpose and obtain (2).

That (3) holds if $\alpha \geq 0$ follows from Proposition 1.24(*c*). It is easy to verify that (3) holds if $\alpha = -1$, using relations like $(-u)^+ = u^-$. The case $\alpha = i$ is also easy: If $f = u + iv$, then

$$\int (if) = \int (iu - v) = \int (-v) + i \int u = -\int v + i \int u = i\left(\int u + i \int v\right)$$
$$= i \int f.$$

Combining these cases with (2), we obtain (3) for any complex α. ////

1.33 Theorem *If $f \in L^1(\mu)$, then*

$$\left| \int_X f \, d\mu \right| \leq \int_X |f| \, d\mu.$$

PROOF Put $z = \int_X f \, d\mu$. Since z is a complex number, there is a complex number α, with $|\alpha| = 1$, such that $\alpha z = |z|$. Let u be the real part of αf. Then $u \leq |\alpha f| = |f|$. Hence

$$\left| \int_X f \, d\mu \right| = \alpha \int_X f \, d\mu = \int_X \alpha f \, d\mu = \int_X u \, d\mu \leq \int_X |f| \, d\mu.$$

The third of the above equalities holds since the preceding ones show that $\int \alpha f \, d\mu$ is real. ////

We conclude this section with another important convergence theorem.

1.34 Lebesgue's Dominated Convergence Theorem *Suppose $\{f_n\}$ is a sequence of complex measurable functions on X such that*

$$f(x) = \lim_{n \to \infty} f_n(x) \tag{1}$$

exists for every $x \in X$. If there is a function $g \in L^1(\mu)$ such that

$$|f_n(x)| \leq g(x) \qquad (n = 1, 2, 3, \ldots; x \in X), \tag{2}$$

then $f \in L^1(\mu)$,

$$\lim_{n \to \infty} \int_X |f_n - f| \, d\mu = 0, \tag{3}$$

and

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu. \tag{4}$$

PROOF Since $|f| \leq g$ and f is measurable, $f \in L^1(\mu)$. Since $|f_n - f| \leq 2g$, Fatou's lemma applies to the functions $2g - |f_n - f|$ and yields

$$\begin{aligned}\int_X 2g\, d\mu &\leq \liminf_{n\to\infty} \int_X (2g - |f_n - f|)\, d\mu \\ &= \int_X 2g\, d\mu + \liminf_{n\to\infty} \left(- \int_X |f_n - f|\, d\mu \right) \\ &= \int_X 2g\, d\mu - \limsup_{n\to\infty} \int_X |f_n - f|\, d\mu.\end{aligned}$$

Since $\int 2g\, d\mu$ is finite, we may subtract it and obtain

$$\limsup_{n\to\infty} \int_X |f_n - f|\, d\mu \leq 0. \tag{5}$$

If a sequence of nonnegative real numbers fails to converge to 0, then its upper limit is positive. Thus (5) implies (3). By Theorem 1.33, applied to $f_n - f$, (3) implies (4). ////

The Role Played by Sets of Measure Zero

1.35 Definition Let P be a property which a point x may or may not have. For instance, P might be the property "$f(x) > 0$" if f is a given function, or it might be "$\{f_n(x)\}$ converges" if $\{f_n\}$ is a given sequence of functions.

If μ is a measure on a σ-algebra $\mathfrak{M}$ and if $E \in \mathfrak{M}$, the statement "P holds almost everywhere on E" (abbreviated to "P holds a.e. on E") means that there exists an $N \in \mathfrak{M}$ such that $\mu(N) = 0$, $N \subset E$, and P holds at every point of $E - N$. This concept of a.e. depends of course very strongly on the given measure, and we shall write "a.e. $[\mu]$" whenever clarity requires that the measure be indicated.

For example, if f and g are measurable functions and if

$$\mu(\{x : f(x) \neq g(x)\}) = 0, \tag{1}$$

we say that $f = g$ a.e. $[\mu]$ on X, and we may write $f \sim g$. This is easily seen to be an equivalence relation. The transitivity ($f \sim g$ and $g \sim h$ implies $f \sim h$) is a consequence of the fact that the union of two sets of measure 0 has measure 0.

Note that if $f \sim g$, then, for every $E \in \mathfrak{M}$,

$$\int_E f\, d\mu = \int_E g\, d\mu. \tag{2}$$

To see this, let N be the set which appears in (1); then E is the union of the disjoint sets $E - N$ and $E \cap N$; on $E - N$, $f = g$, and $\mu(E \cap N) = 0$.

Thus, generally speaking, sets of measure 0 are negligible in integration. It ought to be true that every subset of a negligible set is negligible. But it may happen that some set $N \in \mathfrak{M}$ with $\mu(N) = 0$ has a subset E which is not a member of $\mathfrak{M}$. Of course we can *define* $\mu(E) = 0$ in this case. But will this extension of μ still be a measure, i.e., will it still be defined on a σ-algebra? It is a pleasant fact that the answer is affirmative:

1.36 Theorem *Let $(X, \mathfrak{M}, \mu)$ be a measure space, let $\mathfrak{M}^*$ be the collection of all $E \subset X$ for which there exist sets A and $B \in \mathfrak{M}$ such that $A \subset E \subset B$ and $\mu(B - A) = 0$, and define $\mu(E) = \mu(A)$ in this situation. Then $\mathfrak{M}^*$ is a σ-algebra, and μ is a measure on $\mathfrak{M}^*$.*

This extended measure μ is called *complete*, since all subsets of sets of measure 0 are now measurable; the σ-algebra $\mathfrak{M}^*$ is called the *μ-completion* of $\mathfrak{M}$. The theorem says that every measure can be completed, so, whenever it is convenient, we may assume that any given measure is complete; this just gives us more measurable sets, hence more measurable functions. Most measures that one meets in the ordinary course of events are already complete, but there are exceptions; one of these will occur in the proof of Fubini's theorem in Chap. 8.

PROOF We begin by checking that μ is well defined for every $E \in \mathfrak{M}^*$. Suppose $A \subset E \subset B$, $A_1 \subset E \subset B_1$, and $\mu(B - A) = \mu(B_1 - A_1) = 0$. (The letters A and B will denote members of $\mathfrak{M}$ throughout this proof.) Since

$$A - A_1 \subset E - A_1 \subset B_1 - A_1$$

we have $\mu(A - A_1) = 0$, hence $\mu(A) = \mu(A \cap A_1)$. For the same reason, $\mu(A_1) = \mu(A_1 \cap A)$. We conclude that indeed $\mu(A_1) = \mu(A)$.

Next, let us verify that $\mathfrak{M}^*$ has the three defining properties of a σ-algebra.

(i) $X \in \mathfrak{M}^*$, because $X \in \mathfrak{M}$ and $\mathfrak{M} \subset \mathfrak{M}^*$.

(ii) If $A \subset E \subset B$ then $B^c \subset E^c \subset A^c$. Thus $E \in \mathfrak{M}^*$ implies $E^c \in \mathfrak{M}^*$, because $A^c - B^c = A^c \cap B = B - A$.

(iii) If $A_i \subset E_i \subset B_i$, $E = \bigcup E_i$, $A = \bigcup A_i$, $B = \bigcup B_i$, then $A \subset E \subset B$ and

$$B - A = \bigcup_1^\infty (B_i - A) \subset \bigcup_1^\infty (B_i - A_i).$$

Since countable unions of sets of measure zero have measure zero, it follows that $E \in \mathfrak{M}^*$ if $E_i \in \mathfrak{M}^*$ for $i = 1, 2, 3, \ldots$.

Finally, if the sets E_i are disjoint in step (iii), the same is true of the sets A_i, and we conclude that

$$\mu(E) = \mu(A) = \sum_1^\infty \mu(A_i) = \sum_1^\infty \mu(E_i).$$

This proves that μ is countably additive on $\mathfrak{M}^*$. ////

1.37 The fact that functions which are equal a.e. are indistinguishable as far as integration is concerned suggests that our definition of measurable function might profitably be enlarged. Let us call a function f defined on a set $E \in \mathfrak{M}$ *measurable on* X if $\mu(E^c) = 0$ and if $f^{-1}(V) \cap E$ is measurable for every open set V. If we define $f(x) = 0$ for $x \in E^c$, we obtain a measurable function on X, in the old sense. If our measure happens to be complete, we can define f on E^c in a perfectly arbitrary manner, and we still get a measurable function. The integral of f over any set $A \in \mathfrak{M}$ is independent of the definition of f on E^c; therefore this definition need not even be specified at all.

There are many situations where this occurs naturally. For instance, a function f on the real line may be differentiable only almost everywhere (with respect to Lebesgue measure), but under certain conditions it is still true that f is the integral of its derivative; this will be discussed in Chap. 7. Or a sequence $\{f_n\}$ of measurable functions on X may converge only almost everywhere; with our new definition of measurability, the limit is still a measurable function on X, and we do not have to cut down to the set on which convergence actually occurs.

To illustrate, let us state a corollary of Lebesgue's dominated convergence theorem in a form in which exceptional sets of measure zero are admitted:

1.38 Theorem *Suppose $\{f_n\}$ is a sequence of complex measurable functions defined a.e. on X such that*

$$\sum_{n=1}^{\infty} \int_X |f_n| \, d\mu < \infty. \tag{1}$$

Then the series

$$f(x) = \sum_{n=1}^{\infty} f_n(x) \tag{2}$$

converges for almost all x, $f \in L^1(\mu)$, and

$$\int_X f \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu. \tag{3}$$

PROOF Let S_n be the set on which f_n is defined, so that $\mu(S_n^c) = 0$. Put $\varphi(x) = \sum |f_n(x)|$, for $x \in S = \bigcap S_n$. Then $\mu(S^c) = 0$. By (1) and Theorem 1.27,

$$\int_S \varphi \, d\mu < \infty. \tag{4}$$

If $E = \{x \in S: \varphi(x) < \infty\}$, it follows from (4) that $\mu(E^c) = 0$. The series (2) converges absolutely for every $x \in E$, and if $f(x)$ is defined by (2) for $x \in E$, then $|f(x)| \leq \varphi(x)$ on E, so that $f \in L^1(\mu)$ on E, by (4). If $g_n = f_1 + \cdots + f_n$, then $|g_n| \leq \varphi$, $g_n(x) \to f(x)$ for all $x \in E$, and Theorem 1.34 gives (3) with E in place of X. This is equivalent to (3), since $\mu(E^c) = 0$. ////

Note that even if the f_n were defined at *every* point of X, (1) would only imply that (2) converges *almost everywhere.* Here are some other situations in which we can draw conclusions only almost everywhere:

1.39 Theorem

(*a*) *Suppose* $f: X \to [0, \infty]$ *is measurable,* $E \in \mathfrak{M}$, *and* $\int_E f\,d\mu = 0$. *Then* $f = 0$ *a.e. on* E.
(*b*) *Suppose* $f \in L^1(\mu)$ *and* $\int_E f\,d\mu = 0$ *for every* $E \in \mathfrak{M}$. *Then* $f = 0$ *a.e. on* X.
(*c*) *Suppose* $f \in L^1(\mu)$ *and*

$$\left| \int_X f\,d\mu \right| = \int_X |f|\,d\mu.$$

Then there is a constant α *such that* $\alpha f = |f|$ *a.e. on* X.

Note that (*c*) describes the condition under which equality holds in Theorem 1.33.

PROOF

(*a*) If $A_n = \{x \in E: f(x) > 1/n\}$, $n = 1, 2, 3, \ldots$, then

$$\frac{1}{n}\,\mu(A_n) \leq \int_{A_n} f\,d\mu \leq \int_E f\,d\mu = 0,$$

so that $\mu(A_n) = 0$. Since $\{x \in E: f(x) > 0\} = \bigcup A_n$, (*a*) follows.

(*b*) Put $f = u + iv$, let $E = \{x: u(x) \geq 0\}$. The real part of $\int_E f\,d\mu$ is then $\int_E u^+\,d\mu$. Hence $\int_E u^+\,d\mu = 0$, and (*a*) implies that $u^+ = 0$ a.e. We conclude similarly that

$$u^- = v^+ = v^- = 0 \qquad \text{a.e.}$$

(*c*) Examine the proof of Theorem 1.33. Our present assumption implies that the last inequality in the proof of Theorem 1.33 must actually be an equality. Hence $\int (|f| - u)\,d\mu = 0$. Since $|f| - u \geq 0$, (*a*) shows that $|f| = u$ a.e. This says that the real part of αf is equal to $|\alpha f|$ a.e., hence $\alpha f = |\alpha f| = |f|$ a.e., which is the desired conclusion. ////

1.40 Theorem *Suppose* $\mu(X) < \infty$, $f \in L^1(\mu)$, S *is a closed set in the complex plane, and the averages*

$$A_E(f) = \frac{1}{\mu(E)} \int_E f\,d\mu$$

lie in S *for every* $E \in \mathfrak{M}$ *with* $\mu(E) > 0$. *Then* $f(x) \in S$ *for almost all* $x \in X$.

PROOF Let Δ be a closed circular disc (with center at α and radius $r > 0$, say) in the complement of S. Since S^c is the union of countably many such discs, it is enough to prove that $\mu(E) = 0$, where $E = f^{-1}(\Delta)$.

If we had $\mu(E) > 0$, then

$$|A_E(f) - \alpha| = \frac{1}{\mu(E)} \left| \int_E (f - \alpha)\, d\mu \right| \leq \frac{1}{\mu(E)} \int_E |f - \alpha|\, d\mu \leq r,$$

which is impossible, since $A_E(f) \in S$. Hence $\mu(E) = 0$. ////

1.41 Theorem *Let $\{E_k\}$ be a sequence of measurable sets in X, such that*

$$\sum_{k=1}^{\infty} \mu(E_k) < \infty. \tag{1}$$

Then almost all $x \in X$ lie in at most finitely many of the sets E_k.

PROOF If A is the set of all x which lie in infinitely many E_k, we have to prove that $\mu(A) = 0$. Put

$$g(x) = \sum_{k=1}^{\infty} \chi_{E_k}(x) \qquad (x \in X). \tag{2}$$

For each x, each term in this series is either 0 or 1. Hence $x \in A$ if and only if $g(x) = \infty$. By Theorem 1.27, the integral of g over X is equal to the sum in (1). Thus $g \in L^1(\mu)$, and so $g(x) < \infty$ a.e. ////

Exercises

1 Does there exist an infinite σ-algebra which has only countably many members?

2 Prove an analogue of Theorem 1.8 for n functions.

3 Prove that if f is a real function on a measurable space X such that $\{x: f(x) \geq r\}$ is measurable for every rational r, then f is measurable.

4 Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, \infty]$, and prove the following assertions:

(*a*) $$\limsup_{n \to \infty} (-a_n) = -\liminf_{n \to \infty} a_n.$$

(*b*) $$\limsup_{n \to \infty} (a_n + b_n) \leq \limsup_{n \to \infty} a_n + \limsup_{n \to \infty} b_n$$

provided none of the sums is of the form $\infty - \infty$.

(*c*) If $a_n \leq b_n$ for all n, then

$$\liminf_{n \to \infty} a_n \leq \liminf_{n \to \infty} b_n.$$

Show by an example that strict inequality can hold in (*b*).

5 (*a*) Suppose $f: X \to [-\infty, \infty]$ and $g: X \to [-\infty, \infty]$ are measurable. Prove that the sets

$$\{x: f(x) < g(x)\}, \{x: f(x) = g(x)\}$$

are measurable.

(*b*) Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

6 Let X be an uncountable set, let $\mathfrak{M}$ be the collection of all sets $E \subset X$ such that either E or E^c is at most countable, and define $\mu(E) = 0$ in the first case, $\mu(E) = 1$ in the second. Prove that $\mathfrak{M}$ is a σ-algebra in X and that μ is a measure on $\mathfrak{M}$. Describe the corresponding measurable functions and their integrals.

7 Suppose $f_n: X \to [0, \infty]$ is measurable for $n = 1, 2, 3, \ldots$, $f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0$, $f_n(x) \to f(x)$ as $n \to \infty$, for every $x \in X$, and $f_1 \in L^1(\mu)$. Prove that then

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu$$

and show that this conclusion does *not* follow if the condition "$f_1 \in L^1(\mu)$" is omitted.

8 Put $f_n = \chi_E$ if n is odd, $f_n = 1 - \chi_E$ if n is even. What is the relevance of this example to Fatou's lemma?

9 Suppose μ is a positive measure on X, $f: X \to [0, \infty]$ is measurable, $\int_X f \, d\mu = c$, where $0 < c < \infty$, and α is a constant. Prove that

$$\lim_{n \to \infty} \int_X n \log [1 + (f/n)^\alpha] \, d\mu = \begin{cases} \infty & \text{if } 0 < \alpha < 1, \\ c & \text{if } \alpha = 1, \\ 0 & \text{if } 1 < \alpha < \infty. \end{cases}$$

Hint: If $\alpha \geq 1$, the integrands are dominated by αf. If $\alpha < 1$, Fatou's lemma can be applied.

10 Suppose $\mu(X) < \infty$, $\{f_n\}$ is a sequence of bounded complex measurable functions on X, and $f_n \to f$ uniformly on X. Prove that

$$\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu,$$

and show that the hypothesis "$\mu(X) < \infty$" cannot be omitted.

11 Show that

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

in Theorem 1.41, and hence prove the theorem without any reference to integration.

12 Suppose $f \in L^1(\mu)$. Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| \, d\mu < \epsilon$ whenever $\mu(E) < \delta$.

13 Show that proposition 1.24(*c*) is also true when $c = \infty$.

CHAPTER

TWO

POSITIVE BOREL MEASURES

Vector Spaces

2.1 Definition A *complex vector space* (or a vector space over the complex field) is a set V, whose elements are called *vectors* and in which two operations, called *addition* and *scalar multiplication*, are defined, with the following familiar algebraic properties:

To every pair of vectors x and y there corresponds a vector $x + y$, in such a way that $x + y = y + x$ and $x + (y + z) = (x + y) + z$; V contains a unique vector 0 (the *zero vector* or *origin* of V) such that $x + 0 = x$ for every $x \in V$; and to each $x \in V$ there corresponds a unique vector $-x$ such that $x + (-x) = 0$.

To each pair (α, x), where $x \in V$ and α is a scalar (in this context, the word *scalar* means *complex number*), there is associated a vector $\alpha x \in V$, in such a way that $1x = x$, $\alpha(\beta x) = (\alpha\beta)x$, and such that the two distributive laws

$$\alpha(x + y) = \alpha x + \alpha y, \quad (\alpha + \beta)x = \alpha x + \beta x \tag{1}$$

hold.

A *linear transformation* of a vector space V into a vector space V_1 is a mapping Λ of V into V_1 such that

$$\Lambda(\alpha x + \beta y) = \alpha\Lambda x + \beta\Lambda y \tag{2}$$

for all x and $y \in V$ and for all scalars α and β. In the special case in which V_1 is the field of scalars (this is the simplest example of a vector space, except for the trivial one consisting of 0 alone), Λ is called a *linear functional*. A linear functional is thus a complex function on V which satisfies (2).

Note that one often writes Λx, rather than $\Lambda(x)$, if Λ is linear.

The preceding definitions can of course be made equally well with any field whatsoever in place of the complex field. Unless the contrary is explicitly stated, however, all vector spaces occurring in this book will be complex, with one notable exception: the euclidean spaces R^k are vector spaces over the *real* field.

2.2 Integration as a Linear Functional Analysis is full of vector spaces and linear transformations, and there is an especially close relationship between integration on the one hand and linear functionals on the other.

For instance, Theorem 1.32 shows that $L^1(\mu)$ is a vector space, for any positive measure μ, and that the mapping

$$f \to \int_X f \, d\mu \tag{1}$$

is a linear functional on $L^1(\mu)$. Similarly, if g is any bounded measurable function, the mapping

$$f \to \int_X fg \, d\mu \tag{2}$$

is a linear functional on $L^1(\mu)$; we shall see in Chap. 6 that the functionals (2) are, in a sense, the only interesting ones on $L^1(\mu)$.

For another example, let C be the set of all continuous complex functions on the unit interval $I = [0, 1]$. The sum of the two continuous functions is continuous, and so is any scalar multiple of a continuous function. Hence C is a vector space, and if

$$\Lambda f = \int_0^1 f(x) \, dx \qquad (f \in C), \tag{3}$$

the integral being the ordinary Riemann integral, then Λ is clearly a linear functional on C; Λ has an additional interesting property: it is a *positive linear functional.* This means that $\Lambda f \geq 0$ whenever $f \geq 0$.

One of the tasks which is still ahead of us is the construction of the Lebesgue measure. The construction can be based on the linear functional (3), by the following observation: Consider a segment $(a, b) \subset I$ and consider the class of all $f \in C$ such that $0 \leq f \leq 1$ on I and $f(x) = 0$ for all x not in (a, b). We have $\Lambda f < b - a$ for all such f, but we can choose f so that Λf is as close to $b - a$ as desired. Thus the length (or measure) of (a, b) is intimately related to the values of the functional Λ.

The preceding observation, when looked at from a more general point of view, leads to a remarkable and extremely important theorem of F. Riesz:

To every positive linear functional Λ on C corresponds a finite positive Borel measure μ on I such that

$$\Lambda f = \int_I f \, d\mu \qquad (f \in C). \tag{4}$$

[The converse is obvious: if μ is a finite positive Borel measure on I and if Λ is defined by (4), then Λ is a positive linear functional on C.]

It is clearly of interest to replace the bounded interval I by R^1. We can do this by restricting attention to those continuous functions on R^1 which vanish outside some bounded interval. (These functions are Riemann integrable, for instance.) Next, functions of several variables occur frequently in analysis. Thus we ought to move from R^1 to R^n. It turns out that the proof of the Riesz theorem still goes through, with hardly any changes. Moreover, it turns out that the euclidean properties of R^n (coordinates, orthogonality, etc.) play no role in the proof; in fact, if one thinks of them too much they just get in the way. Essential to the proof are certain *topological* properties of R^n. (Naturally. We are now dealing with *continuous* functions.) The crucial property is that of *local compactness*: Each point of R^n has a neighborhood whose closure is compact.

We shall therefore establish the Riesz theorem in a very general setting (Theorem 2.14). The existence of Lebesgue measure follows then as a special case. Those who wish to concentrate on a more concrete situation may skip lightly over the following section on topological preliminaries (Urysohn's lemma is the item of greatest interest there; see Exercise 3) and may replace locally compact Hausdorff spaces by locally compact metric spaces, or even by euclidean spaces, without missing any of the principal ideas.

It should also be mentioned that there are situations, especially in probability theory, where measures occur naturally on spaces without topology, or on topological spaces that are not locally compact. An example is the so-called Wiener measure which assigns numbers to certain sets of continuous functions and which is a basic tool in the study of Brownian motion. These topics will not be discussed in this book.

Topological Preliminaries

2.3 Definitions Let X be a topological space, as defined in Sec. 1.2.

(*a*) A set $E \subset X$ is *closed* if its complement E^c is open. (Hence $\varnothing$ and X are closed, finite unions of closed sets are closed, and arbitrary intersections of closed sets are closed.)

(*b*) The *closure* $\bar{E}$ of a set $E \subset X$ is the smallest closed set in X which contains E. (The following argument proves the existence of $\bar{E}$: The collection Ω of all closed subsets of X which contain E is not empty, since $X \in \Omega$; let $\bar{E}$ be the intersection of all members of Ω.)

(*c*) A set $K \subset X$ is *compact* if every open cover of K contains a finite subcover. More explicitly, the requirement is that if $\{V_\alpha\}$ is a collection of open sets whose union contains K, then the union of some finite subcollection of $\{V_\alpha\}$ also contains K.
In particular, if X is itself compact, then X is called a *compact space.*

(*d*) A *neighborhood* of a point $p \in X$ is any open subset of X which contains p. (The use of this term is not quite standardized; some use

"neighborhood of p" for any set which contains an open set containing p.)

(*e*) X is a *Hausdorff space* if the following is true: If $p \in X$, $q \in X$, and $p \neq q$, then p has a neighborhood U and q has a neighborhood V such that $U \cap V = \varnothing$.

(*f*) X is *locally compact* if every point of X has a neighborhood whose closure is compact.

Obviously, every compact space is locally compact.

We recall the Heine-Borel theorem: *The compact subsets of a euclidean space R^n are precisely those that are closed and bounded* ([26],† Theorem 2.41). From this it follows easily that R^n is a locally compact Hausdorff space. Also, every metric space is a Hausdorff space.

2.4 Theorem *Suppose K is compact and F is closed, in a topological space X. If $F \subset K$, then F is compact.*

PROOF If $\{V_\alpha\}$ is an open cover of F and $W = F^c$, then $W \cup \bigcup_\alpha V_\alpha$ covers X; hence there is a finite collection $\{V_{\alpha_i}\}$ such that

$$K \subset W \cup V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}.$$

Then $F \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. ////

Corollary *If $A \subset B$ and if B has compact closure, so does A.*

2.5 Theorem *Suppose X is a Hausdorff space, $K \subset X$, K is compact, and $p \in K^c$. Then there are open sets U and W such that $p \in U$, $K \subset W$, and $U \cap W = \varnothing$.*

PROOF If $q \in K$, the Hausdorff separation axiom implies the existence of disjoint open sets U_q and V_q, such that $p \in U_q$ and $q \in V_q$. Since K is compact, there are points $q_1, \ldots, q_n \in K$ such that

$$K \subset V_{q_1} \cup \cdots \cup V_{q_n}.$$

Our requirements are then satisfied by the sets

$$U = U_{q_1} \cap \cdots \cap U_{q_n} \quad \text{and} \quad W = V_{q_1} \cup \cdots \cup V_{q_n}. \qquad ////$$

Corollaries

(*a*) *Compact subsets of Hausdorff spaces are closed.*

(*b*) *If F is closed and K is compact in a Hausdorff space, then $F \cap K$ is compact.*

Corollary (*b*) follows from (*a*) and Theorem 2.4.

† Numbers in brackets refer to the Bibliography.

2.6 Theorem *If $\{K_\alpha\}$ is a collection of compact subsets of a Hausdorff space and if $\bigcap_\alpha K_\alpha = \varnothing$, then some finite subcollection of $\{K_\alpha\}$ also has empty intersection.*

PROOF Put $V_\alpha = K_\alpha^c$. Fix a member K_1 of $\{K_\alpha\}$. Since no point of K_1 belongs to every K_α, $\{V_\alpha\}$ is an open cover of K_1. Hence $K_1 \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$ for some finite collection $\{V_{\alpha_i}\}$. This implies that

$$K_1 \cap K_{\alpha_1} \cap \cdots \cap K_{\alpha_n} = \varnothing. \qquad ////$$

2.7 Theorem *Suppose U is open in a locally compact Hausdorff space X, $K \subset U$, and K is compact. Then there is an open set V with compact closure such that*

$$K \subset V \subset \bar{V} \subset U.$$

PROOF Since every point of K has a neighborhood with compact closure, and since K is covered by the union of finitely many of these neighborhoods, K lies in an open set G with compact closure. If $U = X$, take $V = G$.

Otherwise, let C be the complement of U. Theorem 2.5 shows that to each $p \in C$ there corresponds an open set W_p such that $K \subset W_p$ and $p \notin \overline{W_p}$. Hence $\{C \cap \bar{G} \cap \overline{W_p}\}$, where p ranges over C, is a collection of compact sets with empty intersection. By Theorem 2.6 there are points $p_1, \ldots, p_n \in C$ such that

$$C \cap \bar{G} \cap \overline{W_{p_1}} \cap \cdots \cap \overline{W_{p_n}} = \varnothing.$$

The set

$$V = G \cap W_{p_1} \cap \cdots \cap W_{p_n}$$

then has the required properties, since

$$\bar{V} \subset \bar{G} \cap \overline{W_{p_1}} \cap \cdots \cap \overline{W_{p_n}}. \qquad ////$$

2.8 Definition Let f be a real (or extended-real) function on a topological space. If

$$\{x : f(x) > \alpha\}$$

is open for every real α, f is said to be *lower semicontinuous*. If

$$\{x : f(x) < \alpha\}$$

is open for every real α, f is said to be *upper semicontinuous*.

A real function is obviously continuous if and only if it is both upper and lower semicontinuous.

The simplest examples of semicontinuity are furnished by characteristic functions:

(*a*) Characteristic functions of *open* sets are *lower* semicontinuous.
(*b*) Characteristic functions of *closed* sets are *upper* semicontinuous.

The following property is an almost immediate consequence of the definitions:

(*c*) *The supremum of any collection of lower semicontinuous functions is lower semicontinuous. The infimum of any collection of upper semicontinuous functions is upper semicontinuous.*

2.9 Definition The *support* of a complex function f on a topological space X is the closure of the set

$$\{x: f(x) \neq 0\}.$$

The collection of all continuous complex functions on X whose support is compact is denoted by $C_c(X)$.

Observe that $C_c(X)$ is a vector space. This is due to two facts:

(*a*) The support of $f + g$ lies in the union of the support of f and the support of g, and any finite union of compact sets is compact.
(*b*) The sum of two continuous complex functions is continuous, as are scalar multiples of continuous functions.

(Statement and proof of Theorem 1.8 hold verbatim if "measurable function" is replaced by "continuous function," "measurable space" by "topological space"; take $\Phi(s, t) = s + t$, or $\Phi(s, t) = st$, to prove that sums and products of continuous functions are continuous.)

2.10 Theorem *Let X and Y be topological spaces, and let $f: X \to Y$ be continuous. If K is a compact subset of X, then $f(K)$ is compact.*

PROOF If $\{V_\alpha\}$ is an open cover of $f(K)$, then $\{f^{-1}(V_\alpha)\}$ is an open cover of K, hence $K \subset f^{-1}(V_{\alpha_1}) \cup \cdots \cup f^{-1}(V_{\alpha_n})$ for some $\alpha_1, \ldots, \alpha_n$, and therefore $f(K) \subset V_{\alpha_1} \cup \cdots \cup V_{\alpha_n}$. ////

Corollary *The range of any $f \in C_c(X)$ is a compact subset of the complex plane.*

In fact, if K is the support of $f \in C_c(X)$, then $f(X) \subset f(K) \cup \{0\}$. If X is not compact, then $0 \in f(X)$, but 0 need not lie in $f(K)$, as is seen by easy examples.

2.11 Notation In this chapter the following conventions will be used. The notation

$$K \prec f \tag{1}$$

will mean that K is a compact subset of X, that $f \in C_c(X)$, that $0 \leq f(x) \leq 1$ for all $x \in X$, and that $f(x) = 1$ for all $x \in K$. The notation

$$f \prec V \tag{2}$$

will mean that V is open, that $f \in C_c(X)$, $0 \leq f \leq 1$, and that the support of f lies in V. The notation

$$K \prec f \prec V \tag{3}$$

will be used to indicate that both (1) and (2) hold.

2.12 Urysohn's Lemma *Suppose X is a locally compact Hausdorff space, V is open in X, $K \subset V$, and K is compact. Then there exists an $f \in C_c(X)$, such that*

$$K \prec f \prec V. \tag{1}$$

In terms of characteristic functions, the conclusion asserts the existence of a *continuous function* f which satisfies the inequalities $\chi_K \leq f \leq \chi_V$. Note that it is easy to find *semicontinuous* functions which do this; examples are χ_K and χ_V.

PROOF Put $r_1 = 0$, $r_2 = 1$, and let $r_3, r_4, r_5, \ldots$ be an enumeration of the rationals in (0, 1). By Theorem 2.7, we can find open sets V_0 and then V_1 such that $\bar{V}_0$ is compact and

$$K \subset V_1 \subset \bar{V}_1 \subset V_0 \subset \bar{V}_0 \subset V. \tag{2}$$

Suppose $n \geq 2$ and $V_{r_1}, \ldots, V_{r_n}$ have been chosen in such a manner that $r_i < r_j$ implies $\bar{V}_{r_j} \subset V_{r_i}$. Then one of the numbers $r_1, \ldots, r_n$, say r_i, will be the largest one which is smaller than r_{n+1}, and another, say r_j, will be the smallest one larger than r_{n+1}. Using Theorem 2.7 again, we can find $V_{r_{n+1}}$ so that

$$\bar{V}_{r_j} \subset V_{r_{n+1}} \subset \bar{V}_{r_{n+1}} \subset V_{r_i}.$$

Continuing, we obtain a collection $\{V_r\}$ of open sets, one for every rational $r \in [0, 1]$, with the following properties: $K \subset V_1$, $\bar{V}_0 \subset V$, *each* $\bar{V}_r$ *is compact, and*

$$s > r \text{ implies } \bar{V}_s \subset V_r. \tag{3}$$

Define

$$f_r(x) = \begin{cases} r & \text{if } x \in V_r, \\ 0 & \text{otherwise,} \end{cases} \qquad g_s(x) = \begin{cases} 1 & \text{if } x \in \bar{V}_s, \\ s & \text{otherwise,} \end{cases} \tag{4}$$

and

$$f = \sup_r f_r, \qquad g = \inf_s g_s. \tag{5}$$

The remarks following Definition 2.8 show that f is lower semicontinuous and that g is upper semicontinuous. It is clear that $0 \leq f \leq 1$, that

$f(x) = 1$ if $x \in K$, and that f has its support in $\bar{V}_0$. The proof will be completed by showing that $f = g$.

The inequality $f_r(x) > g_s(x)$ is possible only if $r > s$, $x \in V_r$, and $x \notin \bar{V}_s$. But $r > s$ implies $V_r \subset V_s$. Hence $f_r \leq g_s$ for all r and s, so $f \leq g$.

Suppose $f(x) < g(x)$ for some x. Then there are rationals r and s such that $f(x) < r < s < g(x)$. Since $f(x) < r$, we have $x \notin V_r$; since $g(x) > s$, we have $x \in \bar{V}_s$. By (3), this is a contradiction. Hence $f = g$. ////

2.13 Theorem *Suppose $V_1, \ldots, V_n$ are open subsets of a locally compact Hausdorff space X, K is compact, and*

$$K \subset V_1 \cup \cdots \cup V_n.$$

Then there exist functions $h_i \prec V_i$ $(i = 1, \ldots, n)$ such that

$$h_1(x) + \cdots + h_n(x) = 1 \qquad (x \in K). \tag{1}$$

Because of (1), the collection $\{h_1, \ldots, h_n\}$ is called a *partition of unity on K*, subordinate to the cover $\{V_1, \ldots, V_n\}$.

PROOF By Theorem 2.7, each $x \in K$ has a neighborhood W_x with compact closure $\bar{W}_x \subset V_i$ for some i (depending on x). There are points $x_1, \ldots, x_m$ such that $W_{x_1} \cup \cdots \cup W_{x_m} \supset K$. If $1 \leq i \leq n$, let H_i be the union of those $\bar{W}_{x_j}$ which lie in V_i. By Urysohn's lemma, there are functions g_i such that $H_i \prec g_i \prec V_i$. Define

$$\begin{aligned} h_1 &= g_1 \\ h_2 &= (1 - g_1)g_2 \\ &\cdots\cdots\cdots \\ h_n &= (1 - g_1)(1 - g_2) \cdots (1 - g_{n-1})g_n. \end{aligned} \tag{2}$$

Then $h_i \prec V_i$. It is easily verified, by induction, that

$$h_1 + h_2 + \cdots + h_n = 1 - (1 - g_1)(1 - g_2) \cdots (1 - g_n). \tag{3}$$

Since $K \subset H_1 \cup \cdots \cup H_n$, at least one $g_i(x) = 1$ at each point $x \in K$; hence (3) shows that (1) holds. ////

The Riesz Representation Theorem

2.14 Theorem *Let X be a locally compact Hausdorff space, and let Λ be a positive linear functional on $C_c(X)$. Then there exists a σ-algebra $\mathfrak{M}$ in X which contains all Borel sets in X, and there exists a unique positive measure μ on $\mathfrak{M}$ which represents Λ in the sense that*

(a) $\Lambda f = \int_X f \, d\mu$ *for every* $f \in C_c(X)$,

and which has the following additional properties:

(*b*) $\mu(K) < \infty$ *for every compact set* $K \subset X$.

(*c*) *For every* $E \in \mathfrak{M}$, *we have*

$$\mu(E) = \inf \{\mu(V): E \subset V, V \text{ open}\}.$$

(*d*) *The relation*

$$\mu(E) = \sup \{\mu(K): K \subset E, K \text{ compact}\}$$

holds for every open set E, *and for every* $E \in \mathfrak{M}$ *with* $\mu(E) < \infty$.

(*e*) *If* $E \in \mathfrak{M}$, $A \subset E$, *and* $\mu(E) = 0$, *then* $A \in \mathfrak{M}$.

For the sake of clarity, let us be more explicit about the meaning of the word "positive" in the hypothesis: Λ is assumed to be a linear functional on the complex vector space $C_c(X)$, with the additional property that Λf is a nonnegative real number for every f whose range consists of nonnegative real numbers. Briefly, if $f(X) \subset [0, \infty)$ then $\Lambda f \in [0, \infty)$.

Property (*a*) is of course the one of greatest interest. After we define $\mathfrak{M}$ and μ, (*b*) to (*d*) will be established in the course of proving that $\mathfrak{M}$ is a σ-algebra and that μ is countably additive. We shall see later (Theorem 2.18) that in "reasonable" spaces X *every* Borel measure which satisfies (*b*) also satisfies (*c*) and (*d*) and that (*d*) actually holds for every $E \in \mathfrak{M}$, in those cases. Property (*e*) merely says that $(X, \mathfrak{M}, \mu)$ is a complete measure space, in the sense of Theorem 1.36.

Throughout the proof of this theorem, the letter K will stand for a compact subset of X, and V will denote an open set in X.

Let us begin by proving the uniqueness of μ. If μ satisfies (*c*) and (*d*), it is clear that μ is determined on $\mathfrak{M}$ by its values on compact sets. Hence it suffices to prove that $\mu_1(K) = \mu_2(K)$ for all K, whenever μ_1 and μ_2 are measures for which the theorem holds. So, fix K and $\epsilon > 0$. By (*b*) and (*c*), there exists a $V \supset K$ with $\mu_2(V) < \mu_2(K) + \epsilon$; by Urysohn's lemma, there exists an f so that $K \prec f \prec V$; hence

$$\mu_1(K) = \int_X \chi_K \, d\mu_1 \leq \int_X f \, d\mu_1 = \Lambda f = \int_X f \, d\mu_2$$

$$\leq \int_X \chi_V \, d\mu_2 = \mu_2(V) < \mu_2(K) + \epsilon.$$

Thus $\mu_1(K) \leq \mu_2(K)$. If we interchange the roles of μ_1 and μ_2, the opposite inequality is obtained, and the uniqueness of μ is proved.

Incidentally, the above computation shows that (*a*) forces (*b*).

Construction of μ *and* $\mathfrak{M}$

For every open set V in X, define

$$\mu(V) = \sup \{\Lambda f: f \prec V\}. \tag{1}$$

If $V_1 \subset V_2$, it is clear that (1) implies $\mu(V_1) \leq \mu(V_2)$. Hence

$$\mu(E) = \inf \{\mu(V): E \subset V, V \text{ open}\}, \tag{2}$$

if E is an open set, and it is consistent with (1) to *define* $\mu(E)$ by (2), for *every* $E \subset X$.

Note that although we have defined $\mu(E)$ for every $E \subset X$, the countable additivity of μ will be proved only on a certain σ-algebra $\mathfrak{M}$ in X.

Let $\mathfrak{M}_F$ be the class of all $E \subset X$ which satisfy two conditions: $\mu(E) < \infty$, and

$$\mu(E) = \sup \{\mu(K): K \subset E, K \text{ compact}\}. \tag{3}$$

Finally, let $\mathfrak{M}$ be the class of all $E \subset X$ such that $E \cap K \in \mathfrak{M}_F$ for every compact K.

Proof that μ and $\mathfrak{M}$ have the required properties

It is evident that μ is *monotone*, i.e., that $\mu(A) \leq \mu(B)$ if $A \subset B$ and that $\mu(E) = 0$ implies $E \in \mathfrak{M}_F$ and $E \in \mathfrak{M}$. Thus (*e*) holds, and so does (*c*), by definition.

Since the proof of the other assertions is rather long, it will be convenient to divide it into several steps.

Observe that the positivity of Λ implies that Λ is *monotone*: $f \leq g$ implies $\Lambda f \leq \Lambda g$. This is clear, since $\Lambda g = \Lambda f + \Lambda(g - f)$ and $g - f \geq 0$. This monotonicity will be used in Steps II and X.

STEP I *If $E_1, E_2, E_3, \ldots$ are arbitrary subsets of X, then*

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} \mu(E_i). \tag{4}$$

PROOF We first show that

$$\mu(V_1 \cup V_2) \leq \mu(V_1) + \mu(V_2), \tag{5}$$

if V_1 and V_2 are open. Choose $g \prec V_1 \cup V_2$. By Theorem 2.13 there are functions h_1 and h_2 such that $h_i \prec V_i$ and $h_1(x) + h_2(x) = 1$ for all x in the support of g. Hence $h_i g \prec V_i$, $g = h_1 g + h_2 g$, and so

$$\Lambda g = \Lambda(h_1 g) + \Lambda(h_2 g) \leq \mu(V_1) + \mu(V_2). \tag{6}$$

Since (6) holds for *every* $g \prec V_1 \cup V_2$, (5) follows.

If $\mu(E_i) = \infty$ for some i, then (4) is trivially true. Suppose therefore that $\mu(E_i) < \infty$ for every i. Choose $\epsilon > 0$. By (2) there are open sets $V_i \supset E_i$ such that

$$\mu(V_i) < \mu(E_i) + 2^{-i}\epsilon \qquad (i = 1, 2, 3, \ldots).$$

Put $V = \bigcup_1^\infty V_i$, and choose $f \prec V$. Since f has compact support, we see that $f \prec V_1 \cup \cdots \cup V_n$ for some n. Applying induction to (5), we therefore obtain

$$\Lambda f \leq \mu(V_1 \cup \cdots \cup V_n) \leq \mu(V_1) + \cdots + \mu(V_n) \leq \sum_{i=1}^{\infty} \mu(E_i) + \epsilon.$$

Since this holds for every $f \prec V$, and since $\bigcup E_i \subset V$, it follows that

$$\mu\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \mu(V) \leq \sum_{i=1}^{\infty} \mu(E_i) + \epsilon,$$

which proves (4), since ϵ was arbitrary. ////

STEP II *If K is compact, then $K \in \mathfrak{M}_F$ and*

$$\mu(K) = \inf \{\Lambda f: K \prec f\}. \tag{7}$$

This implies assertion (*b*) of the theorem.

PROOF If $K \prec f$ and $0 < \alpha < 1$, let $V_\alpha = \{x: f(x) > \alpha\}$. Then $K \subset V_\alpha$, and $\alpha g \leq f$ whenever $g \prec V_\alpha$. Hence

$$\mu(K) \leq \mu(V_\alpha) = \sup \{\Lambda g: g \prec V_\alpha\} \leq \alpha^{-1} \Lambda f.$$

Let $\alpha \to 1$, to conclude that

$$\mu(K) \leq \Lambda f. \tag{8}$$

Thus $\mu(K) < \infty$. Since K evidently satisfies (3), $K \in \mathfrak{M}_F$.

If $\epsilon > 0$, there exists $V \supset K$ with $\mu(V) < \mu(K) + \epsilon$. By Urysohn's lemma, $K \prec f \prec V$ for some f. Thus

$$\Lambda f \leq \mu(V) < \mu(K) + \epsilon,$$

which, combined with (8), gives (7). ////

STEP III *Every open set satisfies* (3). *Hence $\mathfrak{M}_F$ contains every open set V with $\mu(V) < \infty$.*

PROOF Let α be a real number such that $\alpha < \mu(V)$. There exists an $f \prec V$ with $\alpha < \Lambda f$. If W is *any* open set which contains the support K of f, then $f \prec W$, hence $\Lambda f \leq \mu(W)$. Thus $\Lambda f \leq \mu(K)$. This exhibits a compact $K \subset V$ with $\alpha < \mu(K)$, so that (3) holds for V. ////

STEP IV *Suppose $E = \bigcup_{i=1}^\infty E_i$, where $E_1, E_2, E_3, \ldots$ are pairwise disjoint members of $\mathfrak{M}_F$. Then*

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i). \tag{9}$$

If, in addition, $\mu(E) < \infty$, then also $E \in \mathfrak{M}_F$.

PROOF We first show that

$$\mu(K_1 \cup K_2) = \mu(K_1) + \mu(K_2) \tag{10}$$

if K_1 and K_2 are disjoint compact sets. Choose $\epsilon > 0$. By Urysohn's lemma, there exists $f \in C_c(X)$ such that $f(x) = 1$ on K_1, $f(x) = 0$ on K_2, and $0 \leq f \leq 1$. By Step II there exists g such that

$$K_1 \cup K_2 \prec g \quad \text{and} \quad \Lambda g < \mu(K_1 \cup K_2) + \epsilon.$$

Note that $K_1 \prec fg$ and $K_2 \prec (1 - f)g$. Since Λ is linear, it follows from (8) that

$$\mu(K_1) + \mu(K_2) \leq \Lambda(fg) + \Lambda(g - fg) = \Lambda g < \mu(K_1 \cup K_2) + \epsilon.$$

Since ϵ was arbitrary, (10) follows now from Step I.

If $\mu(E) = \infty$, (9) follows from Step I. Assume therefore that $\mu(E) < \infty$, and choose $\epsilon > 0$. Since $E_i \in \mathfrak{M}_F$, there are compact sets $H_i \subset E_i$ with

$$\mu(H_i) > \mu(E_i) - 2^{-i}\epsilon \qquad (i = 1, 2, 3, \ldots). \tag{11}$$

Putting $K_n = H_1 \cup \cdots \cup H_n$ and using induction on (10), we obtain

$$\mu(E) \geq \mu(K_n) = \sum_{i=1}^{n} \mu(H_i) > \sum_{i=1}^{n} \mu(E_i) - \epsilon. \tag{12}$$

Since (12) holds for every n and every $\epsilon > 0$, the left side of (9) is not smaller than the right side, and so (9) follows from Step I.

But if $\mu(E) < \infty$ and $\epsilon > 0$, (9) shows that

$$\mu(E) \leq \sum_{i=1}^{N} \mu(E_i) + \epsilon \tag{13}$$

for some N. By (12), it follows that $\mu(E) \leq \mu(K_N) + 2\epsilon$, and this shows that E satisfies (3); hence $E \in \mathfrak{M}_F$. ////

STEP V *If $E \in \mathfrak{M}_F$ and $\epsilon > 0$, there is a compact K and an open V such that $K \subset E \subset V$ and $\mu(V - K) < \epsilon$.*

PROOF Our definitions show that there exist $K \subset E$ and $V \supset E$ so that

$$\mu(V) - \frac{\epsilon}{2} < \mu(E) < \mu(K) + \frac{\epsilon}{2}.$$

Since $V - K$ is open, $V - K \in \mathfrak{M}_F$, by Step III. Hence Step IV implies that

$$\mu(K) + \mu(V - K) = \mu(V) < \mu(K) + \epsilon. \qquad ////$$

STEP VI *If $A \in \mathfrak{M}_F$ and $B \in \mathfrak{M}_F$, then $A - B$, $A \cup B$, and $A \cap B$ belong to $\mathfrak{M}_F$.*

PROOF If $\epsilon > 0$, Step V shows that there are sets K_i and V_i such that $K_1 \subset A \subset V_1$, $K_2 \subset B \subset V_2$, and $\mu(V_i - K_i) < \epsilon$, for $i = 1, 2$. Since

$$A - B \subset V_1 - K_2 \subset (V_1 - K_1) \cup (K_1 - V_2) \cup (V_2 - K_2),$$

Step I shows that

$$\mu(A - B) \leq \epsilon + \mu(K_1 - V_2) + \epsilon. \tag{14}$$

Since $K_1 - V_2$ is a compact subset of $A - B$, (14) shows that $A - B$ satisfies (3), so that $A - B \in \mathfrak{M}_F$.

Since $A \cup B = (A - B) \cup B$, an application of Step IV shows that $A \cup B \in \mathfrak{M}_F$. Since $A \cap B = A - (A - B)$, we also have $A \cap B \in \mathfrak{M}_F$. ////

STEP VII *$\mathfrak{M}$ is a σ-algebra in X which contains all Borel sets.*

PROOF Let K be an arbitrary compact set in X.

If $A \in \mathfrak{M}$, then $A^c \cap K = K - (A \cap K)$, so that $A^c \cap K$ is a difference of two members of $\mathscr{M}_F$. Hence $A^c \cap K \in \mathfrak{M}_F$, and we conclude: $A \in \mathfrak{M}$ implies $A^c \in \mathfrak{M}$.

Next, suppose $A = \bigcup_1^\infty A_i$, where each $A_i \in \mathfrak{M}$. Put $B_1 = A_1 \cap K$, and

$$B_n = (A_n \cap K) - (B_1 \cup \cdots \cup B_{n-1}) \qquad (n = 2, 3, 4, \ldots). \tag{15}$$

Then $\{B_n\}$ is a disjoint sequence of members of $\mathfrak{M}_F$, by Step VI, and $A \cap K = \bigcup_1^\infty B_n$. It follows from Step IV that $A \cap K \in \mathfrak{M}_F$. Hence $A \in \mathfrak{M}$.

Finally, if C is closed, then $C \cap K$ is compact, hence $C \cap K \in \mathfrak{M}_F$, so $C \in \mathfrak{M}$. In particular, $X \in \mathfrak{M}$.

We have thus proved that $\mathfrak{M}$ is a σ-algebra in X which contains all closed subsets of X. Hence $\mathfrak{M}$ contains all Borel sets in X. ////

STEP VIII *$\mathfrak{M}_F$ consists of precisely those sets $E \in \mathfrak{M}$ for which $\mu(E) < \infty$.*

This implies assertion (*d*) of the theorem.

PROOF If $E \in \mathfrak{M}_F$, Steps II and VI imply that $E \cap K \in \mathfrak{M}_F$ for every compact K, hence $E \in \mathfrak{M}$.

Conversely, suppose $E \in \mathfrak{M}$ and $\mu(E) < \infty$, and choose $\epsilon > 0$. There is an open set $V \supset E$ with $\mu(V) < \infty$; by III and V, there is a compact $K \subset V$ with $\mu(V - K) < \epsilon$. Since $E \cap K \in \mathfrak{M}_F$, there is a compact set $H \subset E \cap K$ with

$$\mu(E \cap K) < \mu(H) + \epsilon.$$

Since $E \subset (E \cap K) \cup (V - K)$, it follows that

$$\mu(E) \leq \mu(E \cap K) + \mu(V - K) < \mu(H) + 2\epsilon,$$

which implies that $E \in \mathfrak{M}_F$. ////

STEP IX *μ is a measure on $\mathfrak{M}$.*

PROOF The countable additivity of μ on $\mathfrak{M}$ follows immediately from Steps IV and VIII. ////

STEP X *For every* $f \in C_c(X)$, $\Lambda f = \int_X f\, d\mu$.

This proves (*a*), and completes the theorem.

PROOF Clearly, it is enough to prove this for real f. Also, it is enough to prove the *inequality*

$$\Lambda f \leq \int_X f\, d\mu \tag{16}$$

for every real $f \in C_c(X)$. For once (16) is established, the linearity of Λ shows that

$$-\Lambda f = \Lambda(-f) \leq \int_X (-f)\, d\mu = -\int_X f\, d\mu,$$

which, together with (16), shows that equality holds in (16).

Let K be the support of a real $f \in C_c(X)$, let $[a, b]$ be an interval which contains the range of f (note the Corollary to Theorem 2.10), choose $\epsilon > 0$, and choose y_i, for $i = 0, 1, \ldots, n$, so that $y_i - y_{i-1} < \epsilon$ and

$$y_0 < a < y_1 < \cdots < y_n = b. \tag{17}$$

Put

$$E_i = \{x: y_{i-1} < f(x) \leq y_i\} \cap K \qquad (i = 1, \ldots, n). \tag{18}$$

Since f is continuous, f is Borel measurable, and the sets E_i are therefore disjoint Borel sets whose union is K. There are open sets $V_i \supset E_i$ such that

$$\mu(V_i) < \mu(E_i) + \frac{\epsilon}{n} \qquad (i = 1, \ldots, n) \tag{19}$$

and such that $f(x) < y_i + \epsilon$ for all $x \in V_i$. By Theorem 2.13, there are functions $h_i \prec V_i$ such that $\sum h_i = 1$ on K. Hence $f = \sum h_i f$, and Step II shows that

$$\mu(K) \leq \Lambda(\textstyle\sum h_i) = \sum \Lambda h_i.$$

Since $h_i f \leq (y_i + \epsilon)h_i$, and since $y_i - \epsilon < f(x)$ on E_i, we have

$$\begin{aligned}
\Lambda f &= \sum_{i=1}^{n} \Lambda(h_i f) \leq \sum_{i=1}^{n} (y_i + \epsilon)\Lambda h_i \\
&= \sum_{i=1}^{n} (|a| + y_i + \epsilon)\Lambda h_i - |a| \sum_{i=1}^{n} \Lambda h_i \\
&\leq \sum_{i=1}^{n} (|a| + y_i + \epsilon)[\mu(E_i) + \epsilon/n] - |a|\mu(K) \\
&= \sum_{i=1}^{n} (y_i - \epsilon)\mu(E_i) + 2\epsilon\mu(K) + \frac{\epsilon}{n} \sum_{i=1}^{n} (|a| + y_i + \epsilon) \\
&\leq \int_X f \, d\mu + \epsilon[2\mu(K) + |a| + b + \epsilon].
\end{aligned}$$

Since ϵ was arbitrary, (16) is established, and the proof of the theorem is complete. ////

Regularity Properties of Borel Measures

2.15 Definition A measure μ defined on the σ-algebra of all Borel sets in a locally compact Hausdorff space X is called a *Borel measure on* X. If μ is positive, a Borel set $E \subset X$ is *outer regular* or *inner regular*, respectively, if E has property (*c*) or (*d*) of Theorem 2.14. If every Borel set in X is both outer and inner regular, μ is called *regular*.

In our proof of the Riesz theorem, outer regularity of every set E was built into the construction, but inner regularity was proved only for the open sets and for those $E \in \mathfrak{M}$ for which $\mu(E) < \infty$. It turns out that this flaw is in the nature of things. One cannot prove regularity of μ under the hypothesis of Theorem 2.14; an example is described in Exercise 17.

However, a slight strengthening of the hypotheses does give us a regular measure. Theorem 2.17 shows this. And if we specialize a little more, Theorem 2.18 shows that all regularity problems neatly disappear.

2.16 Definition A set E in a topological space is called *σ-compact* if E is a countable union of compact sets.

A set E in a measure space (with measure μ) is said to have *σ-finite measure* if E is a countable union of sets E_i with $\mu(E_i) < \infty$.

For example, in the situation described in Theorem 2.14, every σ-compact set has σ-finite measure. Also, it is easy to see that if $E \in \mathfrak{M}$ and E has σ-finite measure, then E is inner regular.

2.17 Theorem *Suppose X is a locally compact, σ-compact Hausdorff space. If $\mathfrak{M}$ and μ are as described in the statement of Theorem 2.14, then $\mathfrak{M}$ and μ have the following properties:*

(*a*) *If $E \in \mathfrak{M}$ and $\epsilon > 0$, there is a closed set F and an open set V such that $F \subset E \subset V$ and $\mu(V - F) < \epsilon$.*
(*b*) *μ is a regular Borel measure on X.*
(*c*) *If $E \in \mathfrak{M}$, there are sets A and B such that A is an F_σ, B is a G_δ, $A \subset E \subset B$, and $\mu(B - A) = 0$.*

As a corollary of (*c*) we see that every $E \in \mathfrak{M}$ is the union of an F_σ and a set of measure 0.

PROOF Let $X = K_1 \cup K_2 \cup K_3 \cup \cdots$, where each K_n is compact. If $E \in \mathfrak{M}$ and $\epsilon > 0$, then $\mu(K_n \cap E) < \infty$, and there are open sets $V_n \supset K_n \cap E$ such that

$$\mu(V_n - (K_n \cap E)) < \frac{\epsilon}{2^{n+1}} \qquad (n = 1, 2, 3, \ldots). \tag{1}$$

If $V = \bigcup V_n$, then $V - E \subset \bigcup (V_n - (K_n \cap E))$, so that

$$\mu(V - E) < \frac{\epsilon}{2}.$$

Apply this to E^c in place of E: There is an open set $W \supset E^c$ such that $\mu(W - E^c) < \epsilon/2$. If $F = W^c$, then $F \subset E$, and $E - F = W - E^c$. Now (*a*) follows.

Every closed set $F \subset X$ is σ-compact, because $F = \bigcup (F \cap K_n)$. Hence (*a*) implies that every set $E \in \mathfrak{M}$ is inner regular. This proves (*b*).

If we apply (*a*) with $\epsilon = 1/j$ ($j = 1, 2, 3, \ldots$), we obtain closed sets F_j and open sets V_j such that $F_j \subset E \subset V_j$ and $\mu(V_j - F_j) < 1/j$. Put $A = \bigcup F_j$ and $B = \bigcap V_j$. Then $A \subset E \subset B$, A is an F_σ, B is a G_δ, and $\mu(B - A) = 0$ since $B - A \subset V_j - F_j$ for $j = 1, 2, 3, \ldots$. This proves (*c*). ////

2.18 Theorem *Let X be a locally compact Hausdorff space in which every open set is σ-compact. Let λ be any positive Borel measure on X such that $\lambda(K) < \infty$ for every compact set K. Then λ is regular.*

Note that every euclidean space R^k satisfies the present hypothesis, since every open set in R^k is a countable union of closed balls.

PROOF Put $\Lambda f = \int_X f\,d\lambda$, for $f \in C_c(X)$. Since $\lambda(K) < \infty$ for every compact K, Λ is a positive linear functional on $C_c(X)$, and there is a regular measure μ, satisfying the conclusions of Theorem 2.17, such that

$$\int_X f\,d\lambda = \int_X f\,d\mu \qquad (f \in C_c(X)). \tag{1}$$

We will show that $\lambda = \mu$.

Let V be open in X. Then $V = \bigcup K_i$, where K_i is compact, $i = 1, 2, 3, \ldots$. By Urysohn's lemma we can choose f_i so that $K_i \prec f_i \prec V$. Let $g_n = \max(f_1, \ldots, f_n)$. Then $g_n \in C_c(X)$ and $g_n(x)$ increases to $\chi_V(x)$ at every point $x \in X$. Hence (1) and the monotone convergence theorem imply

$$\lambda(V) = \lim_{n\to\infty} \int_X g_n\,d\lambda = \lim_{n\to\infty} \int_X g_n\,d\mu = \mu(V)\,. \tag{2}$$

Now let E be a Borel set in X, and choose $\epsilon > 0$. Since μ satisfies Theorem 2.17, there is a closed set F and an open set V such that $F \subset E \subset V$ and $\mu(V - F) < \epsilon$. Hence $\mu(V) \leq \mu(F) + \epsilon \leq \mu(E) + \epsilon$.

Since $V - F$ is open, (2) shows that $\lambda(V - F) < \epsilon$, hence $\lambda(V) \leq \lambda(E) + \epsilon$.

Consequently

$$\lambda(E) \leq \lambda(V) = \mu(V) \leq \mu(E) + \epsilon$$

and

$$\mu(E) \leq \mu(V) = \lambda(V) \leq \lambda(E) + \epsilon\,,$$

so that $|\lambda(E) - \mu(E)| < \epsilon$ for every $\epsilon > 0$. Hence $\lambda(E) = \mu(E)$. ////

In Exercise 18 a compact Hausdorff space is described in which the complement of a certain point fails to be σ-compact and in which the conclusion of the preceding theorem is not true.

Lebesgue Measure

2.19 Euclidean Spaces Euclidean k-dimensional space R^k is the set of all points $x = (\xi_1, \ldots, \xi_k)$ whose coordinates ξ_i are real numbers, with the following algebraic and topological structure:

If $x = (\xi_1, \ldots, \xi_k)$, $y = (\eta_1, \ldots, \eta_k)$, and α is a real number, $x + y$ and αx are defined by

$$x + y = (\xi_1 + \eta_1, \ldots, \xi_k + \eta_k), \qquad \alpha x = (\alpha\xi_1, \ldots, \alpha\xi_k)\,. \tag{1}$$

This makes R^k into a real vector space. If $x \cdot y = \sum \xi_i \eta_i$ and $|x| = (x \cdot x)^{1/2}$, the Schwarz inequality $|x \cdot y| \leq |x|\,|y|$ leads to the triangle inequality

$$|x - y| \leq |x - z| + |z - y|\,; \tag{2}$$

hence we obtain a metric by setting $\rho(x, y) = |x - y|$. We assume that these facts are familiar and shall prove them in greater generality in Chap. 4.

If $E \subset R^k$ and $x \in R^k$, the *translate of E by x* is the set

$$E + x = \{y + x: y \in E\}. \tag{3}$$

A set of the form

$$W = \{x: \alpha_i < \xi_i < \beta_i, 1 \leq i \leq k\}, \tag{4}$$

or any set obtained by replacing any or all of the $<$ signs in (4) by $\leq$, is called a *k-cell*; its volume is defined to be

$$\text{vol}\,(W) = \prod_{i=1}^{k} (\beta_i - \alpha_i). \tag{5}$$

If $a \in R^k$ and $\delta > 0$, we shall call the set

$$Q(a; \delta) = \{x: \alpha_i \leq \xi_i < \alpha_i + \delta, 1 \leq i \leq k\} \tag{6}$$

the *δ-box with corner at a*. Here $a = (\alpha_1, \ldots, \alpha_k)$.

For $n = 1, 2, 3, \ldots$, we let P_n be the set of all $x \in R^k$ whose coordinates are integral multiples of 2^{-n}, and we let Ω_n be the collection of all 2^{-n} boxes with corners at points of P_n. We shall need the following four properties of $\{\Omega_n\}$. The first three are obvious by inspection.

(*a*) *If n is fixed, each* $x \in R^k$ *lies in one and only one member of* Ω_n.
(*b*) *If* $Q' \in \Omega_n$, $Q'' \in \Omega_r$, *and* $r < n$, *then either* $Q' \subset Q''$ *or* $Q' \cap Q'' = \varnothing$.
(*c*) *If* $Q \in \Omega_r$, *then* $\text{vol}\,(Q) = 2^{-rk}$; *and if* $n > r$, *the set* P_n *has exactly* $2^{(n-r)k}$ *points in Q.*
(*d*) *Every nonempty open set in* R^k *is a countable union of disjoint boxes belonging to* $\Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \cdots$.

PROOF OF (*d*) If V is open, every $x \in V$ lies in an open ball which lies in V; hence $x \in Q \subset V$ for some Q belonging to some Ω_n. In other words, V is the union of all boxes which lie in V and which belong to some Ω_n. From this collection of boxes, select those which belong to Ω_1, and remove those in Ω_2, $\Omega_3, \ldots$ which lie in any of the selected boxes. From the remaining collection, select those boxes of Ω_2 which lie in V, and remove those in Ω_3, $\Omega_4, \ldots$ which lie in any of the selected boxes. If we proceed in this way, (*a*) and (*b*) show that (*d*) holds. ////

2.20 Theorem *There exists a positive complete measure m defined on a σ-algebra* $\mathfrak{M}$ *in* R^k, *with the following properties:*

(*a*) $m(W) = \text{vol}\,(W)$ *for every k-cell W.*
(*b*) $\mathfrak{M}$ *contains all Borel sets in* R^k; *more precisely,* $E \in \mathfrak{M}$ *if and only if there are sets A and* $B \subset R^k$ *such that* $A \subset E \subset B$, *A is an* F_σ, *B is a* G_δ, *and* $m(B - A) = 0$. *Also, m is regular.*

(*c*) *m is translation-invariant, i.e.,*

$$m(E + x) = m(E)$$

for every $E \in \mathfrak{M}$ *and every* $x \in R^k$.

(*d*) *If* μ *is any positive translation-invariant Borel measure on* R^k *such that* $\mu(K) < \infty$ *for every compact set* K, *then there is a constant* c *such that* $\mu(E) = cm(E)$ *for all Borel sets* $E \subset R^k$.

(*e*) *To every linear transformation* T *of* R^k *into* R^k *corresponds a real number* $\Delta(T)$ *such that*

$$m(T(E)) = \Delta(T)m(E)$$

for every $E \in \mathfrak{M}$. *In particular,* $m(T(E)) = m(E)$ *when* T *is a rotation.*

The members of $\mathfrak{M}$ are the *Lebesgue measurable* sets in R^k; m is the *Lebesgue measure* on R^k. When clarity requires it, we shall write m_k in place of m.

PROOF If f is any complex function on R^k, with compact support, define

$$\Lambda_n f = 2^{-nk} \sum_{x \in P_n} f(x) \qquad (n = 1, 2, 3, \ldots), \tag{1}$$

where P_n is as in Sec. 2.19.

Now suppose $f \in C_c(R^k)$, f is real, W is an open k-cell which contains the support of f, and $\epsilon > 0$. The uniform continuity of f([26], Theorem 4.19) shows that there is an integer N and that there are functions g and h with support in W, such that (i) g and h are constant on each box belonging to Ω_N, (ii) $g \leq f \leq h$, and (iii) $h - g < \epsilon$. If $n > N$, Property 2.19(*c*) shows that

$$\Lambda_N g = \Lambda_n g \leq \Lambda_n f \leq \Lambda_n h = \Lambda_N h. \tag{2}$$

Thus the upper and lower limits of $\{\Lambda_n f\}$ differ by at most ϵ vol (W), and since ϵ was arbitrary, we have proved the existence of

$$\Lambda f = \lim_{n \to \infty} \Lambda_n f \qquad (f \in C_c(R^k)). \tag{3}$$

It is immediate that Λ is a positive linear functional on $C_c(R^k)$. (In fact, Λf is precisely the Riemann integral of f over R^k. We went through the preceding construction in order not to have to rely on any theorems about Riemann integrals in several variables.)

We define m *and* $\mathfrak{M}$ *to be the measure and* σ*-algebra associated with this* Λ *as in Theorem* 2.14.

Since Theorem 2.14 gives us a complete measure and since R^k is σ-compact, Theorem 2.17 implies assertion (*b*) of Theorem 2.20.

To prove (*a*), let W be the open cell 2.19(4), let E_r be the union of those boxes belonging to Ω_r whose closures lie in W, choose f_r so that $\bar{E}_r \prec f_r \prec W$, and put $g_r = \max\{f_1, \ldots, f_r\}$. Our construction of Λ shows that

$$\text{vol}\,(E_r) \leq \Lambda f_r \leq \Lambda g_r \leq \text{vol}\,W. \tag{4}$$

As $r \to \infty$, vol $(E_r) \to$ vol (W), and

$$\Lambda g_r = \int g_r\, dm \to m(W) \tag{5}$$

by the monotone convergence theorem, since $g_r(x) \to \chi_W(x)$ for all $x \in R^k$. Thus $m(W) = \text{vol}\,(W)$ for every open cell W, and since every k-cell is the intersection of a decreasing sequence of open k-cells, we obtain (*a*).

The proofs of (*c*), (*d*), and (*e*) will use the following observation: If λ is a positive Borel measure on R^k and $\lambda(E) = m(E)$ for all boxes E, then the same equality holds for all open sets E, by property 2.19(*d*), and therefore for all Borel sets E, since λ and m are regular (Theorem 2.18).

To prove (*c*), fix $x \in R^k$ and define $\lambda(E) = m(E + x)$. It is clear that λ is then a measure; by (*a*), $\lambda(E) = m(E)$ for all boxes, hence $m(E + x) = m(E)$ for all Borel sets E. The same equality holds for every $E \in \mathfrak{M}$, because of (*b*).

Suppose next that μ satisfies the hypotheses of (*d*). Let Q_0 be a 1-box, put $c = \mu(Q_0)$. Since Q_0 is the union of 2^{nk} disjoint 2^{-n} boxes that are translates of each other, we have

$$2^{nk}\mu(Q) = \mu(Q_0) = cm(Q_0) = c \cdot 2^{nk}m(Q)$$

for every 2^{-n}-box Q. Property 2.19(*d*) implies now that $\mu(E) = cm(E)$ for all open sets $E \subset R^k$. This proves (*d*).

To prove (*e*), let $T: R^k \to R^k$ be linear. If the range of T is a subspace Y of lower dimension, then $m(Y) = 0$ and the desired conclusion holds with $\Delta(T) = 0$. In the other case, elementary linear algebra tells us that T is a one-to-one map of R^k onto R^k whose inverse is also linear. Thus T is a homeomorphism of R^k onto R^k, so that $T(E)$ is a Borel set for every Borel set E, and we can therefore define a positive Borel measure μ on R^k by

$$\mu(E) = m(T(E)).$$

The linearity of T, combined with the translation-invariance of m, gives

$$\mu(E + x) = m(T(E + x)) = m(T(E) + Tx) = m(T(E)) = \mu(E).$$

Thus μ is translation-invariant, and the first assertion of (*e*) follows from (*d*), first for Borel sets E, then for all $E \in \mathfrak{M}$ by (*b*).

To find $\Delta(T)$, we merely need to know $m(T(E))/m(E)$ for one set E with $0 < m(E) < \infty$. If T is a rotation, let E be the unit ball of R^k; then $T(E) = E$, and $\Delta(T) = 1$. ////

2.21 Remarks If m is the Lebesgue measure on R^k, it is customary to write $L^1(R^k)$ in place of $L^1(m)$. If E is a Lebesgue measurable subset of R^k, and if m is restricted to the measurable subsets of E, a new measure space is obtained in an obvious fashion. The phrase "$f \in L^1$ on E" or "$f \in L^1(E)$" is used to indicate that f is integrable on this measure space.

If $k = 1$, if I is any of the sets (a, b), $(a, b]$, $[a, b)$, $[a, b]$, and if $f \in L^1(I)$, it is customary to write

$$\int_a^b f(x)\,dx \quad \text{in place of} \quad \int_I f\,dm.$$

Since the Lebesgue measure of any single point is 0, it makes no difference over which of these four sets the integral is extended.

Everything learned about integration in elementary Calculus courses is still useful in the present context, for *if f is a continuous complex function on $[a, b]$, then the Riemann integral of f and the Lebesgue integral of f over $[a, b]$ coincide.* This is obvious from our construction if $f(a) = f(b) = 0$ and if $f(x)$ is defined to be 0 for $x < a$ and for $x > b$. The general case follows without difficulty. Actually the same thing is true for every Riemann integrable f on $[a, b]$. Since we shall have no occasion to discuss Riemann integrable functions in the sequel, we omit the proof and refer to Theorem 11.33 of [26].

Two natural questions may have occurred to some readers by now: Is every Lebesgue measurable set a Borel set? Is every subset of R^k Lebesgue measurable? The answer is negative in both cases, even when $k = 1$.

The first question can be settled by a cardinality argument which we sketch briefly. Let c be the cardinality of the continuum (the real line or, equivalently, the collection of all sets of integers). We know that R^k has a *countable* base (open balls with rational radii and with centers in some countable dense subset of R^k), and that $\mathscr{B}_k$ (the collection of all Borel sets of R^k) is the σ-algebra generated by this base. It follows from this (we omit the proof) that $\mathscr{B}_k$ has cardinality c. On the other hand, there exist Cantor sets $E \subset R^1$ with $m(E) = 0$. (Exercise 5.) The completeness of m implies that each of the 2^c subsets of E is Lebesgue measurable. Since $2^c > c$, most subsets of E are not Borel sets.

The following theorem answers the second question.

2.22 Theorem *If $A \subset R^1$ and every subset of A is Lebesgue measurable then $m(A) = 0$.*

Corollary *Every set of positive measure has nonmeasurable subsets.*

PROOF We shall use the fact that R^1 is a group, relative to addition. Let Q be the subgroup that consists of the rational numbers, and let E be a set that contains exactly one point from each coset of Q in R^1. (The assertion that

there is such a set is a direct application of the axiom of choice.) Then E has the following two properties.

(*a*) $(E + r) \cap (E + s) = \varnothing$ if $r \in Q, s \in Q, r \neq s$.
(*b*) Every $x \in R^1$ lies in $E + r$ for some $r \in Q$.

To prove (*a*), suppose $x \in (E + r) \cap (E + s)$. Then $x = y + r = z + s$ for some $y \in E$, $z \in E$, $y \neq z$. But $y - z = s - r \in Q$, so that y and z lie in the same coset of Q, a contradiction.

To prove (*b*), let y be the point of E that lies in the same coset as x, put $r = x - y$.

Fix $t \in Q$, for the moment, and put $A_t = A \cap (E + t)$. By hypothesis, A_t is measurable. Let $K \subset A_t$ be compact, let H be the union of the translates $K + r$, where r ranges over $Q \cap [0, 1]$. Then H is bounded, hence $m(H) < \infty$. Since $K \subset E + t$, (*a*) shows that the sets $K + r$ are pairwise disjoint. Thus $m(H) = \sum_r m(K + r)$. But $m(K + r) = m(K)$. It follows that $m(K) = 0$. This holds for every compact $K \subset A_t$. Hence $m(A_t) = 0$.

Finally, (*b*) shows that $A = \bigcup A_t$, where t ranges over Q. Since Q is countable, we conclude that $m(A) = 0$. ////

2.23 Determinants The scale factors $\Delta(T)$ that occur in Theorem 2.20(*e*) can be interpreted algebraically by means of determinants.

Let $\{e_1, \ldots, e_k\}$ be the standard basis for R^k: the ith coordinate of e_j is 1 if $i = j$, 0 if $i \neq j$. If $T: R^k \to R^k$ is linear and

$$Te_j = \sum_{i=1}^{k} \alpha_{ij} e_i \qquad (1 \leq j \leq k) \tag{1}$$

then det T is, by definition, the determinant of the matrix $[T]$ that has α_{ij} in row i and column j.

We claim that

$$\Delta(T) = |\det T|. \tag{2}$$

If $T = T_1 T_2$, it is clear that $\Delta(T) = \Delta(T_1)\Delta(T_2)$. The multiplication theorem for determinants shows therefore that if (2) holds for T_1 and T_2, then (2) also holds for T. Since every linear operator on R^k is a product of finitely many linear operators of the following three types, it suffices to establish (2) for each of these:

(I) $\{Te_1, \ldots, Te_k\}$ is a permutation of $\{e_1, \ldots, e_k\}$.
(II) $Te_1 = \alpha e_1$, $Te_i = e_i$ for $i = 2, \ldots, k$.
(III) $Te_1 = e_1 + e_2$, $Te_i = e_i$ for $i = 2, \ldots, k$.

Let Q be the cube consisting of all $x = (\xi_1, \ldots, \xi_k)$ with $0 \leq \xi_i < 1$ for $i = 1, \ldots, k$.

If T is of type (I), then $[T]$ has exactly one 1 in each row and each column and has 0 in all other places. So $\det T = \pm 1$. Also, $T(Q) = Q$. So $\Delta(T) = 1 = |\det T|$.

If T is of type (II), then clearly $\Delta(T) = |\alpha| = |\det T|$.

If T is of type (III), then $\det T = 1$ and $T(Q)$ is the set of all points $\sum \xi_i e_i$ whose coordinates satisfy

$$\xi_1 \leq \xi_2 < \xi_1 + 1, \qquad 0 \leq \xi_i < 1 \quad \text{if} \quad i \neq 2. \tag{3}$$

If S_1 is the set of points in $T(Q)$ that have $\xi_2 < 1$ and if S_2 is the rest of $T(Q)$, then

$$S_1 \cup (S_2 - e_2) = Q, \tag{4}$$

and $S_1 \cap (S_2 - e_2)$ is empty. Hence $\Delta(T) = m(S_1 \cup S_2) = m(S_1) + m(S_2 - e_2) = m(Q) = 1$, so that we again have $\Delta(T) = |\det T|$.

Continuity Properties of Measurable Functions

Since the continuous functions played such a prominent role in our construction of Borel measures, and of Lebesgue measure in particular, it seems reasonable to expect that there are some interesting relations between continuous functions and measurable functions. In this section we shall give two theorems of this kind.

We shall assume, in both of them, *that μ is a measure on a locally compact Hausdorff space X which has the properties stated in Theorem 2.14.* In particular, μ could be Lebesgue measure on some R^k.

2.24 Lusin's Theorem *Suppose f is a complex measurable function on X, $\mu(A) < \infty$, $f(x) = 0$ if $x \notin A$, and $\epsilon > 0$. Then there exists a $g \in C_c(X)$ such that*

$$\mu(\{x : f(x) \neq g(x)\}) < \epsilon. \tag{1}$$

Furthermore, we may arrange it so that

$$\sup_{x \in X} |g(x)| \leq \sup_{x \in X} |f(x)|. \tag{2}$$

PROOF Assume first that $0 \leq f < 1$ and that A is compact. Attach a sequence $\{s_n\}$ to f, as in the proof of Theorem 1.17, and put $t_1 = s_1$ and $t_n = s_n - s_{n-1}$ for $n = 2, 3, 4, \ldots$. Then $2^n t_n$ is the characteristic function of a set $T_n \subset A$, and

$$f(x) = \sum_{n=1}^{\infty} t_n(x) \qquad (x \in X). \tag{3}$$

Fix an open set V such that $A \subset V$ and $\bar{V}$ is compact. There are compact sets K_n and open sets V_n such that $K_n \subset T_n \subset V_n \subset V$ and $\mu(V_n - K_n) < 2^{-n}\epsilon$. By Urysohn's lemma, there are functions h_n such that $K_n \prec h_n \prec V_n$. Define

$$g(x) = \sum_{n=1}^{\infty} 2^{-n} h_n(x) \qquad (x \in X). \tag{4}$$

This series converges uniformly on X, so g is continuous. Also, the support of g lies in $\bar{V}$. Since $2^{-n}h_n(x) = t_n(x)$ except in $V_n - K_n$, we have $g(x) = f(x)$

except in $\bigcup (V_n - K_n)$, and this latter set has measure less than ϵ. Thus (1) holds if A is compact and $0 \leq f \leq 1$.

It follows that (1) holds if A is compact and f is a bounded measurable function. The compactness of A is easily removed, for if $\mu(A) < \infty$ then A contains a compact set K with $\mu(A - K)$ smaller than any preassigned positive number. Next, if f is a complex measurable function and if $B_n = \{x: |f(x)| > n\}$, then $\bigcap B_n = \varnothing$, so $\mu(B_n) \to 0$, by Theorem 1.19(*e*). Since f coincides with the bounded function $(1 - \chi_{B_n}) \cdot f$ except on B_n, (1) follows in the general case.

Finally, let $R = \sup \{|f(x)|: x \in X\}$, and define $\varphi(z) = z$ if $|z| \leq R$, $\varphi(z) = Rz/|z|$ if $|z| > R$. Then φ is a continuous mapping of the complex plane onto the disc of radius R. If g satisfies (1) and $g_1 = \varphi \circ g$, then g_1 satisfies (1) and (2). ////

Corollary *Assume that the hypotheses of Lusin's theorem are satisfied and that* $|f| \leq 1$. *Then there is a sequence* $\{g_n\}$ *such that* $g_n \in C_c(X)$, $|g_n| \leq 1$, *and*

$$f(x) = \lim_{n \to \infty} g_n(x) \qquad \text{a.e.} \tag{5}$$

PROOF The theorem implies that to each n there corresponds a $g_n \in C_c(X)$, with $|g_n| \leq 1$, such that $\mu(E_n) \leq 2^{-n}$, where E_n is the set of all x at which $f(x) \neq g_n(x)$. For almost every x it is then true that x lies in at most finitely many of the sets E_n (Theorem 1.41). For any such x, it follows that $f(x) = g_n(x)$ for all large enough n. This gives (5). ////

2.25 The Vitali-Carathéodory Theorem *Suppose* $f \in L^1(\mu)$, f *is real-valued, and* $\epsilon > 0$. *Then there exist functions* u *and* v *on* X *such that* $u \leq f \leq v$, u *is upper semicontinuous and bounded above,* v *is lower semicontinuous and bounded below, and*

$$\int_X (v - u)\, d\mu < \epsilon. \tag{1}$$

PROOF Assume first that $f \geq 0$ and that f is not identically 0. Since f is the pointwise limit of an increasing sequence of simple functions s_n, f is the sum of the simple functions $t_n = s_n - s_{n-1}$ (taking $s_0 = 0$), and since t_n is a linear combination of characteristic functions, we see that there are measurable sets E_i (not necessarily disjoint) and constants $c_i > 0$ such that

$$f(x) = \sum_{i=1}^{\infty} c_i \chi_{E_i}(x) \qquad (x \in X). \tag{2}$$

Since

$$\int_X f\,d\mu = \sum_{i=1}^{\infty} c_i\,\mu(E_i), \tag{3}$$

the series in (3) converges. There are compact sets K_i and open sets V_i such that $K_i \subset E_i \subset V_i$ and

$$c_i\,\mu(V_i - K_i) < 2^{-i-1}\epsilon \qquad (i = 1, 2, 3, \ldots). \tag{4}$$

Put

$$v = \sum_{i=1}^{\infty} c_i\,\chi_{V_i}, \qquad u = \sum_{i=1}^{N} c_i\,\chi_{K_i}, \tag{5}$$

where N is chosen so that

$$\sum_{N+1}^{\infty} c_i\,\mu(E_i) < \frac{\epsilon}{2}. \tag{6}$$

Then v is lower semicontinuous, u is upper semicontinuous, $u \le f \le v$, and

$$\begin{aligned} v - u &= \sum_{i=1}^{N} c_i(\chi_{V_i} - \chi_{K_i}) + \sum_{N+1}^{\infty} c_i\,\chi_{V_i} \\ &\le \sum_{i=1}^{\infty} c_i(\chi_{V_i} - \chi_{K_i}) + \sum_{N+1}^{\infty} c_i\,\chi_{E_i} \end{aligned}$$

so that (4) and (6) imply (1).

In the general case, write $f = f^+ - f^-$, attach u_1 and v_1 to f^+, attach u_2 and v_2 to f^-, as above, and put $u = u_1 - v_2$, $v = v_1 - u_2$. Since $-v_2$ is upper semicontinuous and since the sum of two upper semicontinuous functions is upper semicontinuous (similarly for lower semicontinuous; we leave the proof of this as an exercise), u and v have the desired properties. ////

Exercises

1 Let $\{f_n\}$ be a sequence of real nonnegative functions on R^1, and consider the following four statements:

(*a*) If f_1 and f_2 are upper semicontinuous, then $f_1 + f_2$ is upper semicontinuous.
(*b*) If f_1 and f_2 are lower semicontinuous, then $f_1 + f_2$ is lower semicontinuous.
(*c*) If each f_n is upper semicontinuous, then $\sum_1^\infty f_n$ is upper semicontinuous.
(*d*) If each f_n is lower semicontinuous, then $\sum_1^\infty f_n$ is lower semicontinuous.

Show that three of these are true and that one is false. What happens if the word "nonnegative" is omitted? Is the truth of the statements affected if R^1 is replaced by a general topological space?

2 Let f be an arbitrary complex function on R^1, and define

$$\varphi(x, \delta) = \sup\{|f(s) - f(t)| : s, t \in (x - \delta, x + \delta)\},$$

$$\varphi(x) = \inf\{\varphi(x, \delta) : \delta > 0\}.$$

Prove that φ is upper semicontinuous, that f is continuous at a point x if and only if $\varphi(x) = 0$, and hence that the set of points of continuity of an arbitrary complex function is a G_δ.

Formulate and prove an analogous statement for general topological spaces in place of R^1.

3 Let X be a metric space, with metric ρ. For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf \{\rho(x, y): y \in E\}.$$

Show that ρ_E is a uniformly continuous function on X. If A and B are disjoint nonempty closed subsets of X, examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

4 Examine the proof of the Riesz theorem and prove the following two statements:

(*a*) If $E_1 \subset V_1$ and $E_2 \subset V_2$, where V_1 and V_2 are disjoint open sets, then $\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2)$, even if E_1 and E_2 are not in $\mathfrak{M}$.

(*b*) If $E \in \mathfrak{M}_F$, then $E = N \cup K_1 \cup K_2 \cup \cdots$, where $\{K_i\}$ is a disjoint countable collection of compact sets and $\mu(N) = 0$.

In Exercises 5 to 8, m stands for Lebesgue measure on R^1.

5 Let E be Cantor's familiar "middle thirds" set. Show that $m(E) = 0$, even though E and R^1 have the same cardinality.

6 Construct a totally disconnected compact set $K \subset R^1$ such that $m(K) > 0$. (K is to have no connected subset consisting of more than one point.)

If v is lower semicontinuous and $v \leq \chi_K$, show that actually $v \leq 0$. Hence χ_K cannot be approximated from below by lower semicontinuous functions, in the sense of the Vitali-Carathéodory theorem.

7 If $0 < \epsilon < 1$, construct an open set $E \subset [0, 1]$ which is dense in $[0, 1]$, such that $m(E) = \epsilon$. (To say that A is dense in B means that the closure of A contains B.)

8 Construct a Borel set $E \subset R^1$ such that

$$0 < m(E \cap I) < m(I)$$

for every nonempty segment I. Is it possible to have $m(E) < \infty$ for such a set?

9 Construct a sequence of continuous function f_n on $[0, 1]$ such that $0 \leq f_n \leq 1$, such that

$$\lim_{n \to \infty} \int_0^1 f_n(x)\, dx = 0,$$

but such that the sequence $\{f_n(x)\}$ converges for no $x \in [0, 1]$.

10 If $\{f_n\}$ is a sequence of continuous functions on $[0, 1]$ such that $0 \leq f_n \leq 1$ and such that $f_n(x) \to 0$ as $n \to \infty$, for every $x \in [0, 1]$, then

$$\lim_{n \to \infty} \int_0^1 f_n(x)\, dx = 0.$$

Try to prove this without using any measure theory or any theorems about Lebesgue integration. (This is to impress you with the power of the Lebesgue integral. A nice proof was given by W. F. Eberlein in *Communications on Pure and Applied Mathematics*, vol. X, pp. 357–360, 1957.)

11 Let μ be a regular Borel measure on a compact Hausdorff space X; assume $\mu(X) = 1$. Prove that there is a compact set $K \subset X$ (the *carrier* or *support* of μ) such that $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset H of K. *Hint*: Let K be the intersection of all compact K_α with $\mu(K_\alpha) = 1$; show that every open set V which contains K also contains some K_α. Regularity of μ is needed; compare Exercise 18. Show that K^c is the largest open set in X whose measure is 0.

12 Show that every compact subset of R^1 is the support of a Borel measure.

13 Is it true that every compact subset of R^1 is the support of a continuous function? If not, can you describe the class of all compact sets in R^1 which are supports of continuous functions? Is your description valid in other topological spaces?

14 Let f be a real-valued Lebesgue measurable function on R^k. Prove that there exist Borel functions g and h such that $g(x) = h(x)$ a.e. $[m]$, and $g(x) \leq f(x) \leq h(x)$ for every $x \in R^k$.

15 It is easy to guess the limits of

$$\int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2}\, dx \quad \text{and} \quad \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x}\, dx,$$

as $n \to \infty$. Prove that your guesses are correct.

16 Why is $m(Y) = 0$ in the proof of Theorem 2.20(*e*)?

17 Define the distance between points (x_1, y_1) and (x_2, y_2) in the plane to be

$$|y_1 - y_2| \quad \text{if } x_1 = x_2, \qquad 1 + |y_1 - y_2| \quad \text{if } x_1 \neq x_2.$$

Show that this is indeed a metric, and that the resulting metric space X is locally compact.

If $f \in C_c(X)$, let $x_1, \ldots, x_n$ be those values of x for which $f(x, y) \neq 0$ for at least one y (there are only finitely many such x!), and define

$$\Lambda f = \sum_{j=1}^{n} \int_{-\infty}^{\infty} f(x_j, y)\, dy.$$

Let μ be the measure associated with this Λ by Theorem 2.14. If E is the x-axis, show that $\mu(E) = \infty$ although $\mu(K) = 0$ for every compact $K \subset E$.

18 This exercise requires more set-theoretic skill than the preceding ones. Let X be a well-ordered uncountable set which has a last element ω_1, such that every predecessor of ω_1 has at most countably many predecessors. ("Construction": Take any well-ordered set which has elements with uncountably many predecessors, and let ω_1 be the first of these; ω_1 is called the first uncountable ordinal.) For $\alpha \in X$, let $P_\alpha[S_\alpha]$ be the set of all predecessors (successors) of α, and call a subset of X open if it is a P_α or an S_β or a $P_\alpha \cap S_\beta$ or a union of such sets. Prove that X is then a compact Hausdorff space. (*Hint*: No well-ordered set contains an infinite decreasing sequence.)

Prove that the complement of the point ω_1 is an open set which is not σ-compact.

Prove that to every $f \in C(X)$ there corresponds an $\alpha \neq \omega_1$ such that f is constant on S_α.

Prove that the intersection of every countable collection $\{K_n\}$ of uncountable compact subsets of X is uncountable. (*Hint*: Consider limits of increasing countable sequences in X which intersect each K_n in infinitely many points.)

Let $\mathfrak{M}$ be the collection of all $E \subset X$ such that either $E \cup \{\omega_1\}$ or $E^c \cup \{\omega_1\}$ contains an uncountable compact set; in the first case, define $\lambda(E) = 1$; in the second case, define $\lambda(E) = 0$. Prove that $\mathfrak{M}$ is a σ-algebra which contains all Borel sets in X, that λ is a measure on $\mathfrak{M}$ which is *not* regular (every neighborhood of ω_1 has measure 1), and that

$$f(\omega_1) = \int_X f\, d\lambda$$

for every $f \in C(X)$. Describe the regular μ which Theorem 2.14 associates with this linear functional.

19 Go through the proof of Theorem 2.14, assuming X to be compact (or even compact metric) rather than just locally compact, and see what simplifications you can find.

20 Find continuous functions f_n: $[0, 1] \to [0, \infty)$ such that $f_n(x) \to 0$ for all $x \in [0, 1]$ as $n \to \infty$, $\int_0^1 f_n(x)\, dx \to 0$, but $\sup_n f_n$ is not in L^1. (This shows that the conclusion of the dominated convergence theorem may hold even when part of its hypothesis is violated.)

21 If X is compact and f: $X \to (-\infty, \infty)$ is upper semicontinuous, prove that f attains its maximum at some point of X.

22 Suppose that X is a metric space, with metric d, and that $f: X \to [0, \infty]$ is lower semicontinuous $f(p) < \infty$ for at least one $p \in X$. For $n = 1, 2, 3, \ldots, x \in X$, define

$$g_n(x) = \inf \{f(p) + nd(x, p): p \in X\}$$

and prove that

(i) $|g_n(x) - g_n(y)| \leq nd(x, y)$,
(ii) $0 \leq g_1 \leq g_2 \leq \cdots \leq f$,
(iii) $g_n(x) \to f(x)$ as $n \to \infty$, for all $x \in X$.

Thus f is the pointwise limit of an increasing sequence of continuous functions. (Note that the converse is almost trivial.)

23 Suppose V is open in R^k and μ is a finite positive Borel measure on R^k. Is the function that sends x to $\mu(V + x)$ necessarily continuous? lower semicontinuous? upper semicontinuous?

24 A *step function* is, by definition, a finite linear combination of characteristic functions of bounded intervals in R^1. Assume $f \in L^1(R^1)$, and prove that there is a sequence $\{g_n\}$ of step functions so that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} |f(x) - g_n(x)| \, dx = 0.$$

25 (i) Find the smallest constant c such that

$$\log (1 + e^t) < c + t \qquad (0 < t < \infty).$$

(ii) Does

$$\lim_{n \to \infty} \frac{1}{n} \int_0^1 \log \{1 + e^{nf(x)}\} \, dx$$

exist for every real $f \in L^1$? If it exists, what is it?

CHAPTER

THREE

L^p-SPACES

Convex Functions and Inequalities

Many of the most common inequalities in analysis have their origin in the notion of convexity.

3.1 Definition A real function φ defined on a segment (a, b), where $-\infty \leq a < b \leq \infty$, is called *convex* if the inequality

$$\varphi((1-\lambda)x + \lambda y) \leq (1-\lambda)\varphi(x) + \lambda\varphi(y) \tag{1}$$

holds whenever $a < x < b$, $a < y < b$, and $0 \leq \lambda \leq 1$.

Graphically, the condition is that if $x < t < y$, then the point $(t, \varphi(t))$ should lie below or on the line connecting the points $(x, \varphi(x))$ and $(y, \varphi(y))$ in the plane. Also, (1) is equivalent to the requirement that

$$\frac{\varphi(t) - \varphi(s)}{t - s} \leq \frac{\varphi(u) - \varphi(t)}{u - t} \tag{2}$$

whenever $a < s < t < u < b$.

The mean value theorem for differentiation, combined with (2), shows immediately that a real differentiable function φ is convex in (a, b) if and only if $a < s < t < b$ implies $\varphi'(s) \leq \varphi'(t)$, i.e., if and only if the derivative φ' is a monotonically increasing function.

For example, the exponential function is convex on $(-\infty, \infty)$.

3.2 Theorem *If φ is convex on (a, b) then φ is continuous on (a, b).*

PROOF The idea of the proof is most easily conveyed in geometric language. Those who may worry that this is not "rigorous" are invited to transcribe it in terms of epsilons and deltas.

Suppose $a < s < x < y < t < b$. Write S for the point $(s, \varphi(s))$ in the plane, and deal similarly with x, y, and t. Then X is on or below the line SY, hence Y is on or above the line through S and X; also, Y is on or below XT. As $y \to x$, it follows that $Y \to X$, i.e., $\varphi(y) \to \varphi(x)$. Left-hand limits are handled in the same manner, and the continuity of φ follows. ////

Note that this theorem depends on the fact that we are working on an *open* segment. For instance, if $\varphi(x) = 0$ on $[0, 1)$ and $\varphi(1) = 1$, then φ satisfies 3.1(1) on $[0, 1]$ without being continuous.

3.3 Theorem (Jensen's Inequality) *Let μ be a positive measure on a σ-algebra $\mathfrak{M}$ in a set Ω, so that $\mu(\Omega) = 1$. If f is a real function in $L^1(\mu)$, if $a < f(x) < b$ for all $x \in \Omega$, and if φ is convex on (a, b), then*

$$\varphi\left(\int_\Omega f\, d\mu\right) \leq \int_\Omega (\varphi \circ f)\, d\mu. \tag{1}$$

Note: The cases $a = -\infty$ and $b = \infty$ are not excluded. It may happen that $\varphi \circ f$ is not in $L^1(\mu)$; in that case, as the proof will show, the integral of $\varphi \circ f$ exists in the extended sense described in Sec. 1.31, and its value is $+\infty$.

PROOF Put $t = \int_\Omega f\, d\mu$. Then $a < t < b$. If β is the supremum of the quotients on the left of 3.1(2), where $a < s < t$, then β is no larger than any of the quotients on the right of 3.1(2), for any $u \in (t, b)$. It follows that

$$\varphi(s) \geq \varphi(t) + \beta(s - t) \qquad (a < s < b). \tag{2}$$

Hence

$$\varphi(f(x)) - \varphi(t) - \beta(f(x) - t) \geq 0 \tag{3}$$

for every $x \in \Omega$. Since φ is continuous, $\varphi \circ f$ is measurable. If we integrate both sides of (3) with respect to μ, (1) follows from our choice of t and the assumption $\mu(\Omega) = 1$. ////

To give an example, take $\varphi(x) = e^x$. Then (1) becomes

$$\exp\left\{\int_\Omega f\, d\mu\right\} \leq \int_\Omega e^f\, d\mu. \tag{4}$$

If Ω is a finite set, consisting of points $p_1, \ldots, p_n$, say, and if

$$\mu(\{p_i\}) = 1/n, \qquad f(p_i) = x_i,$$

(4) becomes

$$\exp\left\{\frac{1}{n}(x_1 + \cdots + x_n)\right\} \leq \frac{1}{n}(e^{x_1} + \cdots + e^{x_n}), \tag{5}$$

for real x_i. Putting $y_i = e^{x_i}$, we obtain the familiar inequality between the arithmetic and geometric means of n positive numbers:

$$(y_1 y_2 \cdots y_n)^{1/n} \leq \frac{1}{n}(y_1 + y_2 + \cdots + y_n). \tag{6}$$

Going back from this to (4), it should become clear why the left and right sides of

$$\exp\left\{\int_\Omega \log g \, d\mu\right\} \leq \int_\Omega g \, d\mu \tag{7}$$

are often called the geometric and arithmetic means, respectively, of the positive function g.

If we take $\mu(\{p_i\}) = \alpha_i > 0$, where $\sum \alpha_i = 1$, then we obtain

$$y_1^{\alpha_1} y_2^{\alpha_2} \cdots y_n^{\alpha_n} \leq \alpha_1 y_1 + \alpha_2 y_2 + \cdots + \alpha_n y_n \tag{8}$$

in place of (6). These are just a few samples of what is contained in Theorem 3.3.

For a converse, see Exercise 20.

3.4 Definition If p and q are positive real numbers such that $p + q = pq$, or equivalently

$$\frac{1}{p} + \frac{1}{q} = 1, \tag{1}$$

then we call p and q a pair of *conjugate exponents*. It is clear that (1) implies $1 < p < \infty$ and $1 < q < \infty$. An important special case is $p = q = 2$.

As $p \to 1$, (1) forces $q \to \infty$. Consequently 1 and ∞ are also regarded as a pair of conjugate exponents. Many analysts denote the exponent conjugate to p by p', often without saying so explicitly.

3.5 Theorem *Let p and q be conjugate exponents, $1 < p < \infty$. Let X be a measure space, with measure μ. Let f and g be measurable functions on X, with range in* $[0, \infty]$. *Then*

$$\int_X fg \, d\mu \leq \left\{\int_X f^p \, d\mu\right\}^{1/p} \left\{\int_X g^q \, d\mu\right\}^{1/q} \tag{1}$$

and

$$\left\{\int_X (f+g)^p \, d\mu\right\}^{1/p} \leq \left\{\int_X f^p \, d\mu\right\}^{1/p} + \left\{\int_X g^p \, d\mu\right\}^{1/p}. \tag{2}$$

The inequality (1) is Hölder's; (2) is Minkowski's. If $p = q = 2$, (1) is known as the Schwarz inequality.

PROOF Let A and B be the two factors on the right of (1). If $A = 0$, then $f = 0$ a.e. (by Theorem 1.39); hence $fg = 0$ a.e., so (1) holds. If $A > 0$ and $B = \infty$, (1) is again trivial. So we need consider only the case $0 < A < \infty$, $0 < B < \infty$. Put

$$F = \frac{f}{A}, \qquad G = \frac{g}{B}. \tag{3}$$

This gives

$$\int_X F^p \, d\mu = \int_X G^q \, d\mu = 1. \tag{4}$$

If $x \in X$ is such that $0 < F(x) < \infty$ and $0 < G(x) < \infty$, there are real numbers s and t such that $F(x) = e^{s/p}$, $G(x) = e^{t/q}$. Since $1/p + 1/q = 1$, the convexity of the exponential function implies that

$$e^{s/p + t/q} \leq p^{-1}e^s + q^{-1}e^t. \tag{5}$$

It follows that

$$F(x)G(x) \leq p^{-1}F(x)^p + q^{-1}G(x)^q \tag{6}$$

for every $x \in X$. Integration of (6) yields

$$\int_X FG \, d\mu \leq p^{-1} + q^{-1} = 1, \tag{7}$$

by (4); inserting (3) into (7), we obtain (1).

Note that (6) could also have been obtained as a special case of the inequality 3.3(8).

To prove (2), we write

$$(f + g)^p = f \cdot (f + g)^{p-1} + g \cdot (f + g)^{p-1}. \tag{8}$$

Hölder's inequality gives

$$\int f \cdot (f + g)^{p-1} \leq \left\{\int f^p\right\}^{1/p} \left\{\int (f + g)^{(p-1)q}\right\}^{1/q}. \tag{9}$$

Let (9′) be the inequality (9) with f and g interchanged. Since $(p - 1)q = p$, addition of (9) and (9′) gives

$$\int (f + g)^p \leq \left\{\int (f + g)^p\right\}^{1/q} \left[\left\{\int f^p\right\}^{1/p} + \left\{\int g^p\right\}^{1/p}\right]. \tag{10}$$

Clearly, it is enough to prove (2) in the case that the left side is greater than 0 and the right side is less than ∞. The convexity of the function t^p for $0 < t < \infty$ shows that

$$\left(\frac{f + g}{2}\right)^p \leq \frac{1}{2}(f^p + g^p).$$

Hence the left side of (2) is less than ∞, and (2) follows from (10) if we divide by the first factor on the right of (10), bearing in mind that $1 - 1/q = 1/p$. This completes the proof. ////

It is sometimes useful to know the conditions under which equality can hold in an inequality. In many cases this information may be obtained by examining the proof of the inequality.

For instance, equality holds in (7) if and only if equality holds in (6) for almost every x. In (5), equality holds if and only if $s = t$. Hence "$F^p = G^q$ a.e." is a necessary and sufficient condition for equality in (7), if (4) is assumed. In terms of the original functions f and g, the following result is then obtained:

Assuming $A < \infty$ and $B < \infty$, equality holds in (1) if and only if there are constants α and β, not both 0, such that $\alpha f^p = \beta g^q$ a.e.

We leave the analogous discussion of equality in (2) as an exercise.

The L^p-spaces

In this section, X will be an arbitrary measure space with a positive measure μ.

3.6 Definition If $0 < p < \infty$ and if f is a complex measurable function on X, define

$$\|f\|_p = \left\{\int_X |f|^p \, d\mu\right\}^{1/p} \tag{1}$$

and let $L^p(\mu)$ consist of all f for which

$$\|f\|_p < \infty. \tag{2}$$

We call $\|f\|_p$ the L^p-norm of f.

If μ is Lebesgue measure on R^k, we write $L^p(R^k)$ instead of $L^p(\mu)$, as in Sec. 2.21. If μ is the counting measure on a set A, it is customary to denote the corresponding L^p-space by $\ell^p(A)$, or simply by ℓ^p, if A is countable. An element of ℓ^p may be regarded as a complex sequence $x = \{\xi_n\}$, and

$$\|x\|_p = \left\{\sum_{n=1}^{\infty} |\xi_n|^p\right\}^{1/p}.$$

3.7 Definition Suppose $g: X \to [0, \infty]$ is measurable. Let S be the set of all real α such that

$$\mu(g^{-1}((\alpha, \infty])) = 0. \tag{1}$$

If $S = \varnothing$, put $\beta = \infty$. If $S \neq \varnothing$, put $\beta = \inf S$. Since

$$g^{-1}((\beta, \infty]) = \bigcup_{n=1}^{\infty} g^{-1}\left(\left(\beta + \frac{1}{n}, \infty\right]\right), \tag{2}$$

and since the union of a countable collection of sets of measure 0 has measure 0, we see that $\beta \in S$. We call β the *essential supremum* of g.

If f is a complex measurable function on X, we define $\|f\|_\infty$ to be the essential supremum of $|f|$, and we let $L^\infty(\mu)$ consist of all f for which $\|f\|_\infty < \infty$. The members of $L^\infty(\mu)$ are sometimes called *essentially bounded* measurable functions on X.

It follows from this definition that the inequality $|f(x)| \leq \lambda$ *holds for almost all* x *if and only if* $\lambda \geq \|f\|_\infty$.

As in Definition 3.6, $L^\infty(R^k)$ denotes the class of all essentially bounded (with respect to Lebesgue measure) functions on R^k, and $\ell^\infty(A)$ is the class of all bounded functions on A. (Here bounded means the same as essentially bounded, since every nonempty set has positive measure!)

3.8 Theorem *If p and q are conjugate exponents, $1 \leq p \leq \infty$, and if $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then $fg \in L^1(\mu)$, and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q. \tag{1}$$

PROOF For $1 < p < \infty$, (1) is simply Hölder's inequality, applied to $|f|$ and $|g|$. If $p = \infty$, note that

$$|f(x)g(x)| \leq \|f\|_\infty |g(x)| \tag{2}$$

for almost all x; integrating (2), we obtain (1). If $p = 1$, then $q = \infty$, and the same argument applies. ////

3.9 Theorem *Suppose $1 \leq p \leq \infty$, and $f \in L^p(\mu)$, $g \in L^p(\mu)$. Then $f + g \in L^p(\mu)$, and*

$$\|f + g\|_p \leq \|f\|_p + \|g\|_p. \tag{1}$$

PROOF For $1 < p < \infty$, this follows from Minkowski's inequality, since

$$\int_X |f + g|^p \, d\mu \leq \int_X (|f| + |g|)^p \, d\mu.$$

For $p = 1$ or $p = \infty$, (1) is a trivial consequence of the inequality $|f + g| \leq |f| + |g|$. ////

3.10 Remarks Fix p, $1 \leq p \leq \infty$. If $f \in L^p(\mu)$ and α is a complex number, it is clear that $\alpha f \in L^p(\mu)$. In fact,

$$\|\alpha f\|_p = |\alpha| \|f\|_p. \tag{1}$$

In conjunction with Theorem 3.9, this shows that $L^p(\mu)$ is a *complex vector space.*

Suppose f, g, and h are in $L^p(\mu)$. Replacing f by $f-g$ and g by $g-h$ in Theorem 3.9, we obtain

$$\|f-h\|_p \leq \|f-g\|_p + \|g-h\|_p. \tag{2}$$

This suggests that a metric may be introduced in $L^p(\mu)$ by defining the distance between f and g to be $\|f-g\|_p$. Call this distance $d(f, g)$ for the moment. Then $0 \leq d(f, g) < \infty$, $d(f, f) = 0$, $d(f, g) = d(g, f)$, and (2) shows that the triangle inequality $d(f, h) \leq d(f, g) + d(g, h)$ is satisfied. The only other property which d should have to define a metric space is that $d(f, g) = 0$ should imply that $f = g$. In our present situation this need not be so; we have $d(f, g) = 0$ *precisely when* $f(x) = g(x)$ *for almost all* x.

Let us write $f \sim g$ if and only if $d(f, g) = 0$. It is clear that this is an equivalence relation in $L^p(\mu)$ which partitions $L^p(\mu)$ into equivalence classes; each class consists of all functions which are equivalent to a given one. If F and G are two equivalence classes, choose $f \in F$ and $g \in G$, and define $d(F, G) = d(f, g)$; note that $f \sim f_1$ and $g \sim g_1$ implies

$$d(f, g) = d(f_1, g_1),$$

so that $d(F, G)$ is well defined.

With this definition, the set of equivalence classes is now a metric space. Note that it is also a vector space, since $f \sim f_1$ and $g \sim g_1$ implies $f + g \sim f_1 + g_1$ and $\alpha f \sim \alpha f_1$.

When $L^p(\mu)$ is regarded as a metric space, then the space which is really under consideration is therefore *not a space whose elements are functions, but a space whose elements are equivalence classes of functions.* For the sake of simplicity of language, it is, however, customary to relegate this distinction to the status of a tacit understanding and to continue to speak of $L^p(\mu)$ as a space of functions. We shall follow this custom.

If $\{f_n\}$ is a sequence in $L^p(\mu)$, if $f \in L^p(\mu)$, and if $\lim_{n\to\infty} \|f_n - f\|_p = 0$, we say that $\{f_n\}$ *converges to* f *in* $L^p(\mu)$ (or that $\{f_n\}$ converges to f in the mean of order p, or that $\{f_n\}$ is L^p-convergent to f). If to every $\epsilon > 0$ there corresponds an integer N such that $\|f_n - f_m\|_p < \epsilon$ as soon as $n > N$ and $m > N$, we call $\{f_n\}$ a *Cauchy sequence* in $L^p(\mu)$. These definitions are exactly as in any metric space.

It is a very important fact that $L^p(\mu)$ is a *complete* metric space, i.e., that every Cauchy sequence in $L^p(\mu)$ converges to an element of $L^p(\mu)$:

3.11 Theorem *$L^p(\mu)$ is a complete metric space, for $1 \leq p \leq \infty$ and for every positive measure μ.*

PROOF Assume first that $1 \leq p < \infty$. Let $\{f_n\}$ be a Cauchy sequence in $L^p(\mu)$. There is a subsequence $\{f_{n_i}\}$, $n_1 < n_2 < \cdots$, such that

$$\|f_{n_{i+1}} - f_{n_i}\|_p < 2^{-i} \qquad (i = 1, 2, 3, \ldots). \tag{1}$$

Put

$$g_k = \sum_{i=1}^{k} |f_{n_{i+1}} - f_{n_i}|, \qquad g = \sum_{i=1}^{\infty} |f_{n_{i+1}} - f_{n_i}|. \tag{2}$$

Since (1) holds, the Minkowski inequality shows that $\|g_k\|_p < 1$ for $k = 1, 2, 3, \ldots$. Hence an application of Fatou's lemma to $\{g_k^p\}$ gives $\|g\|_p \leq 1$. In particular, $g(x) < \infty$ a.e., so that the series

$$f_{n_1}(x) + \sum_{i=1}^{\infty} (f_{n_{i+1}}(x) - f_{n_i}(x)) \tag{3}$$

converges absolutely for almost every $x \in X$. Denote the sum of (3) by $f(x)$, for those x at which (3) converges; put $f(x) = 0$ on the remaining set of measure zero. Since

$$f_{n_1} + \sum_{i=1}^{k-1} (f_{n_{i+1}} - f_{n_i}) = f_{n_k}, \tag{4}$$

we see that

$$f(x) = \lim_{i \to \infty} f_{n_i}(x) \qquad \text{a.e.} \tag{5}$$

Having found a function f which is the pointwise limit a.e. of $\{f_{n_i}\}$, we now have to prove that this f is the L^p-limit of $\{f_n\}$. Choose $\epsilon > 0$. There exists an N such that $\|f_n - f_m\|_p < \epsilon$ if $n > N$ and $m > N$. For every $m > N$, Fatou's lemma shows therefore that

$$\int_X |f - f_m|^p \, d\mu \leq \liminf_{i \to \infty} \int_X |f_{n_i} - f_m|^p \, d\mu \leq \epsilon^p. \tag{6}$$

We conclude from (6) that $f - f_m \in L^p(\mu)$, hence that $f \in L^p(\mu)$ [since $f = (f - f_m) + f_m$], and finally that $\|f - f_m\|_p \to 0$ as $m \to \infty$. This completes the proof for the case $1 \leq p < \infty$.

In $L^\infty(\mu)$ the proof is much easier. Suppose $\{f_n\}$ is a Cauchy sequence in $L^\infty(\mu)$, let A_k and $B_{m,n}$ be the sets where $|f_k(x)| > \|f_k\|_\infty$ and where $|f_n(x) - f_m(x)| > \|f_n - f_m\|_\infty$, and let E be the union of these sets, for k, m, $n = 1, 2, 3, \ldots$. Then $\mu(E) = 0$, and on the complement of E the sequence $\{f_n\}$ converges uniformly to a bounded function f. Define $f(x) = 0$ for $x \in E$. Then $f \in L^\infty(\mu)$, and $\|f_n - f\|_\infty \to 0$ as $n \to \infty$. ////

The preceding proof contains a result which is interesting enough to be stated separately:

3.12 Theorem *If $1 \leq p \leq \infty$ and if $\{f_n\}$ is a Cauchy sequence in $L^p(\mu)$, with limit f, then $\{f_n\}$ has a subsequence which converges pointwise almost everywhere to $f(x)$.*

The simple functions play an interesting role in $L^p(\mu)$:

3.13 Theorem *Let S be the class of all complex, measurable, simple functions on X such that*

$$\mu(\{x: s(x) \neq 0\}) < \infty. \tag{1}$$

If $1 \leq p < \infty$, then S is dense in $L^p(\mu)$.

PROOF First, it is clear that $S \subset L^p(\mu)$. Suppose $f \geq 0, f \in L^p(\mu)$, and let $\{s_n\}$ be as in Theorem 1.17. Since $0 \leq s_n \leq f$, we have $s_n \in L^p(\mu)$, hence $s_n \in S$. Since $|f - s_n|^p \leq f^p$, the dominated convergence theorem shows that $\|f - s_n\|_p \to 0$ as $n \to \infty$. Thus f is in the L^p-closure of S. The general case (f complex) follows from this. ////

Approximation by Continuous Functions

So far we have considered $L^p(\mu)$ on any measure space. Now let X be a locally compact Hausdorff space, and let μ be a measure on a σ-algebra $\mathfrak{M}$ in X, with the properties stated in Theorem 2.14. For example, X might be R^k, and μ might be Lebesgue measure on R^k.

Under these circumstances, we have the following analogue of Theorem 3.13:

3.14 Theorem *For $1 \leq p < \infty$, $C_c(X)$ is dense in $L^p(\mu)$.*

PROOF Define S as in Theorem 3.13. If $s \in S$ and $\epsilon > 0$, there exists a $g \in C_c(X)$ such that $g(x) = s(x)$ except on a set of measure $< \epsilon$, and $|g| \leq \|s\|_\infty$ (Lusin's theorem). Hence

$$\|g - s\|_p \leq 2\epsilon^{1/p}\|s\|_\infty. \tag{1}$$

Since S is dense in $L^p(\mu)$, this completes the proof. ////

3.15 Remarks Let us discuss the relations between the spaces $L^p(R^k)$ (the L^p-spaces in which the underlying measure is Lebesgue measure on R^k) and the space $C_c(R^k)$ in some detail. We consider a fixed dimension k.

For every $p \in [1, \infty]$ we have a metric on $C_c(R^k)$; the distance between f and g is $\|f - g\|_p$. Note that this is a genuine metric, and that we do not have to pass to equivalence classes. The point is that if two continuous functions on R^k are not identical, then they differ on some nonempty open set V, and $m(V) > 0$, since V contains a k-cell. Thus if two members of $C_c(R^k)$ are equal a.e., they are equal. It is also of interest to note that in $C_c(R^k)$ the essential supremum is the same as the actual supremum: for $f \in C_c(R^k)$

$$\|f\|_\infty = \sup_{x \in R^k} |f(x)|. \tag{1}$$

If $1 \leq p < \infty$, Theorem 3.14 says that $C_c(R^k)$ is dense in $L^p(R^k)$, and Theorem 3.11 shows that $L^p(R^k)$ is complete. Thus *$L^p(R^k)$ is the completion of the metric space which is obtained by endowing $C_c(R^k)$ with the L^p-metric.*

The cases $p = 1$ and $p = 2$ are the ones of greatest interest. Let us state once more, in different words, what the preceding result says if $p = 1$ and $k = 1$; the statement shows that the Lebesgue integral is indeed the "right" generalization of the Riemann integral:

If the distance between two continuous functions f and g, with compact supports in R^1, is defined to be

$$\int_{-\infty}^{\infty} |f(t) - g(t)|\, dt, \tag{2}$$

the completion of the resulting metric space consists precisely of the Lebesgue integrable functions on R^1, provided we identify any two that are equal almost everywhere.

Of course, *every* metric space S has a completion S^* whose elements may be viewed abstractly as equivalence classes of Cauchy sequences in S (see [26], p. 82). The important point in the present situation is that the various L^p-completions of $C_c(R^k)$ again turn out to be spaces of functions on R^k.

The case $p = \infty$ differs from the cases $p < \infty$. *The L^∞-completion of $C_c(R^k)$ is not $L^\infty(R^k)$, but is $C_0(R^k)$, the space of all continuous functions on R^k which "vanish at infinity,"* a concept which will be defined in Sec. 3.16. Since (1) shows that the L^∞-norm coincides with the supremum norm on $C_c(R^k)$, the above assertion about $C_0(R^k)$ is a special case of Theorem 3.17.

3.16 Definition A complex function f on a locally compact Hausdorff space X is said to *vanish at infinity* if to every $\epsilon > 0$ there exists a compact set $K \subset X$ such that $|f(x)| < \epsilon$ for all x not in K.

The class of all continuous f on X which vanish at infinity is called $C_0(X)$.

It is clear that $C_c(X) \subset C_0(X)$, and that the two classes coincide if X is compact. In that case we write $C(X)$ for either of them.

3.17 Theorem *If X is a locally compact Hausdorff space, then $C_0(X)$ is the completion of $C_c(X)$, relative to the metric defined by the supremum norm*

$$\|f\| = \sup_{x \in X} |f(x)|. \tag{1}$$

PROOF An elementary verification shows that $C_0(X)$ satisfies the axioms of a metric space if the distance between f and g is taken to be $\|f - g\|$. We have to show that (*a*) $C_c(X)$ is dense in $C_0(X)$ and (*b*) $C_0(X)$ is a complete metric space.

Given $f \in C_0(X)$ and $\epsilon > 0$, there is a compact set K so that $|f(x)| < \epsilon$ outside K. Urysohn's lemma gives us a function $g \in C_c(X)$ such that $0 \leq g \leq 1$ and $g(x) = 1$ on K. Put $h = fg$. Then $h \in C_c(X)$ and $\|f - h\| < \epsilon$. This proves (*a*).

To prove (*b*), let $\{f_n\}$ be a Cauchy sequence in $C_0(X)$, i.e., assume that $\{f_n\}$ converges uniformly. Then its pointwise limit function f is continuous. Given $\epsilon > 0$, there exists an n so that $\|f_n - f\| < \epsilon/2$ and there is a compact set K so that $|f_n(x)| < \epsilon/2$ outside K. Hence $|f(x)| < \epsilon$ outside K, and we have proved that f vanishes at infinity. Thus $C_0(X)$ is complete. ////

Exercises

1 Prove that the supremum of any collection of convex functions on (a, b) is convex on (a, b) (if it is finite) and that pointwise limits of sequences of convex functions are convex. What can you say about upper and lower limits of sequences of convex functions?

2 If φ is convex on (a, b) and if ψ is convex and nondecreasing on the range of φ, prove that $\psi \circ \varphi$ is convex on (a, b). For $\varphi > 0$, show that the convexity of $\log \varphi$ implies the convexity of φ, but not vice versa.

3 Assume that φ is a continuous real function on (a, b) such that

$$\varphi\left(\frac{x+y}{2}\right) \leq \frac{1}{2}\varphi(x) + \frac{1}{2}\varphi(y)$$

for all x and $y \in (a, b)$. Prove that φ is convex. (The conclusion does *not* follow if continuity is omitted from the hypotheses.)

4 Suppose f is a complex measurable function on X, μ is a positive measure on X, and

$$\varphi(p) = \int_X |f|^p \, d\mu = \|f\|_p^p \qquad (0 < p < \infty).$$

Let $E = \{p: \varphi(p) < \infty\}$. Assume $\|f\|_\infty > 0$.

(*a*) If $r < p < s$, $r \in E$, and $s \in E$, prove that $p \in E$.

(*b*) Prove that $\log \varphi$ is convex in the interior of E and that φ is continuous on E.

(*c*) By (*a*), E is connected. Is E necessarily open? Closed? Can E consist of a single point? Can E be any connected subset of $(0, \infty)$?

(*d*) If $r < p < s$, prove that $\|f\|_p \leq \max(\|f\|_r, \|f\|_s)$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(*e*) Assume that $\|f\|_r < \infty$ for some $r < \infty$ and prove that

$$\|f\|_p \to \|f\|_\infty \qquad \text{as } p \to \infty.$$

5 Assume, in addition to the hypotheses of Exercise 4, that

$$\mu(X) = 1.$$

(*a*) Prove that $\|f\|_r \leq \|f\|_s$ if $0 < r < s \leq \infty$.

(*b*) Under what conditions does it happen that $0 < r < s \leq \infty$ and $\|f\|_r = \|f\|_s < \infty$?

(*c*) Prove that $L^r(\mu) \supset L^s(\mu)$ if $0 < r < s$. Under what conditions do these two spaces contain the same functions?

(*d*) Assume that $\|f\|_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p \to 0} \|f\|_p = \exp\left\{\int_X \log |f| \, d\mu\right\}$$

if $\exp\{-\infty\}$ is defined to be 0.

6 Let m be Lebesgue measure on $[0, 1]$, and define $\|f\|_p$ with respect to m. Find all functions Φ on $[0, \infty)$ such that the relation

$$\Phi(\lim_{p\to 0} \|f\|_p) = \int_0^1 (\Phi \circ f)\, dm$$

holds for every bounded, measurable, positive f. Show first that

$$c\Phi(x) + (1 - c)\Phi(1) = \Phi(x^c) \qquad (x > 0,\ 0 \le c \le 1).$$

Compare with Exercise 5(*d*).

7 For some measures, the relation $r < s$ implies $L^r(\mu) \subset L^s(\mu)$; for others, the inclusion is reversed; and there are some for which $L^r(\mu)$ does not contain $L^s(\mu)$ if $r \neq s$. Give examples of these situations, and find conditions on μ under which these situations will occur.

8 If g is a positive function on $(0, 1)$ such that $g(x) \to \infty$ as $x \to 0$, then there is a convex function h on $(0, 1)$ such that $h \le g$ and $h(x) \to \infty$ as $x \to 0$. True or false? Is the problem changed if $(0, 1)$ is replaced by $(0, \infty)$ and $x \to 0$ is replaced by $x \to \infty$?

9 Suppose f is Lebesgue measurable on $(0, 1)$, and not essentially bounded. By Exercise 4(*e*), $\|f\|_p \to \infty$ as $p \to \infty$. Can $\|f\|_p$ tend to ∞ arbitrarily slowly? More precisely, is it true that to every positive function Φ on $(0, \infty)$ such that $\Phi(p) \to \infty$ as $p \to \infty$ one can find an f such that $\|f\|_p \to \infty$ as $p \to \infty$, but $\|f\|_p \le \Phi(p)$ for all sufficiently large p?

10 Suppose $f_n \in L^p(\mu)$, for $n = 1, 2, 3, \ldots$, and $\|f_n - f\|_p \to 0$ and $f_n \to g$ a.e., as $n \to \infty$. What relation exists between f and g?

11 Suppose $\mu(\Omega) = 1$, and suppose f and g are positive measurable functions on Ω such that $fg \ge 1$. Prove that

$$\int_\Omega f\, d\mu \cdot \int_\Omega g\, d\mu \ge 1.$$

12 Suppose $\mu(\Omega) = 1$ and $h: \Omega \to [0, \infty]$ is measurable. If

$$A = \int_\Omega h\, d\mu,$$

prove that

$$\sqrt{1 + A^2} \le \int_\Omega \sqrt{1 + h^2}\, d\mu \le 1 + A.$$

If μ is Lebesgue measure on $[0, 1]$ and if h is continuous, $h = f'$, the above inequalities have a simple geometric interpretation. From this, conjecture (for general Ω) under what conditions on h equality can hold in either of the above inequalities, and prove your conjecture.

13 Under what conditions on f and g does equality hold in the conclusions of Theorems 3.8 and 3.9? You may have to treat the cases $p = 1$ and $p = \infty$ separately.

14 Suppose $1 < p < \infty$, $f \in L^p = L^p((0, \infty))$, relative to Lebesgue measure, and

$$F(x) = \frac{1}{x} \int_0^x f(t)\, dt \qquad (0 < x < \infty).$$

(*a*) Prove Hardy's inequality

$$\|F\|_p \le \frac{p}{p - 1} \|f\|_p$$

which shows that the mapping $f \to F$ carries L^p into L^p.

(*b*) Prove that equality holds only if $f = 0$ a.e.

(*c*) Prove that the constant $p/(p-1)$ cannot be replaced by a smaller one.

(*d*) If $f > 0$ and $f \in L^1$, prove that $F \notin L^1$.

Suggestions: (*a*) Assume first that $f \geq 0$ and $f \in C_c((0, \infty))$. Integration by parts gives

$$\int_0^\infty F^p(x)\,dx = -p \int_0^\infty F^{p-1}(x) x F'(x)\,dx.$$

Note that $xF' = f - F$, and apply Hölder's inequality to $\int F^{p-1} f$. Then derive the general case. (*c*) Take $f(x) = x^{-1/p}$ on $[1, A]$, $f(x) = 0$ elsewhere, for large A. See also Exercise 14, Chap. 8.

15 Suppose $\{a_n\}$ is a sequence of positive numbers. Prove that

$$\sum_{N=1}^{\infty} \left(\frac{1}{N} \sum_{n=1}^{N} a_n \right)^p \leq \left(\frac{p}{p-1} \right)^p \sum_{n=1}^{\infty} a_n^p$$

if $1 < p < \infty$. *Hint*: If $a_n \geq a_{n+1}$, the result can be made to follow from Exercise 14. This special case implies the general one.

16 Prove Egoroff's theorem: If $\mu(X) < \infty$, if $\{f_n\}$ is a sequence of complex measurable functions which converges pointwise at every point of X, and if $\epsilon > 0$, there is a measurable set $E \subset X$, with $\mu(X - E) < \epsilon$, such that $\{f_n\}$ converges uniformly on E.

(The conclusion is that by redefining the f_n on a set of arbitrarily small measure we can convert a pointwise convergent sequence to a uniformly convergent one; note the similarity with Lusin's theorem.)

Hint: Put

$$S(n, k) = \bigcap_{i, j > n} \left\{ x : |f_i(x) - f_j(x)| < \frac{1}{k} \right\},$$

show that $\mu(S(n, k)) \to \mu(X)$ as $n \to \infty$, for each k, and hence that there is a suitably increasing sequence $\{n_k\}$ such that $E = \bigcap S(n_k, k)$ has the desired property.

Show that the theorem does not extend to σ-finite spaces.

Show that the theorem does extend, with essentially the same proof, to the situation in which the sequence $\{f_n\}$ is replaced by a family $\{f_t\}$, where t ranges over the positive reals; the assumptions are now that, for all $x \in X$,

(i) $\lim_{t \to \infty} f_t(x) = f(x)$ and

(ii) $t \to f_t(x)$ is continuous.

17 (*a*) If $0 < p < \infty$, put $\gamma_p = \max(1, 2^{p-1})$, and show that

$$|\alpha - \beta|^p \leq \gamma_p (|\alpha|^p + |\beta|^p)$$

for arbitrary complex numbers α and β.

(*b*) Suppose μ is a positive measure on X, $0 < p < \infty$, $f \in L^p(\mu)$, $f_n \in L^p(\mu)$, $f_n(x) \to f(x)$ a.e., and $\|f_n\|_p \to \|f\|_p$ as $n \to \infty$. Show that then $\lim \|f - f_n\|_p = 0$, by completing the two proofs that are sketched below.

(i) By Egoroff's theorem, $X = A \cup B$ in such a way that $\int_A |f|^p < \epsilon$, $\mu(B) < \infty$, and $f_n \to f$ uniformly on B. Fatou's lemma, applied to $\int_B |f_n|^p$, leads to

$$\limsup \int_A |f_n|^p\,d\mu \leq \epsilon.$$

(ii) Put $h_n = \gamma_p(|f|^p + |f_n|^p) - |f - f_n|^p$, and use Fatou's lemma as in the proof of Theorem 1.34.

(*c*) Show that the conclusion of (*b*) is false if the hypothesis $\|f_n\|_p \to \|f\|_p$ is omitted, even if $\mu(X) < \infty$.

18 Let μ be a positive measure on X. A sequence $\{f_n\}$ of complex measurable functions on X is said to *converge in measure* to the measurable function f if to every $\epsilon > 0$ there corresponds an N such that

$$\mu(\{x: |f_n(x) - f(x)| > \epsilon\}) < \epsilon$$

for all $n > N$. (This notion is of importance in probability theory.) Assume $\mu(X) < \infty$ and prove the following statements:

(*a*) If $f_n(x) \to f(x)$ a.e., then $f_n \to f$ in measure.

(*b*) If $f_n \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$, then $f_n \to f$ in measure; here $1 \leq p \leq \infty$.

(*c*) If $f_n \to f$ in measure, then $\{f_n\}$ has a subsequence which converges to f a.e.

Investigate the converses of (*a*) and (*b*). What happens to (*a*), (*b*), and (*c*) if $\mu(X) = \infty$, for instance, if μ is Lebesgue measure on R^1?

19 Define the *essential range* of a function $f \in L^\infty(\mu)$ to be the set R_f consisting of all complex numbers w such that

$$\mu(\{x: |f(x) - w| < \epsilon\}) > 0$$

for every $\epsilon > 0$. Prove that R_f is compact. What relation exists between the set R_f and the number $\|f\|_\infty$?

Let A_f be the set of all averages

$$\frac{1}{\mu(E)} \int_E f \, d\mu$$

where $E \in \mathfrak{M}$ and $\mu(E) > 0$. What relations exist between A_f and R_f? Is A_f always closed? Are there measures μ such that A_f is convex for every $f \in L^\infty(\mu)$? Are there measures μ such that A_f fails to be convex for some $f \in L^\infty(\mu)$?

How are these results affected if $L^\infty(\mu)$ is replaced by $L^1(\mu)$, for instance?

20 Suppose φ is a real function on R^1 such that

$$\varphi\left(\int_0^1 f(x)\, dx\right) \leq \int_0^1 \varphi(f)\, dx$$

for every real bounded measurable f. Prove that φ is then convex.

21 Call a metric space Y a *completion* of a metric space X if X is dense in Y and Y is complete. In Sec. 3.15 reference was made to "the" completion of a metric space. State and prove a uniqueness theorem which justifies this terminology.

22 Suppose X is a metric space in which every Cauchy sequence has a convergent subsequence. Does it follow that X is complete? (See the proof of Theorem 3.11.)

23 Suppose μ is a positive measure on X, $\mu(X) < \infty$, $f \in L^\infty(\mu)$, $\|f\|_\infty > 0$, and

$$\alpha_n = \int_X |f|^n \, d\mu \qquad (n = 1, 2, 3, \ldots).$$

Prove that

$$\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty.$$

24 Suppose μ is a positive measure, $f \in L^p(\mu)$, $g \in L^p(\mu)$.

(*a*) If $0 < p < 1$, prove that

$$\int \big| |f|^p - |g|^p \big| \, d\mu \leq \int |f - g|^p \, d\mu$$

that $\Delta(f, g) = \int |f - g|^p \, d\mu$ defines a metric on $L^p(\mu)$, and that the resulting metric space is complete.

(*b*) If $1 \le p < \infty$ and $\|f\|_p \le R$, $\|g\|_p \le R$, prove that

$$\int \big| |f|^p - |g|^p \big| \, d\mu \le 2pR^{p-1}\|f-g\|_p.$$

Hint: Prove first, for $x \ge 0$, $y \ge 0$, that

$$|x^p - y^p| \le \begin{cases} |x-y|^p & \text{if } 0 < p < 1, \\ p|x-y|(x^{p-1}+y^{p-1}) & \text{if } 1 \le p < \infty. \end{cases}$$

Note that (*a*) and (*b*) establish the continuity of the mapping $f \to |f|^p$ that carries $L^p(\mu)$ into $L^1(\mu)$.

25 Suppose μ is a positive measure on X and $f: X \to (0, \infty)$ satisfies $\int_X f\,d\mu = 1$. Prove, for every $E \subset X$ with $0 < \mu(E) < \infty$, that

$$\int_E (\log f)\, d\mu \le \mu(E) \log \frac{1}{\mu(E)}$$

and, when $0 < p < 1$,

$$\int_E f^p \, d\mu \le \mu(E)^{1-p}.$$

26 If f is a positive measurable function on $[0, 1]$, which is larger,

$$\int_0^1 f(x) \log f(x)\, dx \qquad \text{or} \qquad \int_0^1 f(s)\, ds \int_0^1 \log f(t)\, dt?$$

CHAPTER

FOUR

ELEMENTARY HILBERT SPACE THEORY

Inner Products and Linear Functionals

4.1 Definition A complex vector space H is called an *inner product space* (or *unitary space*) if to each ordered pair of vectors x and $y \in H$ there is associated a complex number (x, y), the so-called "inner product" (or "scalar product") of x and y, such that the following rules hold:

(*a*) $(y, x) = \overline{(x, y)}$. (The bar denotes complex conjugation.)
(*b*) $(x + y, z) = (x, z) + (y, z)$ if x, y, and $z \in H$.
(*c*) $(\alpha x, y) = \alpha(x, y)$ if x and $y \in H$ and α is a scalar.
(*d*) $(x, x) \geq 0$ for all $x \in H$.
(*e*) $(x, x) = 0$ only if $x = 0$.

Let us list some immediate consequences of these axioms:

(*c*) implies that $(0, y) = 0$ for all $y \in H$.
(*b*) and (*c*) may be combined into the statement: *For every* $y \in H$, *the mapping* $x \to (x, y)$ *is a linear functional on* H.
(*a*) and (*c*) show that $(x, \alpha y) = \bar{\alpha}(x, y)$.
(*a*) and (*b*) imply the second distributive law:

$$(z, x + y) = (z, x) + (z, y).$$

By (*d*), we may define $\|x\|$, the *norm* of the vector $x \in H$, to be the non-negative square root of (x, x). Thus

(*f*) $$\|x\|^2 = (x, x).$$

4.2 The Schwarz Inequality *The properties* 4.1 (*a*) *to* (*d*) *imply that*

$$|(x, y)| \leq \|x\| \, \|y\|$$

for all x *and* $y \in H$.

PROOF Put $A = \|x\|^2$, $B = |(x, y)|$, and $C = \|y\|^2$. There is a complex number α such that $|\alpha| = 1$ and $\alpha(y, x) = B$. For any real r, we then have

$$(x - r\alpha y, x - r\alpha y) = (x, x) - r\alpha(y, x) - r\bar{\alpha}(x, y) + r^2(y, y). \tag{1}$$

The expression on the left is real and not negative. Hence

$$A - 2Br + Cr^2 \geq 0 \tag{2}$$

for every real r. If $C = 0$, we must have $B = 0$; otherwise (2) is false for large positive r. If $C > 0$, take $r = B/C$ in (2), and obtain $B^2 \leq AC$. ////

4.3 The Triangle Inequality *For* x *and* $y \in H$, *we have*

$$\|x + y\| \leq \|x\| + \|y\|.$$

PROOF By the Schwarz inequality,

$$\begin{aligned} \|x + y\|^2 &= (x + y, x + y) = (x, x) + (x, y) + (y, x) + (y, y) \\ &\leq \|x\|^2 + 2\|x\| \, \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2. \end{aligned}$$ ////

4.4 Definition It follows from the triangle inequality that

$$\|x - z\| \leq \|x - y\| + \|y - z\| \qquad (x, y, z \in H). \tag{1}$$

If we define the distance between x and y to be $\|x - y\|$, all the axioms for a metric space are satisfied; here, for the first time, we use part (*e*) of Definition 4.1.

Thus H is now a metric space. If this metric space is *complete*, i.e., if every Cauchy sequence converges in H, then H is called a *Hilbert space*.

Throughout the rest of this chapter, the letter H will denote a Hilbert space.

4.5 Examples

(*a*) For any fixed n, the set C^n of all n-tuples

$$x = (\xi_1, \ldots, \xi_n),$$

where $\xi_1, \ldots, \xi_n$ are complex numbers, is a Hilbert space if addition and scalar multiplication are defined componentwise, as usual, and if

$$(x, y) = \sum_{j=1}^{n} \xi_j \bar{\eta}_j \qquad (y = (\eta_1, \ldots, \eta_n)).$$

(*b*) If μ is any positive measure, $L^2(\mu)$ is a Hilbert space, with inner product

$$(f, g) = \int_X f\bar{g}\, d\mu.$$

The integrand on the right is in $L^1(\mu)$, by Theorem 3.8, so that (f, g) is well defined. Note that

$$\|f\| = (f,f)^{1/2} = \left\{\int_X |f|^2\, d\mu\right\}^{1/2} = \|f\|_2.$$

The completeness of $L^2(\mu)$ (Theorem 3.11) shows that $L^2(\mu)$ is indeed a Hilbert space. [We recall that $L^2(\mu)$ should be regarded as a space of *equivalence classes* of functions; compare the discussion in Sec. 3.10.]

For $H = L^2(\mu)$, the inequalities 4.2 and 4.3 turn out to be special cases of the inequalities of Hölder and Minkowski.

Note that Example (*a*) is a special case of (*b*). What is the measure in (*a*)?

(*c*) The vector space of all continuous complex functions on [0, 1] is an inner product space if

$$(f, g) = \int_0^1 f(t)\overline{g(t)}\, dt$$

but is not a Hilbert space.

4.6 Theorem *For any fixed $y \in H$, the mappings*

$$x \to (x, y), \quad x \to (y, x), \quad x \to \|x\|$$

are continuous functions on H.

PROOF The Schwarz inequality implies that

$$|(x_1, y) - (x_2, y)| = |(x_1 - x_2, y)| \leq \|x_1 - x_2\|\, \|y\|,$$

which proves that $x \to (x, y)$ is, in fact, uniformly continuous, and the same is true for $x \to (y, x)$. The triangle inequality $\|x_1\| \leq \|x_1 - x_2\| + \|x_2\|$ yields

$$\|x_1\| - \|x_2\| \leq \|x_1 - x_2\|,$$

and if we interchange x_1 and x_2 we see that

$$\big|\, \|x_1\| - \|x_2\| \,\big| \leq \|x_1 - x_2\|$$

for all x_1 and $x_2 \in H$. Thus $x \to \|x\|$ is also uniformly continuous. ////

4.7 Subspaces A subset M of a vector space V is called a *subspace* of V if M is itself a vector space, relative to the addition and scalar multiplication which are defined in V. A necessary and sufficient condition for a set $M \subset V$ to be a subspace is that $x + y \in M$ and $\alpha x \in M$ whenever x and $y \in M$ and α is a scalar.

In the vector space context, the word "subspace" will always have this meaning. Sometimes, for emphasis, we may use the term "linear subspace" in place of subspace.

For example, if V is the vector space of all complex functions on a set S, the set of all bounded complex functions on S is a subspace of V, but the set of all $f \in V$ with $|f(x)| \leq 1$ for all $x \in S$ is not. The real vector space R^3 has the following subspaces, and no others: (*a*) R^3, (*b*) all planes through the origin 0, (*c*) all straight lines through 0, and (*d*) $\{0\}$.

A *closed subspace* of H is a subspace that is a closed set relative to the topology induced by the metric of H.

Note that *if M is a subspace of H, so is its closure $\bar{M}$.* To see this, pick x and y in $\bar{M}$ and let α be a scalar. There are sequences $\{x_n\}$ and $\{y_n\}$ in M that converge to x and y, respectively. It is then easy to verify that $x_n + y_n$ and αx_n converge to $x + y$ and αx, respectively. Thus $x + y \in \bar{M}$ and $\alpha x \in \bar{M}$.

4.8 Convex Sets A set E in a vector space V is said to be *convex* if it has the following geometric property: Whenever $x \in E$, $y \in E$, and $0 < t < 1$, the point

$$z_t = (1 - t)x + ty$$

also lies in E. As t runs from 0 to 1, one may visualize z_t as describing a straight line segment in V, from x to y. Convexity requires that E contain the segments between any two of its points.

It is clear that every subspace of V is convex.

Also, if E is convex, so is each of its translates

$$E + x = \{y + x : y \in E\}.$$

4.9 Orthogonality If $(x, y) = 0$ for some x and $y \in H$, we say that x is orthogonal to y, and sometimes write $x \perp y$. Since $(x, y) = 0$ implies $(y, x) = 0$, the relation $\perp$ is symmetric.

Let $x^\perp$ denote the set of all $y \in H$ which are orthogonal to x; and if M is a subspace of H, let $M^\perp$ be the set of all $y \in H$ which are orthogonal to every $x \in M$.

Note that $x^\perp$ is a subspace of H, since $x \perp y$ and $x \perp y'$ implies $x \perp (y + y')$ and $x \perp \alpha y$. Also, $x^\perp$ is precisely the set of points where the continuous function $y \to (x, y)$ is 0. Hence $x^\perp$ is a *closed* subspace of H. Since

$$M^\perp = \bigcap_{x \in M} x^\perp,$$

$M^\perp$ is an intersection of closed subspaces, and it follows that *$M^\perp$ is a closed subspace of H.*

4.10 Theorem *Every nonempty, closed, convex set E in a Hilbert space H contains a unique element of smallest norm.*

In other words, there is one and only one $x_0 \in E$ such that $\|x_0\| \leq \|x\|$ for every $x \in E$.

PROOF An easy computation, using only the properties listed in Definition 4.1, establishes the identity

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2 \qquad (x \text{ and } y \in H). \tag{1}$$

This is known as the *parallelogram law*: If we interpret $\|x\|$ to be the length of the vector x, (1) says that the sum of the squares of the diagonals of a parallelogram is equal to the sum of the squares of its sides, a familiar proposition in plane geometry.

Let $\delta = \inf\{\|x\| : x \in E\}$. For any x and $y \in E$, we apply (1) to $\frac{1}{2}x$ and $\frac{1}{2}y$ and obtain

$$\tfrac{1}{4}\|x - y\|^2 = \tfrac{1}{2}\|x\|^2 + \tfrac{1}{2}\|y\|^2 - \left\|\frac{x + y}{2}\right\|^2. \tag{2}$$

Since E is convex, $(x + y)/2 \in E$. Hence

$$\|x - y\|^2 \leq 2\|x\|^2 + 2\|y\|^2 - 4\delta^2 \qquad (x \text{ and } y \in E). \tag{3}$$

If also $\|x\| = \|y\| = \delta$, then (3) implies $x = y$, and we have proved the uniqueness assertion of the theorem.

The definition of δ shows that there is a sequence $\{y_n\}$ in E so that $\|y_n\| \to \delta$ as $n \to \infty$. Replace x and y in (3) by y_n and y_m. Then, as $n \to \infty$ and $m \to \infty$, the right side of (3) will tend to 0. This shows that $\{y_n\}$ is a Cauchy sequence. Since H is complete, there exists an $x_0 \in H$ so that $y_n \to x_0$, i.e., $\|y_n - x_0\| \to 0$, as $n \to \infty$. Since $y_n \in E$ and E is closed, $x_0 \in E$. Since the norm is a continuous function on H (Theorem 4.6), it follows that

$$\|x_0\| = \lim_{n \to \infty} \|y_n\| = \delta. \qquad ////$$

4.11 Theorem *Let M be a closed subspace of a Hilbert space H.*

(a) Every $x \in H$ has then a unique decomposition

$$x = Px + Qx$$

into a sum of $Px \in M$ and $Qx \in M^\perp$.

(b) Px and Qx are the nearest points to x in M and in $M^\perp$, respectively.

(c) The mappings $P: H \to M$ and $Q: H \to M^\perp$ are linear.

(d) $\|x\|^2 = \|Px\|^2 + \|Qx\|^2$.

Corollary *If $M \neq H$, then there exists $y \in H$, $y \neq 0$, such that $y \perp M$.*

P and Q are called the *orthogonal projections* of H onto M and $M^\perp$.

PROOF As regards the uniqueness in (*a*), suppose that $x' + y' = x'' + y''$ for some vectors x', x'' in M and y', y'' in $M^\perp$. Then

$$x' - x'' = y'' - y'.$$

Since $x' - x'' \in M$, $y'' - y' \in M^\perp$, and $M \cap M^\perp = \{0\}$ [an immediate consequence of the fact that $(x, x) = 0$ implies $x = 0$], we have $x'' = x'$, $y'' = y'$.

To prove the existence of the decomposition, note that the set

$$x + M = \{x + y : y \in M\}$$

is closed and convex. Define Qx to be the element of smallest norm in $x + M$; this exists, by Theorem 4.10. Define $Px = x - Qx$.

Since $Qx \in x + M$, it is clear that $Px \in M$. Thus P maps H into M.

To prove that Q maps H into $M^\perp$ we show that $(Qx, y) = 0$ for all $y \in M$. Assume $\|y\| = 1$, without loss of generality, and put $z = Qx$. The minimizing property of Qx shows that

$$(z, z) = \|z\|^2 \leq \|z - \alpha y\|^2 = (z - \alpha y, z - \alpha y)$$

for every scalar α. This simplifies to

$$0 \leq -\alpha(y, z) - \bar{\alpha}(z, y) + \alpha\bar{\alpha}.$$

With $\alpha = (z, y)$, this gives $0 \leq -|(z, y)|^2$, so that $(z, y) = 0$. Thus $Qx \in M^\perp$.

We have already seen that $Px \in M$. If $y \in M$, it follows that

$$\|x - y\|^2 = \|Qx + (Px - y)\|^2 = \|Qx\|^2 + \|Px - y\|^2$$

which is obviously minimized when $y = Px$.

We have now proved (*a*) and (*b*). If we apply (*a*) to x, to y, and to $\alpha x + \beta y$, we obtain

$$P(\alpha x + \beta y) - \alpha Px - \beta Py = \alpha Qx + \beta Qy - Q(\alpha x + \beta y).$$

The left side is in M, the right side in $M^\perp$. Hence both are 0, so P and Q are linear.

Since $Px \perp Qx$, (*d*) follows from (*a*).

To prove the corollary, take $x \in H$, $x \notin M$, and put $y = Qx$. Since $Px \in M$, $x \neq Px$, hence $y = x - Px \neq 0$. ////

We have already observed that $x \to (x, y)$ is, for each $y \in H$, a continuous linear functional on H. It is a very important fact that *all* continuous linear functionals on H are of this type.

4.12 Theorem *If L is a continuous linear functional on H, then there is a unique $y \in H$ such that*

$$Lx = (x, y) \qquad (x \in H). \tag{1}$$

PROOF If $Lx = 0$ for all x, take $y = 0$. Otherwise, define

$$M = \{x: Lx = 0\}. \tag{2}$$

The linearity of L shows that M is a subspace. The continuity of L shows that M is closed. Since $Lx \neq 0$ for some $x \in H$, Theorem 4.11 shows that $M^\perp$ does not consist of 0 alone.

Hence there exists $z \in M^\perp$, with $\|z\| = 1$. Put

$$u = (Lx)z - (Lz)x. \tag{3}$$

Since $Lu = (Lx)(Lz) - (Lz)(Lx) = 0$, we have $u \in M$. Thus $(u, z) = 0$. This gives

$$Lx = (Lx)(z, z) = (Lz)(x, z). \tag{4}$$

Thus (1) holds with $y = \alpha z$, where $\bar{\alpha} = Lz$.

The uniqueness of y is easily proved, for if $(x, y) = (x, y')$ for all $x \in H$, set $z = y - y'$; then $(x, z) = 0$ for all $x \in H$; in particular, $(z, z) = 0$, hence $z = 0$. ////

Orthonormal Sets

4.13 Definitions If V is a vector space, if $x_1, \ldots, x_k \in V$, and if $c_1, \ldots, c_k$ are scalars, then $c_1x_1 + \cdots + c_kx_k$ is called a *linear combination* of $x_1, \ldots, x_k$. The set $\{x_1, \ldots, x_k\}$ is called *independent* if $c_1x_1 + \cdots + c_kx_k = 0$ implies that $c_1 = \cdots = c_k = 0$. A set $S \subset V$ is independent if every finite subset of S is independent. The set $[S]$ of all linear combinations of all finite subsets of S (also called the set of all *finite linear combinations* of members of S) is clearly a vector space; $[S]$ is the smallest subspace of V which contains S; $[S]$ is called the *span* of S, or the space spanned by S.

A set of vectors u_α in a Hilbert space H, where α runs through some index set A, is called *orthonormal* if it satisfies the orthogonality relations $(u_\alpha, u_\beta) = 0$ for all $\alpha \neq \beta$, $\alpha \in A$, and $\beta \in A$, and if it is normalized so that $\|u_\alpha\| = 1$ for each $\alpha \in A$. In other words, $\{u_\alpha\}$ is orthonormal provided that

$$(u_\alpha, u_\beta) = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases} \tag{1}$$

If $\{u_\alpha: \alpha \in A\}$ is orthonormal, we associate with each $x \in H$ a complex function $\hat{x}$ on the index set A, defined by

$$\hat{x}(\alpha) = (x, u_\alpha) \qquad (\alpha \in A). \tag{2}$$

One sometimes calls the numbers $\hat{x}(\alpha)$ the *Fourier coefficients* of x, relative to the set $\{u_\alpha\}$.

We begin with some simple facts about *finite* orthonormal sets.

4.14 Theorem *Suppose that* $\{u_\alpha: \alpha \in A\}$ *is an orthonormal set in* H *and that* F *is a finite subset of* A. *Let* M_F *be the span of* $\{u_\alpha: \alpha \in F\}$.

(*a*) *If* φ *is a complex function on* A *that is* 0 *outside* F, *then there is a vector* $y \in M_F$, *namely*

$$y = \sum_{\alpha \in F} \varphi(\alpha) u_\alpha \tag{1}$$

that has $\hat{y}(\alpha) = \varphi(\alpha)$ *for every* $\alpha \in A$. *Also,*

$$\|y\|^2 = \sum_{\alpha \in F} |\varphi(\alpha)|^2. \tag{2}$$

(*b*) *If* $x \in H$ *and*

$$s_F(x) = \sum_{\alpha \in F} \hat{x}(\alpha) u_\alpha \tag{3}$$

then

$$\|x - s_F(x)\| < \|x - s\| \tag{4}$$

for every $s \in M_F$, *except for* $s = s_F(x)$, *and*

$$\sum_{\alpha \in F} |\hat{x}(\alpha)|^2 \leq \|x\|^2. \tag{5}$$

PROOF Part (*a*) is an immediate consequence of the orthogonality relations 4.13(1).

In the proof of (*b*), let us write s_F in place of $s_F(x)$, and note that $\hat{s}_F(\alpha) = \hat{x}(\alpha)$ for all $\alpha \in F$. This says that $(x - s_F) \perp u_\alpha$ if $\alpha \in F$, hence $(x - s_F) \perp (s_F - s)$ for every $s \in M_F$, and therefore

$$\|x - s\|^2 = \|(x - s_F) + (s_F - s)\|^2 = \|x - s_F\|^2 + \|s_F - s\|^2. \tag{6}$$

This gives (4). With $s = 0$, (6) gives $\|s_F\|^2 \leq \|x\|^2$, which is the same as (5), because of (2). ////

The inequality (4) states that the "partial sum" $s_F(x)$ of the "Fourier series" $\sum \hat{x}(\alpha) u_\alpha$ of x is the unique best approximation to x in M_F, relative to the metric defined by the Hilbert space norm.

4.15 We want to drop the finiteness condition that appears in Theorem 4.14 (thus obtaining Theorems 4.17 and 4.18) without even restricting ourselves to sets that are necessarily countable. For this reason it seems advisable to clarify the meaning of the symbol $\sum_{\alpha \in A} \varphi(\alpha)$ when α ranges over an arbitrary set A.

Assume $0 \leq \varphi(\alpha) \leq \infty$ for each $\alpha \in A$. Then

$$\sum_{\alpha \in A} \varphi(\alpha) \tag{1}$$

denotes the supremum of the set of all finite sums $\varphi(\alpha_1) + \cdots + \varphi(\alpha_n)$, where $\alpha_1, \ldots, \alpha_n$ are distinct members of A.

A moment's consideration will show that *the sum* (1) *is thus precisely the Lebesgue integral of* φ *relative to the counting measure* μ *on* A.

In this context one usually writes $\ell^p(A)$ for $L^p(\mu)$. A complex function φ with domain A is thus in $\ell^2(A)$ if and only if

$$\sum_{\alpha \in A} |\varphi(\alpha)|^2 < \infty. \tag{2}$$

Example 4.5(*b*) shows that $\ell^2(A)$ is a Hilbert space, with inner product

$$(\varphi, \psi) = \sum_{\alpha \in A} \varphi(\alpha)\overline{\psi(\alpha)}. \tag{3}$$

Here, again, the sum over A stands for the integral of $\varphi\bar{\psi}$ with respect to the counting measure; note that $\varphi\bar{\psi} \in \ell^1(A)$ because φ and ψ are in $\ell^2(A)$.

Theorem 3.13 shows that *the functions* φ *that are zero except on some finite subset of* A *are dense in* $\ell^2(A)$.

Moreover, if $\varphi \in \ell^2(A)$, then $\{\alpha \in A: \varphi(\alpha) \neq 0\}$ *is at most countable.* For if A_n is the set of all α where $|\varphi(\alpha)| > 1/n$, then the number of elements of A is at most

$$\sum_{\alpha \in A_n} |n\varphi(\alpha)|^2 \leq n^2 \sum_{\alpha \in A} |\varphi(\alpha)|^2.$$

Each A_n ($n = 1, 2, 3, \ldots$) is thus a finite set.

The following lemma about complete metric spaces will make it easy to pass from finite orthonormal sets to infinite ones.

4.16 Lemma *Suppose that*

(*a*) X *and* Y *are metric spaces,* X *is complete,*
(*b*) $f: X \to Y$ *is continuous,*
(*c*) X *has a dense subset* X_0 *on which* f *is an isometry, and*
(*d*) $f(X_0)$ *is dense in* Y.

Then f *is an isometry of* X *onto* Y.

The most important part of the conclusion is that f maps X *onto all of* Y.

Recall that an isometry is simply a mapping that preserves distances. Thus, by assumption, the distance between $f(x_1)$ and $f(x_2)$ in Y is equal to that between x_1 and x_2 in X, for all points x_1, x_2 in X_0.

PROOF The fact that f is an isometry on X is an immediate consequence of the continuity of f, since X_0 is dense in X.

Pick $y \in Y$. Since $f(X_0)$ is dense in Y, there is a sequence $\{x_n\}$ in X_0 such that $f(x_n) \to y$ as $n \to \infty$. Thus $\{f(x_n)\}$ is a Cauchy sequence in Y. Since f is an isometry on X_0, it follows that $\{x_n\}$ is also a Cauchy sequence. The completeness of X implies now that $\{x_n\}$ converges to some $x \in X$, and the continuity of f shows that $f(x) = \lim f(x_n) = y$. ////

4.17 Theorem *Let $\{u_\alpha: \alpha \in A\}$ be an orthonormal set in H, and let P be the space of all finite linear combinations of the vectors u_α.*

The inequality

$$\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 \leq \|x\|^2 \tag{1}$$

holds then for every $x \in H$, and $x \to \hat{x}$ is a continuous linear mapping of H onto $\ell^2(A)$ whose restriction to the closure $\bar{P}$ of P is an isometry of $\bar{P}$ onto $\ell^2(A)$.

PROOF Since the inequality 4.14(5) holds for every finite set $F \subset A$, we have (1), the so-called *Bessel inequality.*

Define f on H by $f(x) = \hat{x}$. Then (1) shows explicitly that f maps H into $\ell^2(A)$. The linearity of f is obvious. If we apply (1) to $x - y$ we see that

$$\|f(y) - f(x)\|_2 = \|\hat{y} - \hat{x}\|_2 \leq \|y - x\|.$$

Thus f is continuous. Theorem 4.14(a) shows that f is an isometry of P onto the dense subspace of $\ell^2(A)$ consisting of those functions whose support is a finite set $F \subset A$. The theorem follows therefore from Lemma 4.16, applied with $X = \bar{P}$, $X_0 = P$, $Y = \ell^2(A)$; note that $\bar{P}$, being a closed subset of the complete metric space H, is itself complete. ////

The fact that the mapping $x \to \hat{x}$ carries H *onto* $\ell^2(A)$ is known as the *Riesz-Fischer theorem.*

4.18 Theorem *Let $\{u_\alpha: \alpha \in A\}$ be an orthonormal set in H. Each of the following four conditions on $\{u_\alpha\}$ implies the other three:*

(i) *$\{u_\alpha\}$ is a maximal orthonormal set in H.*
(ii) *The set P of all finite linear combinations of members of $\{u_\alpha\}$ is dense in H.*
(iii) *The equality*

$$\sum_{\alpha \in A} |\hat{x}(\alpha)|^2 = \|x\|^2$$

holds for every $x \in H$.
(iv) *The equality*

$$\sum_{\alpha \in A} \hat{x}(\alpha)\overline{\hat{y}(\alpha)} = (x, y)$$

holds for all $x \in H$ and $y \in H$.

The last formula is known as *Parseval's identity.* Observe that $\hat{x}$ and $\hat{y}$ are in $\ell^2(A)$, hence $\hat{x}\bar{\hat{y}}$ is in $\ell^1(A)$, so that the sum in (iv) is well defined. Of course, (iii) is the special case $x = y$ of (iv).

Maximal orthonormal sets are often called *complete orthonormal sets* or *orthonormal bases.*

PROOF To say that $\{u_\alpha\}$ is maximal means simply that no vector of H can be adjoined to $\{u_\alpha\}$ in such a way that the resulting set is still orthonormal. This happens precisely when there is no $x \neq 0$ in H that is orthogonal to every u_α.

We shall prove that (i)$\rightarrow$ (ii)$\rightarrow$ (iii)$\rightarrow$ (iv)$\rightarrow$ (i).

If P is not dense in H, then its closure $\bar{P}$ is not all of H, and the corollary to Theorem 4.11 implies that $P^\perp$ contains a nonzero vector. Thus $\{u_\alpha\}$ is not maximal when P is not dense, and (i) implies (ii).

If (ii) holds, so does (iii), by Theorem 4.17.

The implication (iii)$\rightarrow$ (iv) follows from the easily proved Hilbert space identity (sometimes called the "polarization identity")

$$4(x, y) = \|x + y\|^2 - \|x - y\|^2 + i\|x + iy\|^2 - i\|x - iy\|^2$$

which expresses the inner product (x, y) in terms of norms and which is equally valid with $\hat{x}$, $\hat{y}$ in place of x, y, simply because $\ell^2(A)$ is also a Hilbert space. (See Exercise 19 for other identities of this type.) Note that the sums in (iii) and (iv) are $\|\hat{x}\|_2^2$ and $(\hat{x}, \hat{y})$, respectively.

Finally, if (i) is false, there exists $u \neq 0$ in H so that $(u, u_\alpha) = 0$ for all $\alpha \in A$. If $x = y = u$, then $(x, y) = \|u\|^2 > 0$ but $\hat{x}(\alpha) = 0$ for all $\alpha \in A$, hence (iv) fails. Thus (iv) implies (i), and the proof is complete. ////

4.19 Isomorphisms Speaking informally, two algebraic systems of the same nature are said to be *isomorphic* if there is a one-to-one mapping of one onto the other which preserves all relevant properties. For instance, we may ask whether two groups are isomorphic or whether two fields are isomorphic. Two vector spaces are isomorphic if there is a one-to-one *linear* mapping of one onto the other. The linear mappings are the ones which preserve the relevant concepts in a vector space, namely, addition and scalar multiplication.

In the same way, two Hilbert spaces H_1 and H_2 are isomorphic if there is a one-to-one linear mapping Λ of H_1 onto H_2 which also preserves inner products: $(\Lambda x, \Lambda y) = (x, y)$ for all x and $y \in H_1$. Such a Λ is an isomorphism (or, more specifically, a *Hilbert space isomorphism*) of H_1 onto H_2. Using this terminology, Theorems 4.17 and 4.18 yield the following statement:

If $\{u_\alpha: \alpha \in A\}$ is a maximal orthonormal set in a Hilbert space H, and if $\hat{x}(\alpha) = (x, u_\alpha)$, then the mapping $x \rightarrow \hat{x}$ is a Hilbert space isomorphism of H onto $\ell^2(A)$.

One can prove (we shall omit this) that $\ell^2(A)$ and $\ell^2(B)$ are isomorphic if and only if the sets A and B have the same cardinal number. But we shall prove that every nontrivial Hilbert space (this means that the space does not consist of 0 alone) is isomorphic to some $\ell^2(A)$, by proving that every such space contains a maximal orthonormal set (Theorem 4.22). The proof will depend on a property of partially ordered sets which is equivalent to the axiom of choice.

4.20 Partially Ordered Sets A set $\mathscr{P}$ is said to be *partially ordered* by a binary relation $\leq$ if

(a) $a \leq b$ and $b \leq c$ implies $a \leq c$.
(b) $a \leq a$ for every $\alpha \in \mathscr{P}$.
(c) $a \leq b$ and $b \leq a$ implies $a = b$.

A subset $\mathscr{Q}$ of a partially ordered set $\mathscr{P}$ is said to be *totally ordered* (or *linearly ordered*) if every pair $a, b \in \mathscr{Q}$ satisfies either $\alpha \leq b$ or $b \leq a$.

For example, every collection of subsets of a given set is partially ordered by the inclusion relation $\subset$.

To give a more specific example, let $\mathscr{P}$ be the collection of all open subsets of the plane, partially ordered by set inclusion, and let $\mathscr{Q}$ be the collection of all open circular discs with center at the origin. Then $\mathscr{Q} \subset \mathscr{P}$, $\mathscr{Q}$ is totally ordered by $\subset$, and $\mathscr{Q}$ is a *maximal* totally ordered subset of $\mathscr{P}$. This means that if any member of $\mathscr{P}$ not in $\mathscr{Q}$ is adjoined to $\mathscr{Q}$, the resulting collection of sets is no longer totally ordered by $\subset$.

4.21 The Hausdorff Maximality Theorem *Every nonempty partially ordered set contains a maximal totally ordered subset.*

This is a consequence of the axiom of choice and is, in fact, equivalent to it; another (very similar) form of it is known as Zorn's lemma. We give the proof in the Appendix.

If now H is a nontrivial Hilbert space, then there exists a $u \in H$ with $\|u\| = 1$, so that there is a nonempty orthonormal set in H. The existence of a maximal orthonormal set is therefore a consequence of the following theorem:

4.22 Theorem *Every orthonormal set B in a Hilbert space H is contained in a maximal orthonormal set in H.*

PROOF Let $\mathscr{P}$ be the class of all orthonormal sets in H which contain the given set B. Partially order $\mathscr{P}$ by set inclusion. Since $B \in \mathscr{P}$, $\mathscr{P} \neq \varnothing$. Hence $\mathscr{P}$ contains a maximal totally ordered class Ω. Let S be the union of all members of Ω. It is clear that $B \subset S$. We claim that S is a maximal orthonormal set:

If u_1 and $u_2 \in S$, then $u_1 \in A_1$ and $u_2 \in A_2$ for some A_1 and $A_2 \in \Omega$. Since Ω is total ordered, $A_1 \subset A_2$ (or $A_2 \subset A_1$), so that $u_1 \in A_2$ and $u_2 \in A_2$. Since A_2 is orthonormal, $(u_1, u_2) = 0$ if $u_1 \neq u_2$, $(u_1, u_2) = 1$ if $u_1 = u_2$. Thus S is an orthonormal set.

Suppose S is not maximal. Then S is a proper subset of an orthonormal set S^*. Clearly, $S^* \notin \Omega$, and S^* contains every member of Ω. Hence we may adjoin S^* to Ω and still have a total order. This contradicts the maximality of Ω. ////

Trigonometric Series

4.23 Definitions Let T be the unit circle in the complex plane, i.e., the set of all complex numbers of absolute value 1. If F is any function on T and if f is defined on R^1 by

$$f(t) = F(e^{it}), \tag{1}$$

then f is a periodic function of period 2π. This means that $f(t + 2\pi) = f(t)$ for all real t. Conversely, if f is a function on R^1, with period 2π, then there is a function F on T such that (1) holds. Thus we may identify functions on T with 2π-periodic functions on R^1; and, for simplicity of notation, we shall sometimes write $f(t)$ rather than $f(e^{it})$, even if we think of f as being defined on T.

With these conventions in mind, we define $L^p(T)$, for $1 \leq p < \infty$, to be the class of all complex, Lebesgue measurable, 2π-periodic functions on R^1 for which the norm

$$\|f\|_p = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p \, dt\right\}^{1/p} \tag{2}$$

is finite.

In other words, we are looking at $L^p(\mu)$, where μ is Lebesgue measure on $[0, 2\pi]$ (or on T), divided by 2π. $L^\infty(T)$ will be the class of all 2π-periodic members of $L^\infty(R^1)$, with the essential supremum norm, and $C(T)$ consists of all continuous complex functions on T (or, equivalently, of all continuous, complex, 2π-periodic functions on R^1), with norm

$$\|f\|_\infty = \sup_t |f(t)|, \tag{3}$$

The factor $1/(2\pi)$ in (2) simplifies the formalism we are about to develop. For instance, the L^p-norm of the constant function 1 is 1.

A *trigonometric polynomial* is a finite sum of the form

$$f(t) = a_0 + \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt) \qquad (t \in R^1) \tag{4}$$

where $a_0, a_1, \ldots, a_N$ and $b_1, \ldots, b_N$ are complex numbers. On account of the Euler identities, (4) can also be written in the form

$$f(t) = \sum_{n=-N}^{N} c_n e^{int} \tag{5}$$

which is more convenient for most purposes. It is clear that every trigonometric polynomial has period 2π.

We shall denote the set of all integers (positive, zero, and negative) by Z, and put

$$u_n(t) = e^{int} \qquad (n \in Z). \tag{6}$$

If we define the inner product in $L^2(T)$ by

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\overline{g(t)}\, dt \tag{7}$$

[note that this is in agreement with (2)], an easy computation shows that

$$(u_n, u_m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(n-m)t}\, dt = \begin{cases} 1 & \text{if } n = m, \\ 0 & \text{if } n \neq m. \end{cases} \tag{8}$$

Thus $\{u_n: n \in Z\}$ is an orthonormal set in $L^2(T)$, usually called the *trigonometric system.* We shall now prove that this system is maximal, and shall then derive concrete versions of the abstract theorems previously obtained in the Hilbert space context.

4.24 The Completeness of the Trigonometric System Theorem 4.18 shows that the maximality (or completeness) of the trigonometric system will be proved as soon as we can show that the set of all trigonometric polynomials is dense in $L^2(T)$. Since $C(T)$ is dense in $L^2(T)$, by Theorem 3.14 (note that T is compact), it suffices to show that to every $f \in C(T)$ and to every $\epsilon > 0$ there is a trigonometric polynomial P such that $\|f - P\|_2 < \epsilon$. Since $\|g\|_2 \leq \|g\|_\infty$ for every $g \in C(T)$, the estimate $\|f - P\|_2 < \epsilon$ will follow from $\|f - P\|_\infty < \epsilon$, and it is this estimate which we shall prove.

Suppose we had trigonometric polynomials $Q_1, Q_2, Q_3, \ldots$, with the following properties:

(*a*) $$Q_k(t) \geq 0 \text{ for } t \in R^1.$$

(*b*) $$\frac{1}{2\pi} \int_{-\pi}^{\pi} Q_k(t)\, dt = 1.$$

(*c*) *If* $\eta_k(\delta) = \sup \{Q_k(t): \delta \leq |t| \leq \pi\}$, *then*

$$\lim_{k \to \infty} \eta_k(\delta) = 0$$

for every $\delta > 0$.

Another way of stating (*c*) is to say: for every $\delta > 0$, $Q_k(t) \to 0$ uniformly on $[-\pi, -\delta] \cup [\delta, \pi]$.

To each $f \in C(T)$ we associate the functions P_k defined by

$$P_k(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t - s)Q_k(s)\, ds \qquad (k = 1, 2, 3, \ldots). \tag{1}$$

If we replace s by $-s$ (using Theorem 2.20(*e*)) and then by $s - t$, the periodicity of f and Q_k shows that the value of the integral is not affected. Hence

$$P_k(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) Q_k(t - s)\, ds \qquad (k = 1, 2, 3, \ldots). \tag{2}$$

Since each Q_k is a trigonometric polynomial, Q_k is of the form

$$Q_k(t) = \sum_{n=-N_k}^{N_k} a_{n,k} e^{int}, \tag{3}$$

and if we replace t by $t - s$ in (3) and substitute the result in (2), we see that each P_k is a trigonometric polynomial.

Let $\epsilon > 0$ be given. Since f is uniformly continuous on T, there exists a $\delta > 0$ such that $|f(t) - f(s)| < \epsilon$ whenever $|t - s| < \delta$. By (*b*), we have

$$P_k(t) - f(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \{f(t - s) - f(t)\} Q_k(s)\, ds,$$

and (*a*) implies, for all t, that

$$|P_k(t) - f(t)| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(t - s) - f(t)|\, Q_k(s)\, ds = A_1 + A_2,$$

where A_1 is the integral over $[-\delta, \delta]$ and A_2 is the integral over $[-\pi, -\delta] \cup [\delta, \pi]$. In A_1, the integrand is less than $\epsilon Q_k(s)$, so $A_1 < \epsilon$, by (*b*). In A_2, we have $Q_k(s) \leq \eta_k(\delta)$, hence

$$A_2 \leq 2\|f\|_\infty \cdot \eta_k(\delta) < \epsilon \tag{4}$$

for sufficiently large k, by (*c*). Since these estimates are independent of t, we have proved that

$$\lim_{k \to \infty} \|f - P_k\|_\infty = 0. \tag{5}$$

It remains to construct the Q_k. This can be done in many ways. Here is a simple one. Put

$$Q_k(t) = c_k \left\{\frac{1 + \cos t}{2}\right\}^k, \tag{6}$$

where c_k is chosen so that (*b*) holds. Since (*a*) is clear, we only need to show (*c*). Since Q_k is even, (*b*) shows that

$$1 = \frac{c_k}{\pi} \int_0^{\pi} \left\{\frac{1 + \cos t}{2}\right\}^k dt > \frac{c_k}{\pi} \int_0^{\pi} \left\{\frac{1 + \cos t}{2}\right\}^k \sin t\, dt = \frac{2c_k}{\pi(k + 1)}.$$

Since Q_k is decreasing on $[0, \pi]$, it follows that

$$Q_k(t) \leq Q_k(\delta) \leq \frac{\pi(k + 1)}{2} \left(\frac{1 + \cos \delta}{2}\right)^k \qquad (0 < \delta \leq |t| \leq \pi). \tag{7}$$

This implies (c), since $1 + \cos\delta < 2$ if $0 < \delta \leq \pi$.

We have proved the following important result:

4.25 Theorem *If $f \in C(T)$ and $\epsilon > 0$, there is a trigonometric polynomial P such that*

$$|f(t) - P(t)| < \epsilon$$

for every real t.

A more precise result was proved by Fejér (1904): *The arithmetic means of the partial sums of the Fourier series of any $f \in C(T)$ converge uniformly to f.* For a proof (quite similar to the above) see Theorem 3.1 of [45], or p. 89 of [36], vol. I.

4.26 Fourier Series For any $f \in L^1(T)$, we define the *Fourier coefficients* of f by the formula

$$\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt \qquad (n \in Z), \tag{1}$$

where, we recall, Z is the set of all integers. We thus associate with each $f \in L^1(T)$ a function $\hat{f}$ on Z. The *Fourier series* of f is

$$\sum_{-\infty}^{\infty} \hat{f}(n)e^{int}, \tag{2}$$

and its *partial sums* are

$$s_N(t) = \sum_{-N}^{N} \hat{f}(n)e^{int} \qquad (N = 0, 1, 2, \ldots). \tag{3}$$

Since $L^2(T) \subset L^1(T)$, (1) can be applied to every $f \in L^2(T)$. Comparing the definitions made in Secs. 4.23 and 4.13, we can now restate Theorems 4.17 and 4.18 in concrete terms:

The *Riesz-Fischer theorem* asserts that if $\{c_n\}$ is a sequence of complex numbers such that

$$\sum_{n=-\infty}^{\infty} |c_n|^2 < \infty, \tag{4}$$

then there exists an $f \in L^2(T)$ such that

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}\,dt \qquad (n \in Z). \tag{5}$$

The *Parseval theorem* asserts that

$$\sum_{n=-\infty}^{\infty} \hat{f}(n)\overline{\hat{g}(n)} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}\,dt \tag{6}$$

whenever $f \in L^2(T)$ and $g \in L^2(T)$; the series on the left of (6) converges absolutely; and if s_N is as in (3), then

$$\lim_{N \to \infty} \| f - s_N \|_2 = 0, \tag{7}$$

since a special case of (6) yields

$$\| f - s_N \|_2^2 = \sum_{|n| > N} |\hat{f}(n)|^2. \tag{8}$$

Note that (7) says that every $f \in L^2(T)$ is the L^2-limit of the partial sums of its Fourier series; i.e., the Fourier series of f converges to f, in the L^2-sense. Pointwise convergence presents a more delicate problem, as we shall see in Chap. 5.

The Riesz-Fischer theorem and the Parseval theorem may be summarized by saying that the mapping $f \leftrightarrow \hat{f}$ is a Hilbert space isomorphism of $L^2(T)$ onto $\ell^2(Z)$.

The theory of Fourier series in other function spaces, for instance in $L^1(T)$, is much more difficult than in $L^2(T)$, and we shall touch only a few aspects of it.

Observe that the crucial ingredient in the proof of the Riesz-Fischer theorem is the fact that L^2 is complete. This is so well recognized that the name "Riesz-Fischer theorem" is sometimes given to the theorem which asserts the completeness of L^2, or even of any L^p.

Exercises

In this set of exercises, H always denotes a Hilbert space.

1 If M is a closed subspace of H, prove that $M = (M^\perp)^\perp$. Is there a similar true statement for subspaces M which are not necessarily closed?

2 Let $\{x_n : n = 1, 2, 3, \ldots\}$ be a linearly independent set of vectors in H. Show that the following construction yields an orthonormal set $\{u_n\}$ such that $\{x_1, \ldots, x_N\}$ and $\{u_1, \ldots, u_N\}$ have the same span for all N.

Put $u_1 = x_1/\|x_1\|$. Having $u_1, \ldots, u_{n-1}$ define

$$v_n = x_n - \sum_{i=1}^{n-1} (x_n, u_i)u_i, \qquad u_n = v_n/\|v_n\|.$$

Note that this leads to a proof of the existence of a maximal orthonormal set in separable Hilbert spaces which makes no appeal to the Hausdorff maximality principle. (A space is *separable* if it contains a countable dense subset.)

3 Show that $L^p(T)$ is separable if $1 \leq p < \infty$, but that $L^\infty(T)$ is not separable.

4 Show that H is separable if and only if H contains a maximal orthonormal system which is at most countable.

5 If $M = \{x : Lx = 0\}$, where L is a continuous linear functional on H, prove that $M^\perp$ is a vector space of dimension 1 (unless $M = H$).

6 Let $\{u_n\}$ $(n = 1, 2, 3, \ldots)$ be an orthonormal set in H. Show that this gives an example of a closed and bounded set which is not compact. Let Q be the set of all $x \in H$ of the form

$$x = \sum_1^\infty c_n u_n \qquad \left(\text{where } |c_n| \leq \frac{1}{n}\right).$$

Prove that Q is compact. (Q is called the Hilbert cube.)

More generally, let $\{\delta_n\}$ be a sequence of positive numbers, and let S be the set of all $x \in H$ of the form

$$x = \sum_1^\infty c_n u_n \qquad (\text{where } |c_n| \le \delta_n).$$

Prove that S is compact if and only if $\sum_1^\infty \delta_n^2 < \infty$.

Prove that H is not locally compact.

7 Suppose $\{a_n\}$ is a sequence of positive numbers such that $\sum a_n b_n < \infty$ whenever $b_n \ge 0$ and $\sum b_n^2 < \infty$. Prove that $\sum a_n^2 < \infty$.

Suggestion: If $\sum a_n^2 = \infty$ then there are disjoint sets E_k $(k = 1, 2, 3, \ldots)$ so that

$$\sum_{n \in E_k} a_n^2 > 1.$$

Define b_n so that $b_n = c_k a_n$ for $n \in E_k$. For suitably chosen c_k, $\sum a_n b_n = \infty$ although $\sum b_n^2 < \infty$.

8 If H_1 and H_2 are two Hilbert spaces, prove that one of them is isomorphic to a subspace of the other. (Note that every closed subspace of a Hilbert space is a Hilbert space.)

9 If $A \subset [0, 2\pi]$ and A is measurable, prove that

$$\lim_{n \to \infty} \int_A \cos nx \, dx = \lim_{n \to \infty} \int_A \sin nx \, dx = 0.$$

10 Let $n_1 < n_2 < n_3 < \cdots$ be positive integers, and let E be the set of all $x \in [0, 2\pi]$ at which $\{\sin n_k x\}$ converges. Prove that $m(E) = 0$. *Hint*: $2 \sin^2 \alpha = 1 - \cos 2\alpha$, so $\sin n_k x \to \pm 1/\sqrt{2}$ a.e. on E, by Exercise 9.

11 Find a nonempty closed set E in $L^2(T)$ that contains no element of smallest norm.

12 The constants c_k in Sec. 4.24 were shown to be such that $k^{-1}c_k$ is bounded. Estimate the relevant integral more precisely and show that

$$0 < \lim_{k \to \infty} k^{-1/2} c_k < \infty.$$

13 Suppose f is a continuous function on R^1, with period 1. Prove that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n\alpha) = \int_0^1 f(t) \, dt$$

for every irrational real number α. *Hint*: Do it first for

$$f(t) = \exp(2\pi i k t), \qquad k = 0, \pm 1, \pm 2, \ldots.$$

14 Compute

$$\min_{a,b,c} \int_{-1}^{1} |x^3 - a - bx - cx^2|^2 \, dx$$

and find

$$\max \int_{-1}^{1} x^3 g(x) \, dx,$$

where g is subject to the restrictions

$$\int_{-1}^{1} g(x) \, dx = \int_{-1}^{1} x g(x) \, dx = \int_{-1}^{1} x^2 g(x) \, dx = 0; \qquad \int_{-1}^{1} |g(x)|^2 \, dx = 1.$$

15 Compute

$$\min_{a,b,c} \int_0^\infty |x^3 - a - bx - cx^2|^2 e^{-x}\, dx.$$

State and solve the corresponding maximum problem, as in Exercise 14.

16 If $x_0 \in H$ and M is a closed linear subspace of H, prove that

$$\min \{\|x - x_0\| : x \in M\} = \max \{|(x_0, y)| : y \in M^\perp, \|y\| = 1\}.$$

17 Show that there is a continuous one-to-one mapping γ of $[0, 1]$ into H such that $\gamma(b) - \gamma(a)$ is orthogonal to $\gamma(d) - \gamma(c)$ whenever $0 \le a \le b \le c \le d \le 1$. ($\gamma$ may be called a "curve with orthogonal increments.") *Hint*: Take $H = L^2$, and consider characteristic functions of certain subsets of $[0, 1]$.

18 Define $u_s(t) = e^{ist}$ for all $s \in R^1$, $t \in R^1$. Let X be the complex vector space consisting of all finite linear combinations of these functions u_s. If $f \in X$ and $g \in X$, show that

$$(f, g) = \lim_{A \to \infty} \frac{1}{2A} \int_{-A}^{A} f(t)\overline{g(t)}\, dt$$

exists. Show that this inner product makes X into a unitary space whose completion is a non-separable Hilbert space H. Show also that $\{u_s : s \in R^1\}$ is a maximal orthonormal set in H.

19 Fix a positive integer N, put $\omega = e^{2\pi i/N}$, prove the orthogonality relations

$$\frac{1}{N} \sum_{n=1}^{N} \omega^{nk} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } 1 \le k \le N - 1 \end{cases}$$

and use them to derive the identities

$$(x, y) = \frac{1}{N} \sum_{n=1}^{N} \|x + \omega^n y\|^2 \omega^n$$

that hold in every inner product space if $N \ge 3$. Show also that

$$(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \|x + e^{i\theta} y\|^2 e^{i\theta}\, d\theta.$$

CHAPTER

FIVE

EXAMPLES OF BANACH SPACE TECHNIQUES

Banach Spaces

5.1 In the preceding chapter we saw how certain analytic facts about trigonometric series can be made to emerge from essentially goemetric considerations about general Hilbert spaces, involving the notions of convexity, subspaces, orthogonality, and completeness. There are many problems in analysis that can be attacked with greater ease when they are placed within a suitably chosen abstract framework. The theory of Hilbert spaces is not always suitable since orthogonality is something rather special. The class of all Banach spaces affords greater variety. In this chapter we shall develop some of the basic properties of Banach spaces and illustrate them by applications to concrete problems.

5.2 Definition A complex vector space X is said to be a *normed linear space* if to each $x \in X$ there is associated a nonnegative real number $\|x\|$, called the *norm* of x, such that

(*a*) $\|x + y\| \leq \|x\| + \|y\|$ for all x and $y \in X$,
(*b*) $\|ax\| = |\alpha|\,\|x\|$ if $x \in X$ and α is a scalar,
(*c*) $\|x\| = 0$ implies $x = 0$.

By (*a*), the triangle inequality

$$\|x - y\| \leq \|x - z\| + \|z - y\| \qquad (x, y, z \in X)$$

holds. Combined with (*b*) (take $\alpha = 0$, $\alpha = -1$) and (*c*) this shows that every normed linear space may be regarded as a metric space, the distance between x and y being $\|x - y\|$.

A *Banach space* is a normed linear space which is *complete* in the metric defined by its norm.

For instance, every Hilbert space is a Banach space, so is every $L^p(\mu)$ normed by $\|f\|_p$ (provided we identify functions which are equal a.e.) if $1 \leq p \leq \infty$, and so is $C_0(X)$ with the supremum norm. The simplest Banach space is of course the complex field itself, normed by $\|x\| = |x|$.

One can equally well discuss *real* Banach spaces; the definition is exactly the same, except that all scalars are assumed to be real.

5.3 Definition Consider a linear transformation Λ from a normed linear space X into a normed linear space Y, and define its *norm* by

$$\|\Lambda\| = \sup \{\|\Lambda x\| : x \in X, \|x\| \leq 1\}. \tag{1}$$

If $\|\Lambda\| < \infty$, then Λ is called a *bounded linear transformation.*

In (1), $\|x\|$ is the norm of x in X, $\|\Lambda x\|$ is the norm of Λx in Y; it will frequently happen that several norms occur together, and the context will make it clear which is which.

Observe that we could restrict ourselves to *unit vectors* x in (1), i.e., to x with $\|x\| = 1$, without changing the supremum, since

$$\|\Lambda(\alpha x)\| = \|\alpha \Lambda x\| = |\alpha| \, \|\Lambda x\|. \tag{2}$$

Observe also that $\|\Lambda\|$ is the smallest number such that the inequality

$$\|\Lambda x\| \leq \|\Lambda\| \, \|x\| \tag{3}$$

holds for *every* $x \in X$.

The following geometric picture is helpful: Λ maps the *closed unit ball* in X, i.e., the set

$$\{x \in X : \|x\| \leq 1\}, \tag{4}$$

into the closed ball in Y with center at 0 and radius $\|\Lambda\|$.

An important special case is obtained by taking the complex field for Y; in that case we talk about *bounded linear functionals.*

5.4 Theorem *For a linear transformation Λ of a normed linear space X into a normed linear space Y, each of the following three conditions implies the other two:*

(*a*) *Λ is bounded.*
(*b*) *Λ is continuous.*
(*c*) *Λ is continuous at one point of X.*

PROOF Since $\|\Lambda(x_1 - x_2)\| \leq \|\Lambda\| \, \|x_1 - x_2\|$, it is clear that (*a*) implies (*b*), and (*b*) implies (*c*) trivially. Suppose Λ is continuous at x_0. To each $\epsilon > 0$ one can then find a $\delta > 0$ so that $\|x - x_0\| < \delta$ implies $\|\Lambda x - \Lambda x_0\| < \epsilon$. In other words, $\|x\| < \delta$ implies

$$\|\Lambda(x_0 + x) - \Lambda x_0\| < \epsilon.$$

But then the linearity of Λ shows that $\|\Lambda x\| < \epsilon$. Hence $\|\Lambda\| \leq \epsilon/\delta$, and (c) implies (a). ////

Consequences of Baire's Theorem

5.5 The manner in which the completeness of Banach spaces is frequently exploited depends on the following theorem about complete metric spaces, which also has many applications in other parts of mathematics. It implies two of the three most important theorems which make Banach spaces useful tools in analysis, the *Banach-Steinhaus theorem* and the *open mapping theorem.* The third is the *Hahn-Banach extension theorem,* in which completeness plays no role.

5.6 Baire's Theorem *If X is a complete metric space, the intersection of every countable collection of dense open subsets of X is dense in X.*

In particular (except in the trivial case $X = \varnothing$), the intersection is not empty. This is often the principal significance of the theorem.

PROOF Suppose $V_1, V_2, V_3, \ldots$ are dense and open in X. Let W be any open set in X. We have to show that $\bigcap V_n$ has a point in W if $W \neq \varnothing$.

Let ρ be the metric of X; let us write

$$S(x, r) = \{y \in X : \rho(x, y) < r\} \tag{1}$$

and let $\bar{S}(x, r)$ be the closure of $S(x, r)$. [*Note*: There exist situations in which $\bar{S}(x, r)$ does *not* contain all y with $\rho(x, y) \leq r$!]

Since V_1 is dense, $W \cap V_1$ is a nonempty open set, and we can therefore find x_1 and r_1 such that

$$\bar{S}(x_1, r_1) \subset W \cap V_1 \quad \text{and} \quad 0 < r_1 < 1. \tag{2}$$

If $n \geq 2$ and x_{n-1} and r_{n-1} are chosen, the denseness of V_n shows that $V_n \cap S(x_{n-1}, r_{n-1})$ is not empty, and we can therefore find x_n and r_n such that

$$\bar{S}(x_n, r_n) \subset V_n \cap S(x_{n-1}, r_{n-1}) \quad \text{and} \quad 0 < r_n < \frac{1}{n}. \tag{3}$$

By induction, this process produces a sequence $\{x_n\}$ in X. If $i > n$ and $j > n$, the construction shows that x_i and x_j both lie in $S(x_n, r_n)$, so that $\rho(x_i, x_j) < 2r_n < 2/n$, and hence $\{x_n\}$ is a Cauchy sequence. Since X is complete, there is a point $x \in X$ such that $x = \lim_{n \to \infty} x_n$.

Since x_i lies in the closed set $\bar{S}(x_n, r_n)$ if $i > n$, it follows that x lies in each $\bar{S}(x_n, r_n)$, and (3) shows that x lies in each V_n. By (2), $x \in W$. This completes the proof. ////

Corollary *In a complete metric space, the intersection of any countable collection of dense G_δ's is again a dense G_δ.*

This follows from the theorem, since every G_δ is the intersection of a countable collection of open sets, and since the union of countably many countable sets is countable.

5.7 Baire's theorem is sometimes called the *category theorem*, for the following reason.

Call a set $E \subset X$ *nowhere dense* if its closure $\bar{E}$ contains no nonempty open subset of X. Any countable union of nowhere dense sets is called a set of the *first category*; all other subsets of X are of the *second category* (Baire's terminology). Theorem 5.6 is equivalent to the statement that *no complete metric space is of the first category.* To see this, just take complements in the statement of Theorem 5.6.

5.8 The Banach-Steinhaus Theorem *Suppose X is a Banach space, Y is a normed linear space, and $\{\Lambda_\alpha\}$ is a collection of bounded linear transformations of X into Y, where α ranges over some index set A. Then either there exists an $M < \infty$ such that*

$$\|\Lambda_\alpha\| \leq M \tag{1}$$

for every $\alpha \in A$, or

$$\sup_{\alpha \in A} \|\Lambda_\alpha x\| = \infty \tag{2}$$

for all x belonging to some dense G_δ in X.

In geometric terminology, the alternatives are as follows: *Either* there is a ball B in Y (with radius M and center at 0) such that every Λ_α maps the unit ball of X into B, *or* there exist $x \in X$ (in fact, a whole dense G_δ of them) such that no ball in Y contains $\Lambda_\alpha x$ for all α.

The theorem is sometimes referred to as the *uniform boundedness principle.*

PROOF Put

$$\varphi(x) = \sup_{\alpha \in A} \|\Lambda_\alpha x\| \qquad (x \in X) \tag{3}$$

and let

$$V_n = \{x: \varphi(x) > n\} \qquad (n = 1, 2, 3, \ldots). \tag{4}$$

Since each Λ_α is continuous and since the norm of Y is a continuous function on Y (an immediate consequence of the triangle inequality, as in the proof of Theorem 4.6), each function $x \to \|\Lambda_\alpha x\|$ is continuous on X. Hence φ is lower semicontinuous, and each V_n is open.

If one of these sets, say V_N, fails to be dense in X, then there exist an $x_0 \in X$ and an $r > 0$ such that $\|x\| \leq r$ implies $x_0 + x \notin V_N$; this means that $\varphi(x_0 + x) \leq N$, or

$$\|\Lambda_\alpha(x_0 + x)\| \leq N \tag{5}$$

for all $\alpha \in A$ and all x with $\|x\| \leq r$. Since $x = (x_0 + x) - x_0$, we then have

$$\|\Lambda_\alpha x\| \leq \|\Lambda_\alpha(x_0 + x)\| + \|\Lambda_\alpha x_0\| \leq 2N, \tag{6}$$

and it follows that (1) holds with $M = 2N/r$.

The other possibility is that every V_n is dense in X. In that case, $\bigcap V_n$ is a dense G_δ in X, by Baire's theorem; and since $\varphi(x) = \infty$ for every $x \in \bigcap V_n$, the proof is complete. ////

5.9 The Open Mapping Theorem *Let U and V be the open unit balls of the Banach spaces X and Y. To every bounded linear transformation Λ of X onto Y there corresponds a $\delta > 0$ so that*

$$\Lambda(U) \supset \delta V. \tag{1}$$

Note the word "onto" in the hypothesis. The symbol δV stands for the set $\{\delta y\colon y \in V\}$, i.e., the set of all $y \in Y$ with $\|y\| < \delta$.

It follows from (1) and the linearity of Λ that the image of every open ball in X, with center at x_0, say, contains an open ball in Y with center at Λx_0. Hence the image of every open set is open. This explains the name of the theorem.

Here is another way of stating (1): *To every y with $\|y\| < \delta$ there corresponds an x with $\|x\| < 1$ so that $\Lambda x = y$.*

PROOF Given $y \in Y$, there exists an $x \in X$ such that $\Lambda x = y$; if $\|x\| < k$, it follows that $y \in \Lambda(kU)$. Hence Y is the union of the sets $\Lambda(kU)$, for $k = 1, 2, 3, \ldots$. Since Y is complete, the Baire theorem implies that there is a nonempty open set W in the closure of some $\Lambda(kU)$. This means that every point of W is the limit of a sequence $\{\Lambda x_i\}$, where $x_i \in kU$; from now on, k and W are fixed.

Choose $y_0 \in W$, and choose $\eta > 0$ so that $y_0 + y \in W$ if $\|y\| < \eta$. For any such y there are sequences $\{x_i'\}$, $\{x_i''\}$ in kU such that

$$\Lambda x_i' \to y_0, \qquad \Lambda x_i'' \to y_0 + y \qquad (i \to \infty). \tag{2}$$

Setting $x_i = x_i'' - x_i'$, we have $\|x_i\| < 2k$ and $\Lambda x_i \to y$. Since this holds for every y with $\|y\| < \eta$, the linearity of Λ shows that the following is true, if $\delta = \eta/2k$:

To each $y \in Y$ and to each $\epsilon > 0$ there corresponds an $x \in X$ such that

$$\|x\| \leq \delta^{-1}\|y\| \quad \textit{and} \quad \|y - \Lambda x\| < \epsilon. \tag{3}$$

This is almost the desired conclusion, as stated just before the start of the proof, except that there we had $\epsilon = 0$.

Fix $y \in \delta V$, and fix $\epsilon > 0$. By (3) there exists an x_1 with $\|x_1\| < 1$ and

$$\|y - \Lambda x_1\| < \tfrac{1}{2}\delta\epsilon. \tag{4}$$

Suppose $x_1, \ldots, x_n$ are chosen so that

$$\|y - \Lambda x_1 - \cdots - \Lambda x_n\| < 2^{-n}\delta\epsilon. \tag{5}$$

Use (3), with y replaced by the vector on the left side of (5), to obtain an x_{n+1} so that (5) holds with $n + 1$ in place of n, and

$$\|x_{n+1}\| < 2^{-n}\epsilon \qquad (n = 1, 2, 3, \ldots). \tag{6}$$

If we set $s_n = x_1 + \cdots + x_n$, (6) shows that $\{s_n\}$ is a Cauchy sequence in X. Since X is complete, there exists an $x \in X$ so that $s_n \to x$. The inequality $\|x_1\| < 1$, together with (6), shows that $\|x\| < 1 + \epsilon$. Since Λ is continuous, $\Lambda s_n \to \Lambda x$. By (5), $\Lambda s_n \to y$. Hence $\Lambda x = y$.

We have now proved that

$$\Lambda((1 + \epsilon)U) \supset \delta V, \tag{7}$$

or

$$\Lambda(U) \supset (1 + \epsilon)^{-1}\delta V, \tag{8}$$

for every $\epsilon > 0$. The union of the sets on the right of (8), taken over all $\epsilon > 0$, is δV. This proves (1). ////

5.10 Theorem *If X and Y are Banach spaces and if Λ is a bounded linear transformation of X onto Y which is also one-to-one, then there is a $\delta > 0$ such that*

$$\|\Lambda x\| \geq \delta\|x\| \qquad (x \in X). \tag{1}$$

In other words, Λ^{-1} is a bounded linear transformation of Y onto X.

PROOF If δ is chosen as in the statement of Theorem 5.9, the conclusion of that theorem, combined with the fact that Λ is now one-to-one, shows that $\|\Lambda x\| < \delta$ implies $\|x\| < 1$. Hence $\|x\| \geq 1$ implies $\|\Lambda x\| \geq \delta$, and (1) is proved.

The transformation Λ^{-1} is defined on Y by the requirement that $\Lambda^{-1}y = x$ if $y = \Lambda x$. A trivial verification shows that Λ^{-1} is linear, and (1) implies that $\|\Lambda^{-1}\| \leq 1/\delta$. ////

Fourier Series of Continuous Functions

5.11 A Convergence Problem *Is it true for every $f \in C(T)$ that the Fourier series of f converges to $f(x)$ at every point x?*

Let us recall that the nth partial sum of the Fourier series of f at the point x is given by

$$s_n(f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) D_n(x - t)\, dt \qquad (n = 0, 1, 2, \ldots), \tag{1}$$

where

$$D_n(t) = \sum_{k=-n}^{n} e^{ikt}. \tag{2}$$

This follows directly from formulas 4.26(1) and 4.26(3).

The problem is to determine whether

$$\lim_{n \to \infty} s_n(f; x) = f(x) \tag{3}$$

for every $f \in C(T)$ and for every real x. We observed in Sec. 4.26 that the partial sums do converge to f in the L^2-norm, and Theorem 3.12 implies therefore that each $f \in L^2(T)$ [hence also each $f \in C(T)$] is the pointwise limit a.e. of some subsequence of the full sequence of the partial sums. But this does not answer the present question.

We shall see that the Banach-Steinhaus theorem answers the question *negatively*. Put

$$s^*(f; x) = \sup_n |s_n(f; x)|. \tag{4}$$

To begin with, take $x = 0$, and define

$$\Lambda_n f = s_n(f; 0) \qquad (f \in C(T);\ n = 1, 2, 3, \ldots). \tag{5}$$

We know that $C(T)$ is a Banach space, relative to the supremum norm $\|f\|_\infty$. It follows from (1) that each Λ_n is a bounded linear functional on $C(T)$, of norm

$$\|\Lambda_n\| \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(t)|\, dt = \|D_n\|_1. \tag{6}$$

We claim that

$$\|\Lambda_n\| \to \infty \qquad \text{as } n \to \infty. \tag{7}$$

This will be proved by showing that equality holds in (6) and that

$$\|D_n\|_1 \to \infty \qquad \text{as } n \to \infty. \tag{8}$$

Multiply (2) by $e^{it/2}$ and by $e^{-it/2}$ and subtract one of the resulting two equations from the other, to obtain

$$D_n(t) = \frac{\sin (n + \frac{1}{2})t}{\sin (t/2)}. \tag{9}$$

Since $|\sin x| \leq |x|$ for all real x, (9) shows that

$$\|D_n\|_1 > \frac{2}{\pi}\int_0^{\pi}\left|\sin\left(n+\frac{1}{2}\right)t\right|\frac{dt}{t} = \frac{2}{\pi}\int_0^{(n+1/2)\pi}|\sin t|\,\frac{dt}{t}$$

$$> \frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi}|\sin t|\,dt = \frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k} \to \infty,$$

which proves (8).

Next, fix n, and put $g(t) = 1$ if $D_n(t) \geq 0$, $g(t) = -1$ if $D_n(t) < 0$. There exist $f_j \in C(T)$ such that $-1 \leq f_j \leq 1$ and $f_j(t) \to g(t)$ for every t, as $j \to \infty$. By the dominated convergence theorem,

$$\lim_{j\to\infty}\Lambda_n(f_j) = \lim_{j\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}f_j(-t)D_n(t)\,dt = \frac{1}{2\pi}\int_{-\pi}^{\pi}g(-t)D_n(t)\,dt = \|D_n\|_1.$$

Thus equality holds in (6), and we have proved (7).

Since (7) holds, the Banach-Steinhaus theorem asserts now that $s^*(f; 0) = \infty$ for every f in some dense G_δ-set in $C(T)$.

We chose $x = 0$ just for convenience. It is clear that the same result holds for every other x:

To each real number x there corresponds a set $E_x \subset C(T)$ which is a dense G_δ in $C(T)$, such that $s^*(f; x) = \infty$ for every $f \in E_x$.

In particular, the Fourier series of each $f \in E_x$ diverges at x, and we have a negative answer to our question. (Exercise 22 shows the answer is positive if mere continuity is replaced by a somewhat stronger smoothness assumption.)

It is interesting to observe that the above result can be strengthened by another application of Baire's theorem. Let us take countably many points x_i, and let E be the intersection of the corresponding sets

$$E_{x_i} \subset C(T).$$

By Baire's theorem, E is a dense G_δ in $C(T)$. Every $f \in E$ has

$$s^*(f; x_i) = \infty$$

at every point x_i.

For each f, $s^*(f; x)$ is a lower semicontinuous function of x, since (4) exhibits it as the supremum of a collection of continuous functions. Hence $\{x: s^*(f; x) = \infty\}$ is a G_δ in R^1, for each f. If the above points x_i are taken so that their union is dense in $(-\pi, \pi)$, we obtain the following result:

5.12 Theorem *There is a set $E \subset C(T)$ which is a dense G_δ in $C(T)$ and which has the following property: For each $f \in E$, the set*

$$Q_f = \{x: s^*(f; x) = \infty\}$$

is a dense G_δ in R^1.

This gains in interest if we realize that E, as well as each Q_f, is an *uncountable* set:

5.13 Theorem *In a complete metric space X which has no isolated points, no countable dense set is a G_δ.*

PROOF Let x_k be the points of a countable dense set E in X. Assume that E is a G_δ. Then $E = \bigcap V_n$, where each V_n is dense and open. Let

$$W_n = V_n - \bigcup_{k=1}^{n} \{x_k\}.$$

Then each W_n is still a dense open set, but $\bigcap W_n = \varnothing$, in contradiction to Baire's theorem. ////

Note: A slight change in the proof of Baire's theorem shows actually that every dense G_δ contains a perfect set if X is as above.

Fourier Coefficients of L^1-functions

5.14 As in Sec. 4.26, we associate to every $f \in L^1(T)$ a function $\hat{f}$ on Z defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-int}\, dt \qquad (n \in Z). \tag{1}$$

It is easy to prove that $\hat{f}(n) \to 0$ as $|n| \to \infty$, for every $f \in L^1$. For we know that $C(T)$ is dense in $L^1(T)$ (Theorem 3.14) and that the trigonometric polynomials are dense in $C(T)$ (Theorem 4.25). If $\epsilon > 0$ and $f \in L^1(T)$, this says that there is a $g \in C(T)$ and a trigonometric polynomial P such that $\|f - g\|_1 < \epsilon$ and $\|g - P\|_\infty < \epsilon$. Since

$$\|g - P\|_1 \leq \|g - P\|_\infty$$

if follows that $\|f - P\|_1 < 2\epsilon$; and if $|n|$ is large enough (depending on P), then

$$|\hat{f}(n)| = \left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \{f(t) - P(t)\} e^{-int}\, dt \right| \leq \|f - P\|_1 < 2\epsilon. \tag{2}$$

Thus $\hat{f}(n) \to 0$ as $n \to \pm\infty$. This is known as the Riemann-Lebesgue lemma.

The question we wish to raise is whether the converse is true. That is to say, if $\{a_n\}$ is a sequence of complex numbers such that $a_n \to 0$ as $n \to \pm\infty$, does it follow that there is an $f \in L^1(T)$ such that $\hat{f}(n) = a_n$ for all $n \in Z$? In other words, is something like the Riesz-Fischer theorem true in this situation?

This can easily be answered (negatively) with the aid of the open mapping theorem.

Let c_0 be the space of all complex functions φ on Z such that $\varphi(n) \to 0$ as $n \to \pm\infty$, with the supremum norm

$$\|\varphi\|_\infty = \sup \{|\varphi(n)| : n \in Z\}. \tag{3}$$

Then c_0 is easily seen to be a Banach space. In fact, if we declare every subset of Z to be open, then Z is a locally compact Hausdorff space, and c_0 is nothing but $C_0(Z)$.

The following theorem contains the answer to our question:

5.15 Theorem *The mapping $f \to \hat{f}$ is a one-to-one bounded linear transformation of $L^1(T)$ into (but not onto) c_0.*

PROOF Define Λ by $\Lambda f = \hat{f}$. It is clear that Λ is linear. We have just proved that Λ maps $L^1(T)$ into c_0, and formula 5.14(1) shows that $|\hat{f}(n)| \leq \|f\|_1$, so that $\|\Lambda\| \leq 1$. (Actually, $\|\Lambda\| = 1$; to see this, take $f = 1$.) Let us now prove that Λ is one-to-one. Suppose $f \in L^1(T)$ and $\hat{f}(n) = 0$ for every $n \in Z$. Then

$$\int_{-\pi}^{\pi} f(t)g(t)\,dt = 0 \tag{1}$$

if g is any trigonometric polynomial. By Theorem 4.25 and the dominated convergence theorem, (1) holds for every $g \in C(T)$. Apply the dominated convergence theorem once more, in conjunction with the Corollary to Lusin's theorem, to conclude that (1) holds if g is the characteristic function of any measurable set in T. Now Theorem 1.39(b) shows that $f = 0$ a.e.

If the range of Λ were all of c_0, Theorem 5.10 would imply the existence of a $\delta > 0$ such that

$$\|\hat{f}\|_\infty \geq \delta \|f\|_1 \tag{2}$$

for every $f \in L^1(T)$. But if $D_n(t)$ is defined as in Sec. 5.11, then $D_n \in L^1(T)$, $\|\hat{D}_n\|_\infty = 1$ for $n = 1, 2, 3, \ldots$, and $\|D_n\|_1 \to \infty$ as $n \to \infty$. Hence there is no $\delta > 0$ such that the inequalities

$$\|\hat{D}_n\|_\infty \geq \delta \|D_n\|_1 \tag{3}$$

hold for every n.

This completes the proof. ////

The Hahn-Banach Theorem

5.16 Theorem *If M is a subspace of a normed linear space X and if f is a bounded linear functional on M, then f can be extended to a bounded linear functional F on X so that $\|F\| = \|f\|$.*

Note that M need not be closed.

Before we turn to the proof, some comments seem called for. First, to say (in the most general situation) that a function F is an *extension* of f means that the domain of F includes that of f and that $F(x) = f(x)$ for all x in the domain of f. Second, the norms $\|F\|$ and $\|f\|$ are computed relative to the domains of F and f; explicitly,

$$\|f\| = \sup \{|f(x)|: x \in M, \|x\| \leq 1\}, \qquad \|F\| = \sup \{|F(x)|: x \in X, \|x\| \leq 1\},$$

The third comment concerns the field of scalars. So far everything has been stated for complex scalars, but the complex field could have been replaced by the real field without any changes in statements or proofs. The Hahn-Banach theorem is also true in both cases; nevertheless, it appears to be essentially a "real" theorem. The fact that the complex case was not yet proved when Banach wrote his classical book "Opérations linéaires" may be the main reason that real scalars are the only ones considered in his work.

It will be helpful to introduce some temporary terminology. Recall that V is a complex (real) vector space if $x + y \in V$ for x and $y \in V$, and if $ax \in V$ for all complex (real) numbers α. It follows trivially that *every complex vector space is also a real vector space.* A complex function φ on a complex vector space V is a *complex-linear functional* if

$$\varphi(x + y) = \varphi(x) + \varphi(y) \quad \text{and} \quad \varphi(\alpha x) = \alpha\varphi(x) \tag{1}$$

for all x and $y \in V$ and all *complex* α. A real-valued function φ on a complex (real) vector space V is a *real-linear functional* if (1) holds for all *real* α.

If u is the real part of a complex-linear functional f, i.e., if $u(x)$ is the real part of the complex number $f(x)$ for all $x \in V$, it is easily seen that u is a real-linear functional. The following relations hold between f and u:

5.17 Proposition *Let V be a complex vector space.*

(*a*) *If u is the real part of a complex-linear functional f on V, then*

$$f(x) = u(x) - iu(ix) \qquad (x \in V). \tag{1}$$

(*b*) *If u is a real-linear functional on V and if f is defined by* (1), *then f is a complex-linear functional on V.*

(*c*) *If V is a normed linear space and f and u are related as in* (1), *then $\|f\| = \|u\|$.*

PROOF If α and β are real numbers and $z = \alpha + i\beta$, the real part of iz is $-\beta$. This gives the identity

$$z = \operatorname{Re} z - i \operatorname{Re} (iz) \tag{2}$$

for all complex numbers z. Since

$$\operatorname{Re} (if(x)) = \operatorname{Re} f(ix) = u(ix), \tag{3}$$

(1) follows from (2) with $z = f(x)$.

Under the hypotheses (*b*), it is clear that $f(x+y)=f(x)+f(y)$ and that $f(\alpha x)=\alpha f(x)$ for all real α. But we also have

$$f(ix)=u(ix)-iu(-x)=u(ix)+iu(x)=if(x), \tag{4}$$

which proves that f is complex-linear.

Since $|u(x)|\le|f(x)|$, we have $\|u\|\le\|f\|$. On the other hand, to every $x\in V$ there corresponds a complex number α, $|\alpha|=1$, so that $\alpha f(x)=|f(x)|$. Then

$$|f(x)|=f(\alpha x)=u(\alpha x)\le\|u\|\cdot\|\alpha x\|=\|u\|\cdot\|x\|, \tag{5}$$

which proves that $\|f\|\le\|u\|$. ////

5.18 Proof of Theorem 5.16 We first assume that X is a real normed linear space and, consequently, that f is a real-linear bounded functional on M. If $\|f\|=0$, the desired extension is $F=0$. Omitting this case, there is no loss of generality in assuming that $\|f\|=1$.

Choose $x_0\in X$, $x_0\notin M$, and let M_1 be the vector space spanned by M and x_0. Then M_1 consists of all vectors of the form $x+\lambda x_0$, where $x\in M$ and λ is a real scalar. If we define $f_1(x+\lambda x_0)=f(x)+\lambda\alpha$, where α is any fixed real number, it is trivial to verify that an extension of f to a linear functional on M_1 is obtained. The problem is to choose α so that the extended functional still has norm 1. This will be the case provided that

$$|f(x)+\lambda\alpha|\le\|x+\lambda x_0\| \qquad (x\in M,\ \lambda \text{ real}). \tag{1}$$

Replace x by $-\lambda x$ and divide both sides of (1) by $|\lambda|$. The requirement is then that

$$|f(x)-\alpha|\le\|x-x_0\| \qquad (x\in M), \tag{2}$$

i.e., that $A_x\le\alpha\le B_x$ for all $x\in M$, where

$$A_x=f(x)-\|x-x_0\| \quad \text{and} \quad B_x=f(x)+\|x-x_0\|. \tag{3}$$

There exists such an α if and only if all the intervals $[A_x, B_x]$ have a common point, i.e., if and only if

$$A_x\le B_y \tag{4}$$

for all x and $y\in M$. But

$$f(x)-f(y)=f(x-y)\le\|x-y\|\le\|x-x_0\|+\|y-x_0\|, \tag{5}$$

and so (4) follows from (3).

We have now proved that there exists a norm-preserving extension f_1 of f on M_1.

Let $\mathscr{P}$ be the collection of all ordered pairs (M', f'), where M' is a subspace of X which contains M and where f' is a real-linear extension of f to M', with $\|f'\|=1$. Partially order $\mathscr{P}$ by declaring $(M', f')\le(M'', f'')$ to mean that $M'\subset M''$ and $f''(x)=f'(x)$ for all $x\in M'$. The axioms of a partial order

are clearly satisfied, $\mathscr{P}$ is not empty since it contains (M, f), and so the Hausdorff maximality theorem asserts the existence of a maximal totally ordered subcollection Ω of $\mathscr{P}$.

Let Φ be the collection of all M' such that $(M', f') \in \Omega$. Then Φ is totally ordered, by set inclusion, and *therefore* the union $\tilde{M}$ of all members of Φ is a subspace of X. (Note that in general the union of two subspaces is not a subspace. An example is two planes through the origin in R^3.) If $x \in \tilde{M}$, then $x \in M'$ for some $M' \in \Phi$; define $F(x) = f'(x)$, where f' is the function which occurs in the pair $(M', f') \in \Omega$. Our definition of the partial order in Ω shows that it is immaterial which $M' \in \Phi$ we choose to define $F(x)$, as long as M' contains x.

It is now easy to check that F is a linear functional on $\tilde{M}$, with $\|F\| = 1$. If $\tilde{M}$ were a proper subspace X, the first part of the proof would give us a further extension of F, and this would contradict the maximality of Ω. Thus $\tilde{M} = X$, and the proof is complete for the case of real scalars.

If now f is a complex-linear functional on the subspace M of the complex normed linear space X, let u be the real part of f, use the real Hahn-Banach theorem to extend u to a real-linear functional U on X, with $\|U\| = \|u\|$, and define

$$F(x) = U(x) - iU(ix) \qquad (x \in X). \tag{6}$$

By Proposition 5.17, F is a complex-linear extension of f, and

$$\|F\| = \|U\| = \|u\| = \|f\|.$$

This completes the proof. ////

Let us mention two important consequences of the Hahn-Banach theorem:

5.19 Theorem *Let M be a linear subspace of a normed linear space X, and let $x_0 \in X$. Then x_0 is in the closure $\bar{M}$ of M if and only if there is no bounded linear functional f on X such that $f(x) = 0$ for all $x \in M$ but $f(x_0) \neq 0$.*

PROOF If $x_0 \in \bar{M}$, f is a bounded linear functional on X, and $f(x) = 0$ for all $x \in M$, the continuity of f shows that we also have $f(x_0) = 0$.

Conversely, suppose $x_0 \notin \bar{M}$. Then there exists a $\delta > 0$ such that $\|x - x_0\| > \delta$ for all $x \in M$. Let M' be the subspace generated by M and x_0, and define $f(x + \lambda x_0) = \lambda$ if $x \in M$ and λ is a scalar. Since

$$\delta|\lambda| \leq |\lambda| \|x_0 + \lambda^{-1}x\| = \|\lambda x_0 + x\|,$$

we see that f is a linear functional on M' whose norm is at most δ^{-1}. Also $f(x) = 0$ on M, $f(x_0) = 1$. The Hahn-Banach theorem allows us to extend this f from M' to X. ////

5.20 Theorem *If X is a normed linear space and if $x_0 \in X$, $x_0 \neq 0$, there is a bounded linear functional f on X, of norm 1, so that $f(x_0) = \|x_0\|$.*

PROOF Let $M = \{\lambda x_0\}$, and define $f(\lambda x_0) = \lambda \|x_0\|$. Then f is a linear functional of norm 1 on M, and the Hahn-Banach theorem can again be applied. ////

5.21 Remarks If X is a normed linear space, let X^* be the collection of all bounded linear functionals on X. If addition and scalar multiplication of linear functionals are defined in the obvious manner, it is easy to see that X^* is again a normed linear space. In fact, X^* is a Banach space; this follows from the fact that the field of scalars is a complete metric space. We leave the verification of these properties of X^* as an exercise.

One of the consequences of Theorem 5.20 is that X^* is not the trivial vector space (i.e., X^* consists of more than 0) if X is not trivial. In fact, X^* *separates points on* X. This means that if $x_1 \neq x_2$ in X there exists an $f \in X^*$ such that $f(x_1) \neq f(x_2)$. To prove this, merely take $x_0 = x_2 - x_1$ in Theorem 5.20.

Another consequence is that, for $x \in X$,

$$\|x\| = \sup \{|f(x)| : f \in X^*, \|f\| = 1\}.$$

Hence, for fixed $x \in X$, the mapping $f \to f(x)$ is a bounded linear functional on X^*, of norm $\|x\|$.

This interplay between X and X^* (the so-called "dual space" of X) forms the basis of a large portion of that part of mathematics which is known as *functional analysis.*

An Abstract Approach to the Poisson Integral

5.22 Successful applications of the Hahn-Banach theorem to concrete problems depend of course on a knowledge of the bounded linear functionals on the normed linear space under consideration. So far we have only determined the bounded linear functionals on a Hilbert space (where a much simpler proof of the Hahn-Banach theorem exists; see Exercise 6), and we know the positive linear functionals on $C_c(X)$.

We shall now describe a general situation in which the last-mentioned functionals occur naturally.

Let K be a compact Hausdorff space, let H be a compact subset of K, and let A be a subspace of $C(K)$ such that $1 \in A$ (1 denotes the function which assigns the number 1 to each $x \in K$) and such that

$$\|f\|_K = \|f\|_H \qquad (f \in A). \tag{1}$$

Here we used the notation

$$\|f\|_E = \sup \{|f(x)| : x \in E\}. \tag{2}$$

Because of the example discussed in Sec. 5.23, H is sometimes called a *boundary* of K, corresponding to the space A.

If $f \in A$ and $x \in K$, (1) says that

$$|f(x)| \leq \|f\|_H. \tag{3}$$

In particular, if $f(y) = 0$ for every $y \in H$, then $f(x) = 0$ for all $x \in K$. Hence if f_1 and $f_2 \in A$ and $f_1(y) = f_2(y)$ for every $y \in H$, then $f_1 = f_2$; to see this, put $f = f_1 - f_2$.

Let M be the set of all functions on H that are restrictions to H of members of A. It is clear that M is a subspace of $C(H)$. The preceding remark shows that each member of M has a unique extension to a member of A. Thus we have a natural one-to-one correspondence between M and A, which is also norm-preserving, by (1). Hence it will cause no confusion if we use the same letter to designate a member of A and its restriction to H.

Fix a point $x \in K$. The inequality (3) shows that the mapping $f \to f(x)$ is a bounded linear functional on M, of norm 1 [since equality holds in (3) if $f = 1$]. By the Hahn-Banach theorem there is a linear functional Λ on $C(H)$, of norm 1, such that

$$\Lambda f = f(x) \qquad (f \in M). \tag{4}$$

We claim that the properties

$$\Lambda 1 = 1, \qquad \|\Lambda\| = 1 \tag{5}$$

imply that Λ is a *positive* linear functional on $C(H)$.

To prove this, suppose $f \in C(H)$, $0 \leq f \leq 1$, put $g = 2f - 1$, and put $\Lambda g = \alpha + i\beta$, where α and β are real. Note that $-1 \leq g \leq 1$, so that $|g + ir|^2 \leq 1 + r^2$ for every real constant r. Hence (5) implies that

$$(\beta + r)^2 \leq |\alpha + i(\beta + r)|^2 = |\Lambda(g + ir)|^2 \leq 1 + r^2. \tag{6}$$

Thus $\beta^2 + 2r\beta \leq 1$ for every real r, which forces $\beta = 0$. Since $\|g\|_H \leq 1$, we have $|\alpha| \leq 1$; hence

$$\Lambda f = \tfrac{1}{2}\Lambda(1 + g) = \tfrac{1}{2}(1 + \alpha) \geq 0. \tag{7}$$

Now Theorem 2.14 can be applied. It shows that there is a regular positive Borel measure μ_x on H such that

$$\Lambda f = \int_H f \, d\mu_x \qquad (f \in C(H)). \tag{8}$$

In particular, we get the representation formula

$$f(x) = \int_H f \, d\mu_x \qquad (f \in A). \tag{9}$$

What we have proved is that *to each $x \in K$ there corresponds a positive measure μ_x on the "boundary" H which "represents" x in the sense that* (9) *holds for every $f \in A$.*

Note that Λ determines μ_x uniquely; but there is no reason to expect the Hahn-Banach extension to be unique. Hence, in general, we cannot say much about the uniqueness of the representing measures. Under special circumstances we do get uniqueness, as we shall see presently.

5.23 To see an example of the preceding situation, let $U = \{z: |z| < 1\}$ be the open unit disc in the complex plane, put $K = \bar{U}$ (the closed unit disc), and take for H the boundary T of U. We claim that every polynomial f, i.e., every function of the form

$$f(z) = \sum_{n=0}^{N} a_n z^n, \tag{1}$$

where $a_0, \ldots, a_N$ are complex numbers, satisfies the relation

$$\|f\|_U = \|f\|_T. \tag{2}$$

(Note that the continuity of f shows that the supremum of $|f|$ over U is the same as that over $\bar{U}$.)

Since $\bar{U}$ is compact, there exists a $z_0 \in \bar{U}$ such that $|f(z_0)| \geq |f(z)|$ for all $z \in \bar{U}$. Assume $z_0 \in U$. Then

$$f(z) = \sum_{n=0}^{N} b_n (z - z_0)^n, \tag{3}$$

and if $0 < r < 1 - |z_0|$, we obtain

$$\sum_{n=0}^{N} |b_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(z_0 + re^{i\theta})|^2 \, d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(z_0)|^2 \, d\theta = |b_0|^2,$$

so that $b_1 = b_2 = \cdots = b_N = 0$; i.e., f is constant. Thus $z_0 \in T$ for every nonconstant polynomial f, and this proves (2).

(We have just proved a special case of the *maximum modulus theorem*; we shall see later that this is an important property of all holomorphic functions.)

5.24 The Poisson Integral Let A be any subspace of $C(\bar{U})$ (where $\bar{U}$ is the closed unit disc, as above) such that A contains all polynomials and such that

$$\|f\|_U = \|f\|_T \tag{1}$$

holds for every $f \in A$. We do not exclude the possibility that A consists of precisely the polynomials, but A might be larger.

The general result obtained in Sec. 5.22 applies to A and shows that to each $z \in U$ there corresponds a positive Borel measure μ_z on T such that

$$f(z) = \int_T f \, d\mu_z \qquad (f \in A). \tag{2}$$

(This also holds for $z \in T$, but is then trivial: μ_z is simply the unit mass concentrated at the point z.)

We now fix $z \in U$ and write $z = re^{i\theta}$, $0 \le r < 1$, θ real.

If $u_n(w) = w^n$, then $u_n \in A$ for $n = 0, 1, 2, \ldots$; hence (2) shows that

$$r^n e^{in\theta} = \int_T u_n \, d\mu_z \qquad (n = 0, 1, 2, \ldots). \tag{3}$$

Since $u_{-n} = \bar{u}_n$ on T, (3) leads to

$$\int_T u_n \, d\mu_z = r^{|n|} e^{in\theta} \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{4}$$

This suggests that we look at the real function

$$P_r(\theta - t) = \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta - t)} \qquad (t \text{ real}), \tag{5}$$

since

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} P_r(\theta - t) e^{int} \, dt = r^{|n|} e^{in\theta} \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{6}$$

Note that the series (5) is dominated by the convergent geometric series $\sum r^{|n|}$, so that it is legitimate to insert the series into the integral (6) and to integrate term by term, which gives (6). Comparison of (4) and (6) gives

$$\int_T f \, d\mu_z = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) P_r(\theta - t) \, dt \tag{7}$$

for $f = u_n$, hence for every trigonometric polynomial f, and Theorem 4.25 now implies that (7) holds for every $f \in C(T)$. [This shows that μ_z was uniquely determined by (2). Why?]

In particular, (7) holds if $f \in A$, and then (2) gives the representation

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) P_r(\theta - t) \, dt \qquad (f \in A). \tag{8}$$

The series (5) can be summed explicitly, since it is the real part of

$$1 + 2 \sum_{1}^{\infty} (ze^{-it})^n = \frac{e^{it} + z}{e^{it} - z} = \frac{1 - r^2 + 2ir \sin(\theta - t)}{|1 - ze^{-it}|^2}.$$

Thus

$$P_r(\theta - t) = \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2}. \tag{9}$$

This is the so-called "Poisson kernel." Note that $P_r(\theta - t) \ge 0$ if $0 \le r < 1$.

We now summarize what we have proved:

5.25 Theorem *Suppose A is a vector space of continuous complex functions on the closed unit disc $\bar{U}$. If A contains all polynomials, and if*

$$\sup_{z \in U} |f(z)| = \sup_{z \in T} |f(z)| \tag{1}$$

for every $f \in A$ (where T is the unit circle, the boundary of U), then the Poisson integral representation

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r \cos(\theta - t) + r^2} f(e^{it})\, dt \qquad (z = re^{i\theta}) \tag{2}$$

is valid for every $f \in A$ and every $z \in U$.

Exercises

1 Let X consist of two points a and b, put $\mu(\{a\}) = \mu(\{b\}) = \frac{1}{2}$, and let $L^p(\mu)$ be the resulting *real* L^p-space. Identify each real function f on X with the point $(f(a), f(b))$ in the plane, and sketch the unit balls of $L^p(\mu)$, for $0 < p \leq \infty$. Note that they are convex if and only if $1 \leq p \leq \infty$. For which p is this unit ball a square? A circle? If $\mu(\{a\}) \neq \mu(b)$, how does the situation differ from the preceding one?

2 Prove that the unit ball (open or closed) is convex in every normed linear space.

3 If $1 < p < \infty$, prove that the unit ball of $L^p(\mu)$ is *strictly convex*; this means that if

$$\|f\|_p = \|g\|_p = 1, \qquad f \neq g, \qquad h = \tfrac{1}{2}(f + g),$$

then $\|h\|_p < 1$. (Geometrically, the surface of the ball contains no straight lines.) Show that this fails in every $L^1(\mu)$, in every $L^\infty(\mu)$, and in every $C(X)$. (Ignore trivialities, such as spaces consisting of only one point.)

4 Let C be the space of all continuous functions on $[0, 1]$, with the supremum norm. Let M consist of all $f \in C$ for which

$$\int_0^{1/2} f(t)\, dt - \int_{1/2}^{1} f(t)\, dt = 1.$$

Prove that M is a closed convex subset of C which contains no element of minimal norm.

5 Let M be the set of all $f \in L^1([0, 1])$, relative to Lebesgue measure, such that

$$\int_0^1 f(t)\, dt = 1.$$

Show that M is a closed convex subset of $L^1([0, 1])$ which contains infinitely many elements of minimal norm. (Compare this and Exercise 4 with Theorem 4.10.)

6 Let f be a bounded linear functional on a subspace M of a Hilbert space H. Prove that f has a *unique* norm-preserving extension to a bounded linear functional on H, and that this extension vanishes on $M^\perp$.

7 Construct a bounded linear functional on some subspace of some $L^1(\mu)$ which has two (hence infinitely many) distinct norm-preserving linear extensions to $L^1(\mu)$.

8 Let X be a normed linear space, and let X^* be its dual space, as defined in Sec. 5.21, with the norm

$$\|f\| = \sup \{|f(x)| : \|x\| \leq 1\}.$$

(*a*) Prove that X^* is a Banach space.

(*b*) Prove that the mapping $f \to f(x)$ is, for each $x \in X$, a bounded linear functional on X^*, of norm $\|x\|$. (This gives a natural imbedding of X in its "second dual" X^{**}, the dual space of X^*.)

(*c*) Prove that $\{\|x_n\|\}$ is bounded if $\{x_n\}$ is a sequence in X such that $\{f(x_n)\}$ is bounded for every $f \in X^*$.

9 Let c_0, ℓ^1, and ℓ^∞ be the Banach spaces consisting of all complex sequences $x = \{\xi_i\}$, $i = 1, 2, 3, \ldots$, defined as follows:

$$x \in \ell^1 \text{ if and only if } \|x\|_1 = \sum |\xi_i| < \infty.$$

$$x \in \ell^\infty \text{ if and only if } \|x\|_\infty = \sup |\xi_i| < \infty.$$

c_0 is the subspace of ℓ^∞ consisting of all $x \in \ell^\infty$ for which $\xi_i \to 0$ as $i \to \infty$.

Prove the following four statements.

(*a*) If $y = \{\eta_i\} \in \ell^1$ and $\Lambda x = \sum \xi_i \eta_i$ for every $x \in c_0$, then Λ is a bounded linear functional on c_0, and $\|\Lambda\| = \|y\|_1$. Moreover, every $\Lambda \in (c_0)^*$ is obtained in this way. In brief, $(c_0)^* = \ell^1$.

(More precisely, these two spaces are not equal; the preceding statement exhibits an isometric vector space isomorphism between them.)

(*b*) In the same sense, $(\ell^1)^* = \ell^\infty$.

(*c*) Every $y \in \ell^1$ induces a bounded linear functional on ℓ^∞, as in (*a*). However, this does *not* give all of $(\ell^\infty)^*$, since $(\ell^\infty)^*$ contains nontrivial functionals that vanish on all of c_0.

(*d*) c_0 and ℓ^1 are separable but ℓ^∞ is not.

10 If $\sum \alpha_i \xi_i$ converges for every sequence (ξ_i) such that $\xi_i \to 0$ as $i \to \infty$, prove that $\sum |\alpha_i| < \infty$.

11 For $0 < \alpha \leq 1$, let Lip α denote the space of all complex functions f on $[a, b]$ for which

$$M_f = \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s - t|^\alpha} < \infty.$$

Prove that Lip α is a Banach space, if $\|f\| = |f(a)| + M_f$; also, if

$$\|f\| = M_f + \sup_x |f(x)|.$$

(The members of Lip α are said to satisfy a *Lipschitz condition* of order α.)

12 Let K be a triangle (two-dimensional figure) in the plane, let H be the set consisting of the vertices of K, and let A be the set of all real functions f on K, of the form

$$f(x, y) = \alpha x + \beta y + \gamma \qquad (\alpha, \beta, \text{ and } \gamma \text{ real}).$$

Show that to each $(x_0, y_0) \in K$ there corresponds a unique measure μ on H such that

$$f(x_0, y_0) = \int_H f \, d\mu.$$

(Compare Sec. 5.22.)

Replace K by a square, let H again be the set of its vertices, and let A be as above. Show that to each point of K there still corresponds a measure on H, with the above property, but that uniqueness is now lost.

Can you extrapolate to a more general theorem? (Think of other figures, higher dimensional spaces.)

13 Let $\{f_n\}$ be a sequence of continuous complex functions on a (nonempty) complete metric space X, such that $f(x) = \lim f_n(x)$ exists (as a complex number) for every $x \in X$.

(*a*) Prove that there is an open set $V \neq \varnothing$ and a number $M < \infty$ such that $|f_n(x)| < M$ for all $x \in V$ and for $n = 1, 2, 3, \ldots$.

(*b*) If $\epsilon > 0$, prove that there is an open set $V \neq \varnothing$ and an integer N such that $|f(x) - f_n(x)| \leq \epsilon$ if $x \in V$ and $n \geq N$.

Hint for (*b*): For $N = 1, 2, 3, \ldots$, put

$$A_N = \{x: |f_m(x) - f_n(x)| \leq \epsilon \text{ if } m \geq N \text{ and } n \geq N\}.$$

Since $X = \bigcup A_N$, some A_N has a nonempty interior.

14 Let C be the space of all real continuous functions on $I = [0, 1]$ with the supremum norm. Let X_n be the subset of C consisting of those f for which there exists a $t \in I$ such that $|f(s) - f(t)| \leq n|s - t|$ for all $s \in I$. Fix n and prove that each open set in C contains an open set which does not intersect X_n. (Each $f \in C$ can be uniformly approximated by a zigzag function g with very large slopes, and if $\|g - h\|$ is small, $h \notin X_n$.) Show that this implies the existence of a dense G_δ in C which consists entirely of nowhere differentiable functions.

15 Let $A = (a_{ij})$ be an infinite matrix with complex entries, where $i, j = 0, 1, 2, \ldots$. A associates with each sequence $\{s_j\}$ a sequence $\{\sigma_i\}$, defined by

$$\sigma_i = \sum_{j=0}^{\infty} a_{ij} s_j \qquad (i = 1, 2, 3, \ldots),$$

provided that these series converge.

Prove that A transforms every convergent sequence $\{s_j\}$ to a sequence $\{\sigma_i\}$ which converges to the same limit if and only if the following conditions are satisfied:

(*a*) $$\lim_{i \to \infty} a_{ij} = 0 \qquad \text{for each } j.$$

(*b*) $$\sup_i \sum_{j=0}^{\infty} |a_{ij}| < \infty.$$

(*c*) $$\lim_{i \to \infty} \sum_{j=0}^{\infty} a_{ij} = 1.$$

The process of passing from $\{s_j\}$ to $\{\sigma_i\}$ is called a *summability method*. Two examples are:

$$a_{ij} = \begin{cases} \dfrac{1}{i+1} & \text{if } 0 \leq j \leq i, \\ 0 & \text{if } i < j, \end{cases}$$

and

$$a_{ij} = (1 - r_i) r_i^j, \qquad 0 < r_i < 1, \qquad r_i \to 1.$$

Prove that each of these also transforms some divergent sequences $\{s_j\}$ (even some unbounded ones) to convergent sequences $\{\sigma_i\}$.

16 Suppose X and Y are Banach spaces, and suppose Λ is a linear mapping of X into Y, with the following property: For every sequence $\{x_n\}$ in X for which $x = \lim x_n$ and $y = \lim \Lambda x_n$ exist, it is true that $y = \Lambda x$. Prove that Λ is continuous.

This is the so-called "closed graph theorem." *Hint*: Let $X \oplus Y$ be the set of all ordered pairs (x, y), $x \in X$ and $y \in Y$, with addition and scalar multiplication defined componentwise. Prove that $X \oplus Y$ is a Banach space, if $\|(x, y)\| = \|x\| + \|y\|$. The graph G of Λ is the subset of $X \oplus Y$ formed by the pairs $(x, \Lambda x)$, $x \in X$. Note that our hypothesis says that G is closed; hence G is a Banach space. Note that $(x, \Lambda x) \to x$ is continuous, one-to-one, and linear and maps G onto X.

Observe that there exist *nonlinear* mappings (of R^1 onto R^1, for instance) whose graph is closed although they are not continuous: $f(x) = 1/x$ if $x \neq 0, f(0) = 0$.

17 If μ is a positive measure, each $f \in L^\infty(\mu)$ defines a multiplication operator M_f on $L^2(\mu)$ into $L^2(\mu)$, such that $M_f(g) = fg$. Prove that $\|M_f\| \leq \|f\|_\infty$. For which measures μ is it true that $\|M_f\| = \|f\|_\infty$ for all $f \in L^\infty(\mu)$? For which $f \in L^\infty(\mu)$ does M_f map $L^2(\mu)$ onto $L^2(\mu)$?

18 Suppose $\{\Lambda_n\}$ is a sequence of bounded linear transformations from a normed linear space X to a Banach space Y, suppose $\|\Lambda_n\| \leq M < \infty$ for all n, and suppose there is a dense set $E \subset X$ such that $\{\Lambda_n x\}$ converges for each $x \in E$. Prove that $\{\Lambda_n x\}$ converges for each $x \in X$.

19 If s_n is the nth partial sum of the Fourier series of a function $f \in C(T)$, prove that $s_n/\log n \to 0$ uniformly, as $n \to \infty$, for each $f \in C(T)$. That is, prove that

$$\lim_{n\to\infty} \frac{\|s_n\|_\infty}{\log n} = 0.$$

On the other hand, if $\lambda_n/\log n \to 0$, prove that there exists an $f \in C(T)$ such that the sequence $\{s_n(f; 0)/\lambda_n\}$ is unbounded. *Hint*: Apply the reasoning of Exercise 18 and that of Sec. 5.11, with a better estimate of $\|D_n\|_1$ than was used there.

20 (*a*) Does there exist a sequence of continuous positive functions f_n on R^1 such that $\{f_n(x)\}$ is unbounded if and only if x is rational?

(*b*) Replace "rational" by "irrational" in (*a*) and answer the resulting question.

(*c*) Replace "$\{f_n(x)\}$ is unbounded" by "$f_n(x) \to \infty$ as $n \to \infty$" and answer the resulting analogues of (*a*) and (*b*).

21 Suppose $E \subset R^1$ is measurable, and $m(E) = 0$. Must there be a translate $E + x$ of E that does not intersect E? Must there be a homeomorphism h of R^1 onto R^1 so that $h(E)$ does not intersect E?

22 Suppose $f \in C(T)$ and $f \in \text{Lip } \alpha$ for some $\alpha > 0$. (See Exercise 11.) Prove that the Fourier series of f converges to $f(x)$, by completing the following outline: It is enough to consider the case $x = 0$, $f(0) = 0$. The difference between the partial sums $s_n(f; 0)$ and the integrals

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\frac{\sin nt}{t}\,dt$$

tends to 0 as $n \to \infty$. The function $f(t)/t$ is in $L^1(T)$. Apply the Riemann-Lebesgue lemma. More careful reasoning shows that the convergence is actually uniform on T.

CHAPTER

SIX

COMPLEX MEASURES

Total Variation

6.1 Introduction Let $\mathfrak{M}$ be a σ-algebra in a set X. Call a countable collection $\{E_i\}$ of members of $\mathfrak{M}$ a *partition of* E if $E_i \cap E_j = \varnothing$ whenever $i \neq j$, and if $E = \bigcup E_i$. A *complex measure* μ on $\mathfrak{M}$ is then a complex function on $\mathfrak{M}$ such that

$$\mu(E) = \sum_{i=1}^{\infty} \mu(E_i) \qquad (E \in \mathfrak{M}) \tag{1}$$

for every partition $\{E_i\}$ of E.

Observe that the convergence of the series in (1) is now part of the requirement (unlike for positive measures, where the series could either converge or diverge to ∞). Since the union of the sets E_i is not changed if the subscripts are permuted, every rearrangement of the series (1) must also converge. Hence ([26], Theorem 3.56) the series actually converges absolutely.

Let us consider the problem of finding a positive measure λ which dominates a given complex measure μ on $\mathfrak{M}$, in the sense that $|\mu(E)| \leq \lambda(E)$ for every $E \in \mathfrak{M}$, and let us try to keep λ as small as we can. Every solution to our problem (if there is one at all) must satisfy

$$\lambda(E) = \sum_{i=1}^{\infty} \lambda(E_i) \geq \sum_{1}^{\infty} |\mu(E_i)|, \tag{2}$$

for every partition $\{E_i\}$ of any set $E \in \mathfrak{M}$, so that $\lambda(E)$ is at least equal to the supremum of the sums on the right of (2), taken over all partitions of E. This suggests that we *define a set function* $|\mu|$ *on* $\mathfrak{M}$ *by*

$$|\mu|(E) = \sup \sum_{i=1}^{\infty} |\mu(E_i)| \qquad (E \in \mathfrak{M}), \tag{3}$$

the supremum being taken over all partitions $\{E_i\}$ *of* E.

This notation is perhaps not the best, but it is the customary one. Note that $|\mu|(E) \geq |\mu(E)|$, but that in general $|\mu|(E)$ is not equal to $|\mu(E)|$.

It turns out, as will be proved below, that $|\mu|$ actually *is* a measure, so that our problem does have a solution. The discussion which led to (3) shows then clearly that $|\mu|$ is the minimal solution, in the sense that any other solution λ has the property $\lambda(E) \geq |\mu|(E)$ for all $E \in \mathfrak{M}$.

The set function $|\mu|$ is called the *total variation* of μ, or sometimes, to avoid misunderstanding, the *total variation measure*. The term "total variation of μ" is also frequently used to denote the number $|\mu|(X)$.

If μ is a positive measure, then of course $|\mu| = \mu$.

Besides being a measure, $|\mu|$ has another unexpected property: $|\mu|(X) < \infty$. Since $|\mu(E)| \leq |\mu|(E) \leq |\mu|(X)$, this implies that every complex measure μ on any σ-algebra is bounded: If the range of μ lies in the complex plane, then it actually lies in some disc of finite radius. This property (proved in Theorem 6.4) is sometimes expressed by saying that μ *is of bounded variation.*

6.2 Theorem *The total variation $|\mu|$ of a complex measure μ on $\mathfrak{M}$ is a positive measure on $\mathfrak{M}$.*

PROOF Let $\{E_i\}$ be a partition of $E \in \mathfrak{M}$. Let t_i be real numbers such that $t_i < |\mu|(E_i)$. Then each E_i has a partition $\{A_{ij}\}$ such that

$$\sum_j |\mu(A_{ij})| > t_i \qquad (i = 1, 2, 3, \ldots). \tag{1}$$

Since $\{A_{ij}\}$ $(i, j = 1, 2, 3, \ldots)$ is a partition of E, it follows that

$$\sum_i t_i \leq \sum_{i,j} |\mu(A_{ij})| \leq |\mu|(E). \tag{2}$$

Taking the supremum of the left side of (2), over all admissible choices of $\{t_i\}$, we see that

$$\sum_i |\mu|(E_i) \leq |\mu|(E). \tag{3}$$

To prove the opposite inequality, let $\{A_j\}$ be any partition of E. Then for any fixed j, $\{A_j \cap E_i\}$ is a partition of A_j, and for any fixed i, $\{A_j \cap E_i\}$ is a partition of E_i. Hence

$$\begin{aligned} \sum_j |\mu(A_j)| &= \sum_j \left| \sum_i \mu(A_j \cap E_i) \right| \\ &\leq \sum_j \sum_i |\mu(A_j \cap E_i)| \\ &= \sum_i \sum_j |\mu(A_j \cap E_i)| \leq \sum_i |\mu|(E_i). \end{aligned} \tag{4}$$

Since (4) holds for every partition $\{A_j\}$ of E, we have

$$|\mu|(E) \leq \sum_i |\mu|(E_i). \tag{5}$$

By (3) and (5), $|\mu|$ is countably additive.

Note that the Corollary to Theorem 1.27 was used in (2) and (4).

That $|\mu|$ is not identically ∞ is a trivial consequence of Theorem 6.4 but can also be seen right now, since $|\mu|(\varnothing) = 0$. ////

6.3 Lemma *If $z_1, \ldots, z_N$ are complex numbers then there is a subset S of $\{1, \ldots, N\}$ for which*

$$\left|\sum_{k \in S} z_k\right| \geq \frac{1}{\pi} \sum_{k=1}^{N} |z_k|.$$

PROOF Write $z_k = |z_k| e^{i\alpha_k}$. For $-\pi \leq \theta \leq \pi$, let $S(\theta)$ be the set of all k for which $\cos(\alpha_k - \theta) > 0$. Then

$$\left|\sum_{S(\theta)} z_k\right| = \left|\sum_{S(\theta)} e^{-i\theta} z_k\right| \geq \operatorname{Re} \sum_{S(\theta)} e^{-i\theta} z_k = \sum_{k=1}^{N} |z_k| \cos^+(\alpha_k - \theta).$$

Choose θ_0 so as to maximize the last sum, and put $S = S(\theta_0)$. This maximum is at least as large as the average of the sum over $[-\pi, \pi]$, and this average is $\pi^{-1} \sum |z_k|$, because

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \cos^+(\alpha - \theta)\, d\theta = \frac{1}{\pi}$$

for every α. ////

6.4 Theorem *If μ is a complex measure on X, then*

$$|\mu|(X) < \infty.$$

PROOF Suppose first that some set $E \in \mathfrak{M}$ has $|\mu|(E) = \infty$. Put $t = \pi(1 + |\mu(E)|)$. Since $|\mu|(E) > t$, there is a partition $\{E_i\}$ of E such that

$$\sum_{i=1}^{N} |\mu(E_i)| > t$$

for some N. Apply Lemma 6.3, with $z_i = \mu(E_i)$, to conclude that there is a set $A \subset E$ (a union of some of the sets E_i) for which

$$|\mu(A)| > t/\pi > 1.$$

Setting $B = E - A$, it follows that

$$|\mu(B)| = |\mu(E) - \mu(A)| \geq |\mu(A)| - |\mu(E)| > \frac{t}{\pi} - |\mu(E)| = 1.$$

We have thus split E into disjoint sets A and B with $|\mu(A)| > 1$ and $|\mu(B)| > 1$. Evidently, at least one of $|\mu|(A)$ and $|\mu|(B)$ is ∞, by Theorem 6.2.

Now if $|\mu|(X) = \infty$, split X into A_1, B_1, as above, with $|\mu(A_1)| > 1$, $|\mu|(B_1) = \infty$. Split B_1 into A_2, B_2, with $|\mu(A_2)| > 1$, $|\mu|(B_2) = \infty$. Continuing in this way, we get a countably infinite disjoint collection $\{A_i\}$, with $|\mu(A_i)| > 1$ for each i. The countable additivity of μ implies that

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i).$$

But this series cannot converge, since $\mu(A_i)$ does not tend to 0 as $i \to \infty$. This contradiction shows that $|\mu|(X) < \infty$. ////

6.5 If μ and λ are complex measures on the same σ-algebra $\mathfrak{M}$, we define $\mu + \lambda$ and $c\mu$ by

$$\begin{aligned}(\mu + \lambda)(E) &= \mu(E) + \lambda(E) \\ (c\mu)(E) &= c\mu(E)\end{aligned} \qquad (E \in \mathfrak{M}) \tag{1}$$

for any scalar c, in the usual manner. It is then trivial to verify that $\mu + \lambda$ and $c\mu$ are complex measures. The collection of all complex measures on $\mathfrak{M}$ is thus a vector space. If we put

$$\|\mu\| = |\mu|(X), \tag{2}$$

it is easy to verify that all axioms of a normed linear space are satisfied.

6.6 Positive and Negative Variations Let us now specialize and consider a *real* measure μ on a σ-algebra $\mathfrak{M}$. (Such measures are frequently called *signed* measures.) Define $|\mu|$ as before, and define

$$\mu^+ = \tfrac{1}{2}(|\mu| + \mu), \qquad \mu^- = \tfrac{1}{2}(|\mu| - \mu). \tag{1}$$

Then both μ^+ and μ^- are positive measures on $\mathfrak{M}$, and they are bounded, by Theorem 6.4. Also,

$$\mu = \mu^+ - \mu^-, \qquad |\mu| = \mu^+ + \mu^-. \tag{2}$$

The measures μ^+ and μ^- are called the *positive* and *negative variations* of μ, respectively. This representation of μ as the difference of the positive measures μ^+ and μ^- is known as the *Jordan decomposition* of μ. Among all representations of μ as a difference of two positive measures, the Jordan decomposition has a certain minimum property which will be established as a corollary to Theorem 6.14.

Absolute Continuity

6.7 Definitions Let μ be a positive measure on a σ-algebra $\mathfrak{M}$, and let λ be an arbitrary measure on $\mathfrak{M}$; λ may be positive or complex. (Recall that a complex measure has its range in the complex plane, but that our usage of the term "positive measure" includes ∞ as an admissible value. Thus the positive measures do not form a subclass of the complex ones.)

We say that λ is *absolutely continuous* with respect to μ, and write

$$\lambda \ll \mu \tag{1}$$

if $\lambda(E) = 0$ for every $E \in \mathfrak{M}$ for which $\mu(E) = 0$.

If there is a set $A \in \mathfrak{M}$ such that $\lambda(E) = \lambda(A \cap E)$ for every $E \in \mathfrak{M}$, we say that λ is *concentrated on* A. This is equivalent to the hypothesis that $\lambda(E) = 0$ whenever $E \cap A = \varnothing$.

Suppose λ_1 and λ_2 are measures on $\mathfrak{M}$, and suppose there exists a pair of disjoint sets A and B such that λ_1 is concentrated on A and λ_2 is concentrated on B. Then we say that λ_1 and λ_2 are *mutually singular*, and write

$$\lambda_1 \perp \lambda_2. \tag{2}$$

Here are some elementary properties of these concepts.

6.8 Proposition *Suppose, μ, λ, λ_1, and λ_2 are measures on a σ-algebra $\mathfrak{M}$, and μ is positive.*

(*a*) *If λ is concentrated on A, so is $|\lambda|$.*
(*b*) *If $\lambda_1 \perp \lambda_2$, then $|\lambda_1| \perp |\lambda_2|$.*
(*c*) *If $\lambda_1 \perp \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 + \lambda_2 \perp \mu$.*
(*d*) *If $\lambda_1 \ll \mu$ and $\lambda_2 \ll \mu$, then $\lambda_1 + \lambda_2 \ll \mu$.*
(*e*) *If $\lambda \ll \mu$, then $|\lambda| \ll \mu$.*
(*f*) *If $\lambda_1 \ll \mu$ and $\lambda_2 \perp \mu$, then $\lambda_1 \perp \lambda_2$.*
(*g*) *If $\lambda \ll \mu$ and $\lambda \perp \mu$, then $\lambda = 0$.*

PROOF

(*a*) If $E \cap A = \varnothing$ and $\{E_j\}$ is any partition of E, then $\lambda(E_j) = 0$ for all j. Hence $|\lambda|(E) = 0$.
(*b*) This follows immediately from (*a*).
(*c*) There are disjoint sets A_1 and B_1 such that λ_1 is concentrated on A_1 and μ on B_1, and there are disjoint sets A_2 and B_2 such that λ_2 is concentrated on A_2 and μ on B_2. Hence $\lambda_1 + \lambda_2$ is concentrated on $A = A_1 \cup A_2$, μ is concentrated on $B = B_1 \cap B_2$, and $A \cap B = \varnothing$.
(*d*) This is obvious.
(*e*) Suppose $\mu(E) = 0$, and $\{E_j\}$ is a partition of E. Then $\mu(E_j) = 0$; and since $\lambda \ll \mu$, $\lambda(E_j) = 0$ for all j, hence $\sum |\lambda(E_j)| = 0$. This implies $|\lambda|(E) = 0$.

(*f*) Since $\lambda_2 \perp \mu$, there is a set A with $\mu(A) = 0$ on which λ_2 is concentrated. Since $\lambda_1 \ll \mu$, $\lambda_1(E) = 0$ for every $E \subset A$. So λ_1 is concentrated on the complement of A.

(*g*) By (*f*), the hypothesis of (*g*) implies, that $\lambda \perp \lambda$, and this clearly forces $\lambda = 0$. ////

We come now to the principal theorem about absolute continuity. In fact, it is probably the most important theorem in measure theory. Its statement will involve σ-finite measures. The following lemma describes one of their significant properties.

6.9 Lemma *If μ is a positive σ-finite measure on a σ-algebra $\mathfrak{M}$ in a set X, then there is a function $w \in L^1(\mu)$ such that $0 < w(x) < 1$ for every $x \in X$.*

PROOF To say that μ is σ-finite means that X is the union of countably many sets $E_n \in \mathfrak{M}$ ($n = 1, 2, 3, \ldots$) for which $\mu(E_n)$ is finite. Put $w_n(x) = 0$ if $x \in X - E_n$ and put

$$w_n(x) = 2^{-n}/(1 + \mu(E_n))$$

if $x \in E_n$. Then $w = \sum_1^\infty w_n$ has the required properties. ////

The point of the lemma is that μ can be replaced by a *finite* measure $\tilde{\mu}$ (namely, $d\tilde{\mu} = w\, d\mu$) which, because of the strict positivity of w, has *precisely* the same sets of measure 0 as μ.

6.10 The Theorem of Lebesgue-Radon-Nikodym *Let μ be a positive σ-finite measure on a σ-algebra $\mathfrak{M}$ in a set X, and let λ be a complex measure on $\mathfrak{M}$.*

(*a*) *There is then a unique pair of complex measures λ_a and λ_s on $\mathfrak{M}$ such that*

$$\lambda = \lambda_a + \lambda_s, \qquad \lambda_a \ll \mu, \qquad \lambda_s \perp \mu. \tag{1}$$

If λ is positive and finite, then so are λ_a and λ_s.

(*b*) *There is a unique $h \in L^1(\mu)$ such that*

$$\lambda_a(E) = \int_E h\, d\mu \tag{2}$$

for every set $E \in \mathfrak{M}$.

The pair (λ_a, λ_s) is called the *Lebesgue decomposition* of λ relative to μ. The uniqueness of the decomposition is easily seen, for if (λ_a', λ_s') is another pair which satisfies (1), then

$$\lambda_a' - \lambda_a = \lambda_s - \lambda_s', \tag{3}$$

$\lambda_a' - \lambda_a \ll \mu$, and $\lambda_s - \lambda_s' \perp \mu$, hence both sides of (3) are 0; we have used 6.8(*c*), 6.8(*d*), and 6.8(*g*).

The *existence* of the decomposition is the significant part of (*a*).

Assertion (*b*) is known as the *Radon-Nikodym* theorem. Again, uniqueness of h is immediate, from Theorem 1.39(*b*). Also, if h is *any* member of $L^1(\mu)$, the integral in (2) defines a measure on $\mathfrak{M}$ (Theorem 1.29) which is clearly absolutely continuous with respect to μ. The point of the Radon-Nikodym theorem is the converse: *Every* $\lambda \ll \mu$ (in which case $\lambda_a = \lambda$) is obtained in this way.

The function h which occurs in (2) is called the *Radon-Nikodym derivative* of λ_a with respect to μ. As noted after Theorem 1.29, we may express (2) in the form $d\lambda_a = h\, d\mu$, or even in the form $h = d\lambda_a/d\mu$.

The idea of the following proof, which yields both (*a*) and (*b*) at one stroke, is due to von Neumann.

PROOF Assume first that λ is a positive bounded measure on $\mathfrak{M}$. Associate w to μ as in Lemma 6.9. Then $d\varphi = d\lambda + w\, d\mu$ defines a positive bounded measure φ on $\mathfrak{M}$. The definition of the sum of two measures shows that

$$\int_X f\, d\varphi = \int_X f\, d\lambda + \int_X fw\, d\mu \tag{4}$$

for $f = \chi_E$, hence for simple f, hence for any nonnegative measurable f. If $f \in L^2(\varphi)$, the Schwarz inequality gives

$$\left|\int_X f\, d\lambda\right| \leq \int_X |f|\, d\lambda \leq \int_X |f|\, d\varphi \leq \left\{\int_X |f|^2\, d\varphi\right\}^{1/2} \{\varphi(X)\}^{1/2}.$$

Since $\varphi(X) < \infty$, we see that

$$f \to \int_X f\, d\lambda \tag{5}$$

is a bounded linear functional on $L^2(\varphi)$. We know that every bounded linear functional on a Hilbert space H is given by an inner product with an element of H. Hence there exists a $g \in L^2(\varphi)$ such that

$$\int_X f\, d\lambda = \int_X fg\, d\varphi \tag{6}$$

for every $f \in L^2(\varphi)$.

Observe how the completeness of $L^2(\varphi)$ was used to guarantee the existence of g. Observe also that although g is defined uniquely as an element of $L^2(\varphi)$, g is determined only a.e. $[\varphi]$ as a point function on X.

Put $f = \chi_E$ in (6), for any $E \in \mathfrak{M}$ with $\varphi(E) > 0$. The left side of (6) is then $\lambda(E)$, and since $0 \leq \lambda \leq \varphi$, we have

$$0 \leq \frac{1}{\varphi(E)} \int_E g\, d\varphi = \frac{\lambda(E)}{\varphi(E)} \leq 1. \tag{7}$$

Hence $g(x) \in [0, 1]$ for almost all x (with respect to φ), by Theorem 1.40. We may therefore assume that $0 \leq g(x) \leq 1$ for every $x \in X$, without affecting (6), and we rewrite (6) in the form

$$\int_X (1 - g) f \, d\lambda = \int_X fgw \, d\mu. \tag{8}$$

Put

$$A = \{x: 0 \leq g(x) < 1\}, \qquad B = \{x: g(x) = 1\}, \tag{9}$$

and define measures λ_a and λ_s by

$$\lambda_a(E) = \lambda(A \cap E), \qquad \lambda_s(E) = \lambda(B \cap E), \tag{10}$$

for all $E \in \mathfrak{M}$.

If $f = \chi_B$ in (8), the left side is 0, the right side is $\int_B w \, d\mu$. Since $w(x) > 0$ for all x, we conclude that $\mu(B) = 0$. Thus $\lambda_s \perp \mu$.

Since g is bounded, (8) holds if f is replaced by

$$(1 + g + \cdots + g^n)\chi_E$$

for $n = 1, 2, 3, \ldots$, $E \in \mathfrak{M}$. For such f, (8) becomes

$$\int_E (1 - g^{n+1}) \, d\lambda = \int_E g(1 + g + \cdots + g^n)w \, d\mu. \tag{11}$$

At every point of B, $g(x) = 1$, hence $1 - g^{n+1}(x) = 0$. At every point of A, $g^{n+1}(x) \to 0$ monotonically. The left side of (11) converges therefore to $\lambda(A \cap E) = \lambda_a(E)$ as $n \to \infty$.

The integrands on the right side of (11) increase monotonically to a non-negative measurable limit h, and the monotone convergence theorem shows that the right side of (11) tends to $\int_E h \, d\mu$ as $n \to \infty$.

We have thus proved that (2) holds for every $E \in \mathfrak{M}$. Taking $E = X$, we see that $h \in L^1(\mu)$, since $\lambda_a(X) < \infty$.

Finally, (2) shows that $\lambda_a \ll \mu$, and the proof is complete for positive λ.

If λ is a complex measure on $\mathfrak{M}$, then $\lambda = \lambda_1 + i\lambda_2$, with λ_1 and λ_2 real, and we can apply the preceding case to the positive and negative variations of λ_1 and λ_2. ////

If both μ and λ are positive and σ-finite, most of Theorem 6.10 is still true. We can now write $X = \bigcup X_n$, where $\mu(X_n) < \infty$ and $\lambda(X_n) < \infty$, for $n = 1, 2, 3, \ldots$. The Lebesgue decompositions of the measures $\lambda(E \cap X_n)$ still give us a Lebesgue decomposition of λ, and we still get a function h which satisfies Eq. 6.10(2); however, it is no longer true that $h \in L^1(\mu)$, although h is "locally in L^1," i.e., $\int_{X_n} h \, d\mu < \infty$ for each n.

Finally, if we go beyond σ-finiteness, we meet situations where the two theorems under consideration actually fail. For example, let μ be Lebesgue measure on (0, 1), and let λ be the counting measure on the σ-algebra of all Lebesgue

measurable sets in (0, 1). *Then λ has no Lebesgue decomposition relative to μ, and although $\mu \ll \lambda$ and μ is bounded, there is no $h \in L^1(\lambda)$ such that $d\mu = h\, d\lambda$.* We omit the easy proof.

The following theorem may explain why the word "continuity" is used in connection with the relation $\lambda \ll \mu$.

6.11 Theorem *Suppose μ and λ are measures on a σ-algebra $\mathfrak{M}$, μ is positive, and λ is complex. Then the following two conditions are equivalent:*

(*a*) $\lambda \ll \mu$.

(*b*) *To every $\epsilon > 0$ corresponds a $\delta > 0$ such that $|\lambda(E)| < \epsilon$ for all $E \in \mathfrak{M}$ with $\mu(E) < \delta$.*

Property (*b*) is sometimes used as the definition of absolute continuity. However, (*a*) does not imply (*b*) if λ is a positive unbounded measure. For instance, let μ be Lebesgue measure on (0, 1), and put

$$\lambda(E) = \int_E t^{-1}\, dt$$

for every Lebesgue measurable set $E \subset (0, 1)$.

PROOF Suppose (*b*) holds. If $\mu(E) = 0$, then $\mu(E) < \delta$ for every $\delta > 0$, hence $|\lambda(E)| < \epsilon$ for every $\epsilon > 0$, so $\lambda(E) = 0$. Thus (*b*) implies (*a*).

Suppose (*b*) is false. Then there exists an $\epsilon > 0$ and there exist sets $E_n \in \mathfrak{M}$ ($n = 1, 2, 3, \ldots$) such that $\mu(E_n) < 2^{-n}$ but $|\lambda(E_n)| \geq \epsilon$. Hence $|\lambda|(E_n) \geq \epsilon$. Put

$$A_n = \bigcup_{i=n}^{\infty} E_i, \qquad A = \bigcap_{n=1}^{\infty} A_n. \tag{1}$$

Then $\mu(A_n) < 2^{-n+1}$, $A_n \supset A_{n+1}$, and so Theorem 1.19(*e*) shows that $\mu(A) = 0$ and that

$$|\lambda|(A) = \lim_{n \to \infty} |\lambda|(A_n) \geq \epsilon > 0,$$

since $|\lambda|(A_n) \geq |\lambda|(E_n)$.

It follows that we do *not* have $|\lambda| \ll \mu$, hence (*a*) is false, by Proposition 6.8(*e*). ////

Consequences of the Radon-Nikodym Theorem

6.12 Theorem *Let μ be a complex measure on a σ-algebra $\mathfrak{M}$ in X. Then there is a measurable function h such that $|h(x)| = 1$ for all $x \in X$ and such that*

$$d\mu = h\, d|\mu|. \tag{1}$$

By analogy with the representation of a complex number as the product of its absolute value and a number of absolute value 1, Eq. (1) is sometimes referred to as the *polar representation* (or *polar decomposition*) of μ.

PROOF It is trivial that $\mu \ll |\mu|$, and therefore the Radon-Nikodym theorem guarantees the existence of some $h \in L^1(|\mu|)$ which satisfies (1).

Let $A_r = \{x: |h(x)| < r\}$, where r is some positive number, and let $\{E_j\}$ be a partition of A_r. Then

$$\sum_j |\mu(E_j)| = \sum_j \left| \int_{E_j} h \, d|\mu| \right| \leq \sum_j r|\mu|(E_j) = r|\mu|(A_r),$$

so that $|\mu|(A_r) \leq r|\mu|(A_r)$. If $r < 1$, this forces $|\mu|(A_r) = 0$. Thus $|h| \geq 1$ a.e.

On the other hand, if $|\mu|(E) > 0$, (1) shows that

$$\left| \frac{1}{|\mu|(E)} \int_E h \, d|\mu| \right| = \frac{|\mu(E)|}{|\mu|(E)} \leq 1.$$

We now apply Theorem 1.40 (with the closed unit disc in place of S) and conclude that $|h| \leq 1$ a.e.

Let $B = \{x \in X: |h(x)| \neq 1\}$. We have shown that $|\mu|(B) = 0$, and if we redefine h on B so that $h(x) = 1$ on B, we obtain a function with the desired properties. ////

6.13 Theorem *Suppose μ is a positive measure on $\mathfrak{M}$, $g \in L^1(\mu)$, and*

$$\lambda(E) = \int_E g \, d\mu \qquad (E \in \mathfrak{M}). \tag{1}$$

Then

$$|\lambda|(E) = \int_E |g| \, d\mu \qquad (E \in \mathfrak{M}). \tag{2}$$

PROOF By Theorem 6.12, there is a function h, of absolute value 1, such that $d\lambda = h \, d|\lambda|$. By hypothesis, $d\lambda = g \, d\mu$. Hence

$$h \, d|\lambda| = g \, d\mu.$$

This gives $d|\lambda| = \bar{h}g \, d\mu$. (Compare with Theorem 1.29.)

Since $|\lambda| \geq 0$ and $\mu \geq 0$, it follows that $\bar{h}g \geq 0$ a.e. $[\mu]$, so that $\bar{h}g = |g|$ a.e. $[\mu]$. ////

6.14 The Hahn Decomposition Theorem *Let μ be a real measure on a σ-algebra $\mathfrak{M}$ in a set X. Then there exist sets A and $B \in \mathfrak{M}$ such that*

$A \cup B = X$, $A \cap B = \varnothing$, and such that the positive and negative variations μ^+ and μ^- of μ satisfy

$$\mu^+(E) = \mu(A \cap E), \quad \mu^-(E) = -\mu(B \cap E) \qquad (E \in \mathfrak{M}). \tag{1}$$

In other words, X is the union of two disjoint measurable sets A and B, such that "A carries all the positive mass of μ" [since (1) implies that $\mu(E) \geq 0$ if $E \subset A$] and "B carries all the negative mass of μ" [since $\mu(E) \leq 0$ if $E \subset B$]. The pair (A, B) is called a Hahn decomposition of X, induced by μ.

PROOF By Theorem 6.12, $d\mu = h\, d|\mu|$, where $|h| = 1$. Since μ is real, it follows that h is real (a.e., and therefore everywhere, by redefining on a set of measure 0), hence $h = \pm 1$. Put

$$A = \{x: h(x) = 1\}, \qquad B = \{x: h(x) = -1\}. \tag{2}$$

Since $\mu^+ = \frac{1}{2}(|\mu| + \mu)$, and since

$$\tfrac{1}{2}(1 + h) = \begin{cases} h & \text{on } A, \\ 0 & \text{on } B, \end{cases} \tag{3}$$

we have, for any $E \in \mathfrak{M}$,

$$\mu^+(E) = \frac{1}{2}\int_E (1 + h)\, d|\mu| = \int_{E \cap A} h\, d|\mu| = \mu(E \cap A). \tag{4}$$

Since $\mu(E) = \mu(E \cap A) + \mu(E \cap B)$ and since $\mu = \mu^+ - \mu^-$, the second half of (1) follows from the first. ////

Corollary *If $\mu = \lambda_1 - \lambda_2$, where λ_1 and λ_2 are positive measures, then $\lambda_1 \geq \mu^+$ and $\lambda_2 \geq \mu^-$.*

This is the minimum property of the Jordan decomposition which was mentioned in Sec. 6.6.

PROOF Since $\mu \leq \lambda_1$, we have

$$\mu^+(E) = \mu(E \cap A) \leq \lambda_1(E \cap A) \leq \lambda_1(E). \qquad ////$$

Bounded Linear Functionals on L^p

6.15 Let μ be a positive measure, suppose $1 \leq p \leq \infty$, and let q be the exponent conjugate to p. The Hölder inequality (Theorem 3.8) shows that if $g \in L^q(\mu)$ and if Φ_g is defined by

$$\Phi_g(f) = \int_X fg\, d\mu, \tag{1}$$

then Φ_g is a bounded linear functional on $L^p(\mu)$, of norm at most $\|g\|_q$. The question naturally arises whether all bounded linear functionals on $L^p(\mu)$ have this form, and whether the representation is unique.

For $p = \infty$, Exercise 13 shows that the answer is negative: $L^1(m)$ does not furnish all bounded linear functionals on $L^\infty(m)$. For $1 < p < \infty$, the answer is affirmative. It is also affirmative for $p = 1$, provided certain measure-theoretic pathologies are excluded. For σ-finite measure spaces, no difficulties arise, and we shall confine ourselves to this case.

6.16 Theorem *Suppose $1 \leq p < \infty$, μ is a σ-finite positive measure on X, and Φ is a bounded linear functional on $L^p(\mu)$. Then there is a unique $g \in L^q(\mu)$, where q is the exponent conjugate to p, such that*

$$\Phi(f) = \int_X fg \, d\mu \qquad (f \in L^p(\mu)). \tag{1}$$

Moreover, if Φ and g are related as in (1), *we have*

$$\|\Phi\| = \|g\|_q. \tag{2}$$

In other words, $L^q(\mu)$ is isometrically isomorphic to the dual space of $L^p(\mu)$, under the stated conditions.

PROOF The uniqueness of g is clear, for if g and g' satisfy (1), then the integral of $g - g'$ over any measurable set E of finite measure is 0 (as we see by taking χ_E for f), and the σ-finiteness of μ implies therefore that $g - g' = 0$ a.e.

Next, if (1) holds, Hölder's inequality implies

$$\|\Phi\| \leq \|g\|_q. \tag{3}$$

So it remains to prove that g exists and that equality holds in (3). If $\|\Phi\| = 0$, (1) and (2) hold with $g = 0$. So assume $\|\Phi\| > 0$.

We first consider the case $\mu(X) < \infty$.

For any measurable set $E \subset X$, define

$$\lambda(E) = \Phi(\chi_E).$$

Since Φ is linear, and since $\chi_{A \cup B} = \chi_A + \chi_B$ if A and B are disjoint, we see that λ is additive. To prove countable additivity, suppose E is the union of countably many disjoint measurable sets E_i, put $A_k = E_1 \cup \cdots \cup E_k$, and note that

$$\|\chi_E - \chi_{A_k}\|_p = [\mu(E - A_k)]^{1/p} \to 0 \qquad (k \to \infty); \tag{4}$$

the continuity of Φ shows now that $\lambda(A_k) \to \lambda(E)$. So λ is a complex measure. [In (4) the assumption $p < \infty$ was used.] It is clear that $\lambda(E) = 0$ if $\mu(E) = 0$,

since then $\|\chi_E\|_p = 0$. Thus $\lambda \ll \mu$, and the Radon-Nikodym theorem ensures the existence of a function $g \in L^1(\mu)$ such that, for every measurable $E \subset X$,

$$\Phi(\chi_E) = \int_E g \, d\mu = \int_X \chi_E g \, d\mu. \tag{5}$$

By linearity it follows that

$$\Phi(f) = \int_X fg \, d\mu \tag{6}$$

holds for every simple measurable f, and so also for every $f \in L^\infty(\mu)$, since every $f \in L^\infty(\mu)$ is a uniform limit of simple functions f_i. Note that the uniform convergence of f_i to f implies $\|f_i - f\|_p \to 0$, hence $\Phi(f_i) \to \Phi(f)$, as $i \to \infty$.

We want to conclude that $g \in L^q(\mu)$ and that (2) holds; it is best to split the argument into two cases.

CASE 1 $p = 1$. Here (5) shows that

$$\left| \int_E g \, d\mu \right| \leq \|\Phi\| \cdot \|\chi_E\|_1 = \|\Phi\| \cdot \mu(E)$$

for every $E \in \mathfrak{M}$. By Theorem 1.40, $|g(x)| \leq \|\Phi\|$ a.e., so that $\|g\|_\infty \leq \|\Phi\|$.

CASE 2 $1 < p < \infty$. There is a measurable function α, $|\alpha| = 1$, such that $\alpha g = |g|$ [Proposition 1.9(*e*)]. Let $E_n = \{x: |g(x)| \leq n\}$, and define $f = \chi_{E_n} |g|^{q-1}\alpha$. Then $|f|^p = |g|^q$ on E_n, $f \in L^\infty(\mu)$, and (6) gives

$$\int_{E_n} |g|^q \, d\mu = \int_X fg \, d\mu = \Phi(f) \leq \|\Phi\| \left\{ \int_{E_n} |g|^q \right\}^{1/p},$$

so that

$$\int_X \chi_{E_n} |g|^q \, d\mu \leq \|\Phi\|^q \qquad (n = 1, 2, 3, \ldots). \tag{7}$$

If we apply the monotone convergence theorem to (7), we obtain $\|g\|_q \leq \|\Phi\|$.

Thus (2) holds and $g \in L^q(\mu)$. It follows that both sides of (6) are continuous functions on $L^p(\mu)$. They coincide on the dense subset $L^\infty(\mu)$ of $L^p(\mu)$; hence they coincide on all of $L^p(\mu)$, and this completes the proof if $\mu(X) < \infty$.

If $\mu(X) = \infty$ but μ is σ-finite, choose $w \in L^1(\mu)$ as in Lemma 6.9. Then $d\tilde{\mu} = w \, d\mu$ defines a finite measure on $\mathfrak{M}$, and

$$F \to w^{1/p} F \tag{8}$$

is a linear isometry of $L^p(\tilde{\mu})$ onto $L^p(\mu)$, because $w(x) > 0$ for every $x \in X$. Hence

$$\Psi(F) = \Phi(w^{1/p} F) \tag{9}$$

defines a bounded linear functional Ψ on $L^p(\tilde{\mu})$, with $\|\Psi\| = \|\Phi\|$.

The first part of the proof shows now that there exists $G \in L^q(\tilde{\mu})$ such that

$$\Psi(F) = \int_X FG \, d\tilde{\mu} \qquad (F \in L^p(\tilde{\mu})). \tag{10}$$

Put $g = w^{1/q}G$. (If $p = 1$, $g = G$.) Then

$$\int_X |g|^q \, d\mu = \int_X |G|^q \, d\tilde{\mu} = \|\Psi\|^q = \|\Phi\|^q \tag{11}$$

if $p > 1$, whereas $\|g\|_\infty = \|G\|_\infty = \|\Psi\| = \|\Phi\|$ if $p = 1$. Thus (2) holds, and since $G \, d\tilde{\mu} = w^{1/p}g \, d\mu$, we finally get

$$\Phi(f) = \Psi(w^{-1/p}f) = \int_X w^{-1/p} fG \, d\tilde{\mu} = \int_X fg \, d\mu \tag{12}$$

for every $f \in L^p(\mu)$. ////

6.17 Remark We have already encountered the special case $p = q = 2$ of Theorem 6.16. In fact, the proof of the general case was based on this special case, for we used the knowledge of the bounded linear functionals on $L^2(\mu)$ in the proof of the Radon-Nikodym theorem, and the latter was the key to the proof of Theorem 6.16. The special case $p = 2$, in turn, depended on the completeness of $L^2(\mu)$, on the fact that $L^2(\mu)$ is therefore a Hilbert space, and on the fact that the bounded linear functionals on a Hilbert space are given by inner products.

We now turn to the complex version of Theorem 2.14.

The Riesz Representation Theorem

6.18 Let X be a locally compact Hausdorff space. Theorem 2.14 characterizes the *positive* linear functionals on $C_c(X)$. We are now in a position to characterize the *bounded* linear functionals Φ on $C_c(X)$. Since $C_c(X)$ is a dense subspace of $C_0(X)$, relative to the supremum norm, every such Φ has a unique extension to a bounded linear functional on $C_0(X)$. Hence we may as well assume to begin with that we are dealing with the Banach space $C_0(X)$.

If μ is a complex Borel measure, Theorem 6.12 asserts that there is a complex Borel function h with $|h| = 1$ such that $d\mu = h \, d|\mu|$. It is therefore reasonable to define integration with respect to a complex measure μ by the formula

$$\int f \, d\mu = \int fh \, d|\mu|. \tag{1}$$

The relation $\int \chi_E \, d\mu = \mu(E)$ is a special case of (1). Thus

$$\int_X \chi_E \, d(\mu + \lambda) = (\mu + \lambda)(E) = \mu(E) + \lambda(E) = \int_X \chi_E \, d\mu + \int_X \chi_E \, d\lambda \tag{2}$$

whenever μ and λ are complex measures on $\mathfrak{M}$ and $E \in \mathfrak{M}$. This leads to the addition formula

$$\int_X f\, d(\mu + \lambda) = \int_X f\, d\mu + \int_X f\, d\lambda, \tag{3}$$

which is valid (for instance) for every bounded measurable f.

We shall call a complex Borel measure μ on X *regular* if $|\mu|$ is regular in the sense of Definition 2.15. If μ is a complex Borel measure on X, it is clear that the mapping

$$f \to \int_X f\, d\mu \tag{4}$$

is a bounded linear functional on $C_0(X)$, whose norm is no larger than $|\mu|(X)$. That *all* bounded linear functionals on $C_0(X)$ are obtained in this way is the content of the Riesz theorem:

6.19 Theorem *If X is a locally compact Hausdorff space, then every bounded linear functional Φ on $C_0(X)$ is represented by a unique regular complex Borel measure μ, in the sense that*

$$\Phi f = \int_X f\, d\mu \tag{1}$$

for every $f \in C_0(X)$. Moreover, the norm of Φ is the total variation of μ:

$$\|\Phi\| = |\mu|(X). \tag{2}$$

PROOF We first settle the uniqueness question. Suppose μ is a regular complex Borel measure on X and $\int f\, d\mu = 0$ for all $f \in C_0(X)$. By Theorem 6.12 there is a Borel function h, with $|h| = 1$, such that $d\mu = h\, d|\mu|$. For any sequence $\{f_n\}$ in $C_0(X)$ we then have

$$|\mu|(X) = \int_X (\bar{h} - f_n)h\, d|\mu| \leq \int_X |\bar{h} - f_n|\, d|\mu|, \tag{3}$$

and since $C_c(X)$ is dense in $L^1(|\mu|)$ (Theorem 3.14), $\{f_n\}$ can be so chosen that the last expression in (3) tends to 0 as $n \to \infty$. Thus $|\mu|(X) = 0$, and $\mu = 0$. It is easy to see that the difference of two regular complex Borel measures on X is regular. This shows that at most one μ corresponds to each Φ.

Now consider a given bounded linear functional Φ on $C_0(X)$. Assume $\|\Phi\| = 1$, without loss of generality. We shall construct a *positive* linear functional Λ on $C_c(X)$, such that

$$|\Phi(f)| \leq \Lambda(|f|) \leq \|f\| \qquad (f \in C_c(X)), \tag{4}$$

where $\|f\|$ denotes the supremum norm.

Once we have this Λ, we associate with it a positive Borel measure λ, as in Theorem 2.14. The conclusion of Theorem 2.14 shows that λ is regular if $\lambda(X) < \infty$. Since

$$\lambda(X) = \sup \{\Lambda f : 0 \leq f \leq 1, f \in C_c(X)\}$$

and since $|\Lambda f| \leq 1$ if $\|f\| \leq 1$, we see that actually $\lambda(X) \leq 1$.

We also deduce from (4) that

$$|\Phi(f)| \leq \Lambda(|f|) = \int_X |f| \, d\lambda = \|f\|_1 \qquad (f \in C_c(X)). \tag{5}$$

The last norm refers to the space $L^1(\lambda)$. Thus Φ is a linear functional on $C_c(X)$ of norm at most 1, *with respect to the $L^1(\lambda)$-norm on $C_c(X)$*. There is a norm-preserving extension of Φ to a linear functional on $L^1(\lambda)$, and therefore Theorem 6.16 (the case $p = 1$) gives a Borel function g, with $|g| \leq 1$, such that

$$\Phi(f) = \int_X fg \, d\lambda \qquad (f \in C_c(X)). \tag{6}$$

Each side of (6) is a continuous functional on $C_0(X)$, and $C_c(X)$ is dense in $C_0(X)$. Hence (6) holds for all $f \in C_0(X)$, and we obtain the representation (1) with $d\mu = g \, d\lambda$.

Since $\|\Phi\| = 1$, (6) shows that

$$\int_X |g| \, d\lambda \geq \sup \{|\Phi(f)| : f \in C_0(X), \|f\| \leq 1\} = 1. \tag{7}$$

We also know that $\lambda(X) \leq 1$ and $|g| \leq 1$. These facts are compatible only if $\lambda(X) = 1$ and $|g| = 1$ a.e. $[\lambda]$. Thus $d|\mu| = |g| \, d\lambda = d\lambda$, by Theorem 6.13, and

$$|\mu|(X) = \lambda(X) = 1 = \|\Phi\|, \tag{8}$$

which proves (2).

So all depends on finding a positive linear functional Λ that satisfies (4). If $f \in C_c^+(X)$ [the class of all nonnegative real members of $C_c(X)$], define

$$\Lambda f = \sup \{|\Phi(h)| : h \in C_c(X), |h| \leq f\}. \tag{9}$$

Then $\Lambda f \geq 0$, Λ satisfies (4), $0 \leq f_1 \leq f_2$ implies $\Lambda f_1 \leq \Lambda f_2$, and $\Lambda(cf) = c\Lambda f$ if c is a positive constant. We have to show that

$$\Lambda(f + g) = \Lambda f + \Lambda g \qquad (f \text{ and } g \in C_c^+(X)), \tag{10}$$

and we then have to extend Λ to a linear functional on $C_c(X)$.

Fix f and $g \in C_c^+(X)$. If $\epsilon > 0$, there exist h_1 and $h_2 \in C_c(X)$ such that $|h_1| \leq f$, $|h_2| \leq g$, and

$$\Lambda f \leq |\Phi(h_1)| + \epsilon, \qquad \Lambda g \leq |\Phi(h_2)| + \epsilon. \tag{11}$$

There are complex numbers α_i, $|\alpha_i| = 1$, so that $\alpha_i \Phi(h_i) = |\Phi(h_i)|$, $i = 1, 2$. Then

$$\begin{aligned} \Lambda f + \Lambda g &\leq |\Phi(h_1)| + |\Phi(h_2)| + 2\epsilon \\ &= \Phi(\alpha_1 h_1 + \alpha_2 h_2) + 2\epsilon \\ &\leq \Lambda(|h_1| + |h_2|) + 2\epsilon \\ &\leq \Lambda(f + g) + 2\epsilon, \end{aligned}$$

so that the inequality $\geq$ holds in (10).

Next, choose $h \in C_c(X)$, subject only to the condition $|h| \leq f + g$, let $V = \{x : f(x) + g(x) > 0\}$, and define

$$\begin{aligned} h_1(x) &= \frac{f(x)h(x)}{f(x) + g(x)}, \qquad h_2(x) = \frac{g(x)h(x)}{f(x) + g(x)} \qquad (x \in V), \\ h_1(x) &= h_2(x) = 0 \qquad (x \notin V). \end{aligned} \tag{12}$$

It is clear that h_1 is continuous at every point of V. If $x_0 \notin V$, then $h(x_0) = 0$; since h is continuous and since $|h_1(x)| \leq |h(x)|$ for all $x \in X$, it follows that x_0 is a point of continuity of h_1. Thus $h_1 \in C_c(X)$, and the same holds for h_2.

Since $h_1 + h_2 = h$ and $|h_1| \leq f$, $|h_2| \leq g$, we have

$$|\Phi(h)| = |\Phi(h_1) + \Phi(h_2)| \leq |\Phi(h_1)| + |\Phi(h_2)| \leq \Lambda f + \Lambda g.$$

Hence $\Lambda(f + g) \leq \Lambda f + \Lambda g$, and we have proved (10).

If f is now a real function, $f \in C_c(X)$, then $2f^+ = |f| + f$, so that $f^+ \in C_c^+(X)$; likewise, $f^- \in C_c^+(X)$; and since $f = f^+ - f^-$, it is natural to define

$$\Lambda f = \Lambda f^+ - \Lambda f^- \qquad (f \in C_c(X), f \text{ real}) \tag{13}$$

and

$$\Lambda(u + iv) = \Lambda u + i\Lambda v. \tag{14}$$

Simple algebraic manipulations, just like those which occur in the proof of Theorem 1.32, show now that our extended functional Λ is linear on $C_c(X)$.

This completes the proof. ////

Exercises

1 If μ is a complex measure on a σ-algebra $\mathfrak{M}$, and if $E \in \mathfrak{M}$, define

$$\lambda(E) = \sup \sum |\mu(E_i)|,$$

the supremum being taken over all *finite* partitions $\{E_i\}$ of E. Does it follow that $\lambda = |\mu|$?

2 Prove that the example given at the end of Sec. 6.10 has the stated properties.

3 Prove that the vector space $M(X)$ of all complex regular Borel measures on a locally compact Hausdorff space X is a Banach space if $\|\mu\| = |\mu|(X)$. *Hint*: Compare Exercise 8, Chap. 5. [That the difference of any two members of $M(X)$ is in $M(X)$ was used in the first paragraph of the proof of Theorem 6.19; supply a proof of this fact.]

4 Suppose $1 \leq p \leq \infty$, and q is the exponent conjugate to p. Suppose μ is a positive σ-finite measure and g is a measurable function such that $fg \in L^1(\mu)$ for every $f \in L^p(\mu)$. Prove that then $g \in L^q(\mu)$.

5 Suppose X consists of two points a and b; define $\mu(\{a\}) = 1$, $\mu(\{b\}) = \mu(X) = \infty$, and $\mu(\varnothing) = 0$. Is it true, for this μ, that $L^\infty(\mu)$ is the dual space of $L^1(\mu)$?

6 Suppose $1 < p < \infty$ and prove that $L^q(\mu)$ is the dual space of $L^p(\mu)$ even if μ is not σ-finite. (As usual, $1/p + 1/q = 1$.)

7 Suppose μ is a complex Borel measure on $[0, 2\pi)$ (or on the unit circle T), and define the Fourier coefficients of μ by

$$\hat{\mu}(n) = \int e^{-int}\, d\mu(t) \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Assume that $\hat{\mu}(n) \to 0$ as $n \to +\infty$ and prove that then $\hat{\mu}(n) \to 0$ as $n \to -\infty$.

Hint: The assumption also holds with $f\, d\mu$ in place of $d\mu$ if f is any trigonometric polynomial, hence if f is continuous, hence if f is any bounded Borel function, hence if $d\mu$ is replaced by $d|\mu|$.

8 In the terminology of Exercise 7, find all μ such that $\hat{\mu}$ is periodic, with period k. [This means that $\hat{\mu}(n + k) = \hat{\mu}(n)$ for all integers n; of course, k is also assumed to be an integer.]

9 Suppose that $\{g_n\}$ is a sequence of positive continuous functions on $I = [0, 1]$, that μ is a positive Borel measure on I, and that

(i) $\lim_{n\to\infty} g_n(x) = 0$ a.e. $[m]$,

(ii) $\int_I g_n\, dm = 1$ for all n,

(iii) $\lim_{n\to\infty} \int_I f g_n\, dm = \int_I f\, d\mu$ for every $f \in C(I)$.

Does it follow that $\mu \perp m$?

10 Let $(X, \mathfrak{M}, \mu)$ be a positive measure space. Call a set $\Phi \subset L^1(\mu)$ *uniformly integrable* if to each $\epsilon > 0$ corresponds a $\delta > 0$ such that

$$\left| \int_E f\, d\mu \right| < \epsilon$$

whenever $f \in \Phi$ and $\mu(E) < \delta$.

(*a*) Prove that every finite subset of $L^1(\mu)$ is uniformly integrable.

(*b*) Prove the following convergence theorem of Vitali:

If (i) $\mu(X) < \infty$, (ii) $\{f_n\}$ *is uniformly integrable*, (iii) $f_n(x) \to f(x)$ *a.e. as* $n \to \infty$, *and* (iv) $|f(x)| < \infty$ *a.e., then* $f \in L^1(\mu)$ *and*

$$\lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0.$$

Suggestion: Use Egoroff's theorem.

(*c*) Show that (*b*) fails if μ is Lebesgue measure on $(-\infty, \infty)$, even if $\{\|f_n\|_1\}$ is assumed to be bounded. Hypothesis (i) can therefore not be omitted in (*b*).

(*d*) Show that hypothesis (iv) is redundant in (*b*) for some μ (for instance, for Lebesgue measure on a bounded interval), but that there are finite measures for which the omission of (iv) would make (*b*) false.

(*e*) Show that Vitali's theorem implies Lebesgue's dominated convergence theorem, for finite measure spaces. Construct an example in which Vitali's theorem applies although the hypotheses of Lebesgue's theorem do not hold.

(*f*) Construct a sequence $\{f_n\}$, say on $[0, 1]$, so that $f_n(x) \to 0$ for every x, $\int f_n \to 0$, but $\{f_n\}$ is *not* uniformly integrable (with respect to Lebesgue measure).

(*g*) However, the following converse of Vitali's theorem is true:

If $\mu(X) < \infty$, $f_n \in L^1(\mu)$, *and*

$$\lim_{n\to\infty} \int_E f_n\, d\mu$$

exists for every $E \in \mathfrak{M}$, *then* $\{f_n\}$ *is uniformly integrable.*

Prove this by completing the following outline.

Define $\rho(A, B) = \int |\chi_A - \chi_B|\, d\mu$. Then $(\mathfrak{M}, \rho)$ is a complete metric space (modulo sets of measure 0), and $E \to \int_E f_n\, d\mu$ is continuous for each n. If $\epsilon > 0$, there exist E_0, δ, N (Exercise 13, Chap. 5) so that

$$\left| \int_E (f_n - f_N)\, d\mu \right| < \epsilon \quad \text{if} \quad \rho(E, E_0) < \delta, \qquad n > N. \tag{*}$$

If $\mu(A) < \delta$, (*) holds with $B = E_0 - A$ and $C = E_0 \cup A$ in place of E. Thus (*) holds with A in place of E and 2ϵ in place of ϵ. Now apply (*a*) to $\{f_1, \ldots, f_N\}$: There exists $\delta' > 0$ such that

$$\left| \int_A f_n\, d\mu \right| < 3\epsilon \quad \text{if} \quad \mu(A) < \delta', \qquad n = 1, 2, 3, \ldots.$$

11 Suppose μ is a positive measure on X, $\mu(X) < \infty$, $f_n \in L^1(\mu)$ for $n = 1, 2, 3, \ldots$, $f_n(x) \to f(x)$ a.e., and there exists $p > 1$ and $C < \infty$ such that $\int_X |f_n|^p\, d\mu < C$ for all n. Prove that

$$\lim_{n \to \infty} \int_X |f - f_n|\, d\mu = 0.$$

Hint: $\{f_n\}$ is uniformly integrable.

12 Let $\mathfrak{M}$ be the collection of all sets E in the unit interval $[0, 1]$ such that either E or its complement is at most countable. Let μ be the counting measure on this σ-algebra $\mathfrak{M}$. If $g(x) = x$ for $0 \le x \le 1$, show that g is not $\mathfrak{M}$-measurable, although the mapping

$$f \to \sum xf(x) = \int fg\, d\mu$$

makes sense for every $f \in L^1(\mu)$ and defines a bounded linear functional on $L^1(\mu)$. Thus $(L^1)^* \neq L^\infty$ in this situation.

13 Let $L^\infty = L^\infty(m)$, where m is Lebesgue measure on $I = [0, 1]$. Show that there is a bounded linear functional $\Lambda \neq 0$ on L^∞ that is 0 on $C(I)$, and that therefore there is no $g \in L^1(m)$ that satisfies $\Lambda f = \int_I fg\, dm$ for every $f \in L^\infty$. Thus $(L^\infty)^* \neq L^1$.

CHAPTER

SEVEN

DIFFERENTIATION

In elementary Calculus we learn that integration and differentiation are inverses of each other. This fundamental relation persists to a large extent in the context of the Lebesgue integral. We shall see that some of the most important facts about differentiation of integrals and integration of derivatives can be obtained with a minimum of effort by first studying derivatives of measures and the associated maximal functions. The Radon-Nikodym theorem and the Lebesgue decomposition will play a prominent role.

Derivatives of Measures

We begin with a simple theorem whose main purpose is to motivate the definitions that follow.

7.1 Theorem *Suppose μ is a complex Borel measure on R^1 and*

$$f(x) = \mu((-\infty, x)) \qquad (x \in R^1). \tag{1}$$

If $x \in R^1$ and A is a complex number, each of the following two statements implies the other:

(a) f is differentiable at x and $f'(x) = A$.

(b) To every $\epsilon > 0$ corresponds a $\delta > 0$ such that

$$\left| \frac{\mu(I)}{m(I)} - A \right| < \epsilon \tag{2}$$

for every open segment I that contains x and whose length is less than δ. Here m denotes Lebesgue measure on R^1.

7.2 Definitions Theorem 7.1 suggests that one might define the derivative of μ at x to be the limit of the quotients $\mu(I)/m(I)$, as the segments I shrink to x, and that an analogous definition might be appropriate in several variables, i.e., in R^k rather than in R^1.

Accordingly, let us fix a dimension k, denote the open ball with center $x \in R^k$ and radius $r > 0$ by

$$B(x, r) = \{y \in R^k : |y - x| < r\} \tag{1}$$

(the absolute value indicates the euclidean metric, as in Sec. 2.19), associate to any complex Borel measure μ on R^k the quotients

$$(Q_r \mu)(x) = \frac{\mu(B(x, r))}{m(B(x, r))}, \tag{2}$$

where $m = m_k$ is Lebesgue measure on R^k, and define the *symmetric derivative* of μ at x to be

$$(D\mu)(x) = \lim_{r \to 0} (Q_r \mu)(x) \tag{3}$$

at those points $x \in R^k$ at which this limit exists.

We shall study $D\mu$ by means of the *maximal function* $M\mu$. For $\mu \geq 0$, this is defined by

$$(M\mu)(x) = \sup_{0 < r < \infty} (Q_r \mu)(x), \tag{4}$$

and the maximal function of a complex Borel measure μ is, by definition, that of its total variation $|\mu|$.

The functions $M\mu : R^k \to [0, \infty]$ are *lower semicontinuous*, hence measurable.

To see this, assume $\mu \geq 0$, pick $\lambda > 0$, let $E = \{M\mu > \lambda\}$, and fix $x \in E$. Then there is an $r > 0$ such that

$$\mu(B(x, r)) = tm(B(x, r)) \tag{5}$$

for some $t > \lambda$, and there is a $\delta > 0$ that satisfies

$$(r + \delta)^k < r^k t/\lambda. \tag{6}$$

If $|y - x| < \delta$, then $B(y, r + \delta) \supset B(x, r)$, and therefore

$$\mu(B(y, r + \delta)) \geq tm(B(x, r)) = t[r/(r + \delta)]^k m(B(y, r + \delta)) > \lambda m(B(y, r + \delta)).$$

Thus $B(x, \delta) \subset E$. This proves that E is open.

Our first objective is the "maximal theorem" 7.4. The following covering lemma will be used in its proof.

7.3 Lemma *If W is the union of a finite collection of balls $B(x_i, r_i)$, $1 \le i \le N$, then there is a set $S \subset \{1, \dots, N\}$ so that*

(a) the balls $B(x_i, r_i)$ with $i \in S$ are disjoint,
(b) $W \subset \bigcup_{i \in S} B(x_i, 3r_i)$, and
(c) $m(W) \le 3^k \sum_{i \in S} m(B(x_i, r_i))$.

PROOF Order the balls $B_i = B(x_i, r_i)$ so that $r_1 \ge r_2 \ge \cdots \ge r_N$. Put $i_1 = 1$. Discard all B_j that intersect B_{i_1}. Let B_{i_2} be the first of the remaining B_j, if there are any. Discard all B_j with $j > i_2$ that intersect B_{i_2}, let B_{i_3} be the first of the remaining ones, and so on, as long as possible. This process stops after a finite number of steps and gives $S = \{i_1, i_2, \dots\}$.

It is clear that (*a*) holds. Every discarded B_j is a subset of $B(x_i, 3r_i)$ for some $i \in S$, for if $r' \le r$ and $B(x', r')$ intersects $B(x, r)$, then $B(x', r') \subset B(x, 3r)$. This proves (*b*), and (*c*) follows from (*b*) because

$$m(B(x, 3r)) = 3^k m(B(x, r))$$

in R^k. ////

The following theorem says, roughly speaking, that the maximal function of a measure cannot be large on a large set.

7.4 Theorem *If μ is a complex Borel measure on R^k and λ is a positive number, then*

$$m\{M\mu > \lambda\} \le 3^k \lambda^{-1} \|\mu\|. \tag{1}$$

Here $\|\mu\| = |\mu|(R^k)$, and the left side of (1) is an abbreviation for the more cumbersome expression

$$m(\{x \in R^k : (M\mu)(x) > \lambda\}). \tag{2}$$

We shall often simplify notation in this way.

PROOF Fix μ and λ. Let K be a compact subset of the open set $\{M\mu > \lambda\}$. Each $x \in K$ is the center of an open ball B for which

$$|\mu|(B) > \lambda m(B).$$

Some finite collection of these B's covers K, and Lemma 7.3 gives us a disjoint subcollection, say $\{B_1, \dots, B_n\}$, that satisfies

$$m(K) \le 3^k \sum_1^n m(B_i) \le 3^k \lambda^{-1} \sum_1^n |\mu|(B_i) \le 3^k \lambda^{-1} \|\mu\|.$$

The disjointness of $\{B_1, \dots, B_n\}$ was used in the last inequality.

Now (1) follows by taking the supremum over all compact $K \subset \{M\mu > \lambda\}$. ////

7.5 Weak L^1 If $f \in L^1(R^k)$ and $\lambda > 0$, then

$$m\{|f| > \lambda\} \leq \lambda^{-1}\|f\|_1 \tag{1}$$

because, putting $E = \{|f| > \lambda\}$, we have

$$\lambda m(E) \leq \int_E |f|\, dm \leq \int_{R^k} |f|\, dm = \|f\|_1. \tag{2}$$

Accordingly, any measurable function f for which

$$\lambda \cdot m\{|f| > \lambda\} \tag{3}$$

is a bounded function of λ on $(0, \infty)$ is said to belong to *weak* L^1.

Thus weak L^1 contains L^1. That it is actually larger is shown most simply by the function $1/x$ on $(0, 1)$.

We associate to each $f \in L^1(R^k)$ its *maximal function* Mf: $R^k \to [0, \infty]$, by setting

$$(Mf)(x) = \sup_{0<r<\infty} \frac{1}{m(B_r)} \int_{B(x,r)} |f|\, dm. \tag{4}$$

[We wrote B_r in place of $B(x, r)$ because $m(B(x, r))$ depends only on the radius r.] If we identify f with the measure μ given by $d\mu = f\, dm$, we see that (4) agrees with the previously defined $M\mu$. Theorem 7.4 states therefore that the "maximal operator" M sends L^1 to weak L^1, with a bound (namely 3^k) that depends only on the space R^k:

For every $f \in L^1(R^k)$ and every $\lambda > 0$,

$$m\{Mf > \lambda\} \leq 3^k \lambda^{-1}\|f\|_1. \tag{5}$$

7.6 Lebesgue points If $f \in L^1(R^k)$, any $x \in R^k$ for which it is true that

$$\lim_{r\to 0} \frac{1}{m(B_r)} \int_{B(x,r)} |f(y) - f(x)|\, dm(y) = 0 \tag{1}$$

is called a *Lebesgue point* of f.

For example, (1) holds if f is continuous at the point x. In general, (1) means that the averages of $|f - f(x)|$ are small on small balls centered at x. The Lebesgue points of f are thus points where f does not oscillate too much, in an average sense.

It is probably far from obvious that every $f \in L^1$ has Lebesgue points. But the following remarkable theorem shows that they always exist. (See also Exercise 23.)

7.7 Theorem *If $f \in L^1(R^k)$, then almost every $x \in R^k$ is a Lebesgue point of f.*

PROOF Define

$$(T_r f)(x) = \frac{1}{m(B_r)} \int_{B(x,\,r)} |f - f(x)|\, dm \tag{1}$$

for $x \in R^k$, $r > 0$, and put

$$(Tf)(x) = \limsup_{r \to 0} (T_r f)(x). \tag{2}$$

We have to prove that $Tf = 0$ a.e. $[m]$.

Pick $y > 0$. Let n be a positive integer. By Theorem 3.14, there exists $g \in C(R^k)$ so that $\|f - g\|_1 < 1/n$. Put $h = f - g$.

Since g is continuous, $Tg = 0$. Since

$$(T_r h)(x) \leq \frac{1}{m(B_r)} \int_{B(x,\,r)} |h|\, dm + |h(x)| \tag{3}$$

we have

$$Th \leq Mh + |h|. \tag{4}$$

Since $T_r f \leq T_r g + T_r h$, it follows that

$$Tf \leq Mh + |h|. \tag{5}$$

Therefore

$$\{Tf > 2y\} \subset \{Mh > y\} \cup \{|h| > y\}. \tag{6}$$

Denote the union on the right of (6) by $E(y, n)$. Since $\|h\|_1 < 1/n$, Theorem 7.4 and the inequality 7.5(1) show that

$$m(E(y, n)) \leq (3^k + 1)/(yn). \tag{7}$$

The left side of (6) is independent of n. Hence

$$\{Tf > 2y\} \subset \bigcap_{n=1}^{\infty} E(y, n). \tag{8}$$

This intersection has measure 0, by (7), so that $\{Tf > 2y\}$ is a subset of a set of measure 0. Since Lebesgue measure is complete, $\{Tf > 2y\}$ is Lebesgue measurable, and has measure 0. This holds for every positive y. Hence $Tf = 0$ a.e. $[m]$. ////

Theorem 7.7 yields interesting information, with very little effort, about topics such as

(*a*) differentiation of absolutely continuous measures,
(*b*) differentiation using sets other than balls,
(*c*) differentiation of indefinite integrals in R^1,
(*d*) metric density of measurable sets.

We shall now discuss these topics.

7.8 Theorem *Suppose μ is a complex Borel measure on R^k, and $\mu \ll m$. Let f be the Radon-Nikodym derivative of μ with respect to m. Then $D\mu = f$ a.e. $[m]$, and*

$$\mu(E) = \int_E (D\mu)\, dm \tag{1}$$

for all Borel sets $E \subset R^k$.

In other words, the Radon-Nikodym derivative can also be obtained as a limit of the quotients $Q_r \mu$.

PROOF The Radon-Nikodym theorem asserts that (1) holds with f in place of $D\mu$. At any Lebesgue point x of f, it follows that

$$f(x) = \lim_{r \to 0} \frac{1}{m(B_r)} \int_{B(x,r)} f\, dm = \lim_{r \to 0} \frac{\mu(B(x, r))}{m(B(x, r))}. \tag{2}$$

Thus $(D\mu)(x)$ exists and equals $f(x)$ at every Lebesgue point of f, hence a.e. $[m]$. ////

7.9 Nicely shrinking sets Suppose $x \in R^k$. A sequence $\{E_i\}$ of Borel sets in R^k is said to *shrink to x nicely* if there is a number $\alpha > 0$ with the following property: There is a sequence of balls $B(x, r_i)$, with $\lim r_i = 0$, such that $E_i \subset B(x, r_i)$ and

$$m(E_i) \geq \alpha \cdot m(B(x, r_i)) \tag{1}$$

for $i = 1, 2, 3, \ldots$.

Note that it is not required that $x \in E_i$, nor even that x be in the closure of E_i. Condition (1) is a quantitative version of the requirement that each E_i must occupy a substantial portion of some spherical neighborhood of x. For example, a nested sequence of k-cells whose longest edge is at most 1,000 times as long as its shortest edge and whose diameter tends to 0 shrinks nicely. A nested sequence of rectangles (in R^2) whose edges have lengths $1/i$ and $(1/i)^2$ does not shrink nicely.

7.10 Theorem *Associate to each $x \in R^k$ a sequence $\{E_i(x)\}$ that shrinks to x nicely, and let $f \in L^1(R^k)$. Then*

$$f(x) = \lim_{i \to \infty} \frac{1}{m(E_i(x))} \int_{E_i(x)} f\, dm \tag{1}$$

at every Lebesgue point of f, hence a.e. $[m]$.

PROOF Let x be a Lebesgue point of f and let $\alpha(x)$ and $B(x, r_i)$ be the positive number and the balls that are associated to the sequence $\{E_i(x)\}$. Then, because $E_i(x) \subset B(x, r_i)$,

$$\frac{\alpha(x)}{m(E_i(x))} \int_{E_i(x)} |f - f(x)|\, dm \leq \frac{1}{m(B(x, r_i))} \int_{B(x, r_i)} |f - f(x)|\, dm.$$

The right side converges to 0 as $i \to \infty$, because $r_i \to 0$ and x is a Lebesgue point of f. Hence the left side converges to 0, and (1) follows. ////

Note that no relation of any sort was assumed to exist between $\{E_i(x)\}$ and $\{E_i(y)\}$, for different points x and y.

Note also that Theorem 7.10 leads to a correspondingly stronger form of Theorem 7.8. We omit the details.

7.11 Theorem *If $f \in L^1(R^1)$ and*

$$F(x) = \int_{-\infty}^{x} f\, dm \qquad (-\infty < x < \infty),$$

then $F'(x) = f(x)$ at every Lebesgue point of f, hence a.e. $[m]$.

(This is the easy half of the fundamental theorem of Calculus, extended to Lebesgue integrals.)

PROOF Let $\{\delta_i\}$ be a sequence of positive numbers that converges to 0. Theorem 7.10, with $E_i(x) = [x, x + \delta_i]$, shows then that the right-hand derivative of F exists at all Lebesgue points of x of f and that it is equal to $f(x)$ at these points. If we let $E_i(x)$ be $[x - \delta_i, x]$ instead, we obtain the same result for the left-hand derivative of F at x. ////

7.12 Metric density Let E be a Lebesgue measurable subset of R^k. The metric density of E at a point $x \in R^k$ is defined to be

$$\lim_{r \to 0} \frac{m(E \cap B(x, r))}{m(B(x, r))} \tag{1}$$

provided, of course, that this limit exists.

If we let f be the characteristic function of E and apply Theorem 7.8 or Theorem 7.10, we see that *the metric density of E is 1 at almost every point of E, and that it is 0 at almost every point of the complement of E.*

Here is a rather striking consequence of this, which should be compared with Exercise 8 in Chap. 2:

If $\epsilon > 0$, there is no set $E \subset R^1$ that satisfies

$$\epsilon < \frac{m(E \cap I)}{m(I)} < 1 - \epsilon \tag{2}$$

for every segment I.

Having dealt with differentiation of absolutely continuous measures, we now turn to those that are singular with respect to m.

7.13 Theorem *Associate to each $x \in R^k$ a sequence $\{E_i(x)\}$ that shrinks to x nicely. If μ is a complex Borel measure and $\mu \perp m$, then*

$$\lim_{i \to \infty} \frac{\mu(E_i(x))}{m(E_i(x))} = 0 \quad \text{a.e. } [m]. \tag{1}$$

PROOF The Jordan decomposition theorem shows that it suffices to prove (1) under the additional assumption that $\mu \geq 0$. In that case, arguing as in the proof of Theorem 7.10, we have

$$\frac{\alpha(x)\mu(E_i(x))}{m(E_i(x))} \leq \frac{\mu(E_i(x))}{m(B(x, r_i))} \leq \frac{\mu(B(x, r_i))}{m(B(x, r_i))}.$$

Hence (1) is a consequence of the special case

$$(D\mu)(x) = 0 \quad \text{a.e. } [m], \tag{2}$$

which will now be proved.

The upper derivative $\bar{D}\mu$, defined by

$$(\bar{D}\mu)(x) = \lim_{n \to \infty} \left[\sup_{0 < r < 1/n} (Q_r \mu)(x) \right] \qquad (x \in R^k) \tag{3}$$

is a Borel function, because the quantity in brackets decreases as n increases and is, for each n, a lower semicontinuous function of x; the reasoning used in Sec. 7.2 proves this.

Choose $\lambda > 0$, $\epsilon > 0$. Since $\mu \perp m$, μ is concentrated on a set of Lebesgue measure 0. The regularity of μ (Theorem 2.18) shows therefore that there is a compact set K, with $m(K) = 0$, $\mu(K) > \|\mu\| - \epsilon$.

Define $\mu_1(E) = \mu(K \cap E)$, for any Borel set $E \subset R^k$, and put $\mu_2 = \mu - \mu_1$. Then $\|\mu_2\| < \epsilon$, and, for every x outside K,

$$(\bar{D}\mu)(x) = (\bar{D}\mu_2)(x) \leq (M\mu_2)(x). \tag{4}$$

Hence

$$\{\bar{D}\mu > \lambda\} \subset K \cup \{M\mu_2 > \lambda\}, \tag{5}$$

and Theorem 7.4 shows that

$$m\{\bar{D}\mu > \lambda\} \leq 3^k \lambda^{-1} \|\mu_2\| < 3^k \lambda^{-1} \epsilon. \tag{6}$$

Since (6) holds for every $\epsilon > 0$ and for every $\lambda > 0$, we conclude that $\bar{D}\mu = 0$ a.e. $[m]$, i.e., that (2) holds. ////

Theorems 7.10 and 7.13 can be combined in the following way:

7.14 Theorem *Suppose that to each $x \in R^k$ is associated some sequence $\{E_i(x)\}$ that shrinks to x nicely, and that μ is a complex Borel measure on R^k.*

Let $d\mu = f\,dm + d\mu_s$ be the Lebesgue decomposition of μ with respect to m. Then

$$\lim_{i\to\infty} \frac{\mu(E_i(x))}{m(E_i(x))} = f(x) \quad \text{a.e. } [m].$$

In particular, $\mu \perp m$ if and only if $(D\mu)(x) = 0$ a.e. $[m]$.

The following result contrasts strongly with Theorem 7.13:

7.15 Theorem *If μ is a positive Borel measure on R^k and $\mu \perp m$, then*

$$(D\mu)(x) = \infty \quad \text{a.e. } [\mu]. \tag{1}$$

PROOF There is a Borel set $S \subset R^k$ with $m(S) = 0$ and $\mu(R^k - S) = 0$, and there are open sets $V_j \supset S$ with $m(V_j) < 1/j$, for $j = 1, 2, 3, \ldots$.

For $N = 1, 2, 3, \ldots$, let E_N be the set of all $x \in S$ to which correspond radii $r_i = r_i(x)$, with $\lim r_i = 0$, such that

$$\mu(B(x, r_i)) < Nm(B(x, r_i)). \tag{2}$$

Then (1) holds for every $x \in S - \bigcup_N E_N$.

Fix N and j, for the moment. Every $x \in E_N$ is then the center of a ball $B_x \subset V_j$ that satisfies (2). Let β_x be the open ball with center x whose radius is 1/3 of that of B_x. The union of these balls β_x is an open set $W_{j,N}$ that contains E_N and lies in V_j. We claim that

$$\mu(W_{j,N}) < 3^k N/j. \tag{3}$$

To prove (3), let $K \subset W_{j,N}$ be compact. Finitely many β_x cover K. Lemma 7.3 shows therefore that there is a finite set $F \subset E_N$ with the following properties:

(*a*) $\{\beta_x : x \in F\}$ is a disjoint collection, and
(*b*) $K \subset \bigcup_{x\in F} B_x$.

Thus

$$\begin{aligned}\mu(K) &\le \sum_{x\in F} \mu(B_x) < N \sum_{x\in F} m(B_x) \\ &= 3^k N \sum_{x\in F} m(\beta_x) \le 3^k N m(V_j) < 3^k N/j.\end{aligned}$$

This proves (3).

Now put $\Omega_N = \bigcap_j W_{j,N}$. Then $E_N \subset \Omega_N$, Ω_N is a G_δ, $\mu(\Omega_N) = 0$, and $(D\mu)(x) = \infty$ at every point of $S - \bigcup_N \Omega_N$. ////

The Fundamental Theorem of Calculus

7.16 This theorem concerns functions defined on some compact interval $[a, b]$ in R^1. It has two parts. The first asserts, roughly speaking, that the derivative of the indefinite integral of a function is that same function. We dealt with this in Theorem 7.11. The second part goes the other way: one returns to the original function by integrating its derivative. More precisely

$$f(x) - f(a) = \int_a^x f'(t)\, dt \qquad (a \le x \le b). \tag{1}$$

In the elementary version of this theorem, one assumes that f is differentiable at every point of $[a, b]$ and that f' is a continuous function. The proof of (1) is then easy.

In trying to extend (1) to the setting of the Lebesgue integral, questions such as the following come up naturally:

Is it enough to assume that $f' \in L^1$, rather than that f' is continuous?

If f is continuous and differentiable at *almost all* points of $[a, b]$, must (1) then hold?

Before proving any positive results, here are two examples that show how (1) can fail.

(*a*) Put $f(x) = x^2 \sin(x^{-2})$ if $x \neq 0$, $f(0) = 0$. Then f is differentiable at every point, but

$$\int_0^1 |f'(t)|\, dt = \infty, \tag{2}$$

so $f' \notin L^1$.

If we interpret the integral in (1) (with $[0, 1]$ in place of $[a, b]$) as the limit, as $\epsilon \to 0$, of the integrals over $[\epsilon, 1]$, then (1) still holds for this f.

More complicated situations can arise where this kind of passage to the limit is of no use. There are integration processes, due to Denjoy and Perron (see [18], [28]), which are so designed that (1) holds whenever f is differentiable at every point. These fail to have the property that the integrability of f implies that of $|f|$, and therefore do not play such an important role in analysis.

(*b*) Suppose f is continuous on $[a, b]$, f is differentiable at almost every point of $[a, b]$, and $f' \in L^1$ on $[a, b]$. Do these assumptions imply that (1) holds?

Answer: No.

Choose $\{\delta_n\}$ so that $1 = \delta_0 > \delta_1 > \delta_2 > \cdots$, $\delta_n \to 0$. Put $E_0 = [0, 1]$. Suppose $n \ge 0$ and E_n is constructed so that E_n is the union of 2^n disjoint closed intervals, each of length $2^{-n}\delta_n$. Delete a segment in the center of each of these 2^n intervals, so that each of the remaining 2^{n+1} intervals has length

$2^{-n-1}\delta_{n+1}$ (this is possible, since $\delta_{n+1} < \delta_n$), and let E_{n+1} be the union of these 2^{n+1} intervals. Then $E_1 \supset E_2 \supset \cdots$, $m(E_n) = \delta_n$, and if

$$E = \bigcap_{n=1}^{\infty} E_n, \tag{3}$$

then E is compact and $m(E) = 0$. (In fact, E is perfect.) Put

$$g_n = \delta_n^{-1}\chi_{E_n} \quad \text{and} \quad f_n(x) = \int_0^x g_n(t)\,dt \qquad (n = 0, 1, 2, \ldots). \tag{4}$$

Then $f_n(0) = 0$, $f_n(1) = 1$, and each f_n is a monotonic function which is constant on each segment in the complement of E_n. If I is one of the 2^n intervals whose union is E_n, then

$$\int_I g_n(t)\,dt = \int_I g_{n+1}(t)\,dt = 2^{-n}. \tag{5}$$

It follows from (5) that

$$f_{n+1}(x) = f_n(x) \qquad (x \notin E_n) \tag{6}$$

and that

$$|f_n(x) - f_{n+1}(x)| \le \int_I |g_n - g_{n+1}| < 2^{-n+1} \qquad (x \in E_n). \tag{7}$$

Hence $\{f_n\}$ converges uniformly to a continuous monotonic function f, with $f(0) = 0, f(1) = 1$, and $f'(x) = 0$ for all $x \notin E$. Since $m(E) = 0$, we have $f' = 0$ a.e.

Thus (1) fails.

If $\delta_n = (2/3)^n$, the set E is Cantor's "middle thirds" set.

Having seen what can go wrong, assume now that $f' \in L^1$ and that (1) does hold. There is then a measure μ, defined by $d\mu = f'\,dm$. Since $\mu \ll m$, Theorem 6.11 shows that there corresponds to each $\epsilon > 0$ a $\delta > 0$ so that $|\mu|(E) < \epsilon$ whenever E is a union of disjoint segments whose total length is less than δ. Since $f(y) - f(x) = \mu((x, y))$ if $a \le x < y \le b$, it follows that the absolute continuity of f, as defined below, is *necessary* for (1). Theorem 7.20 will show that this necessary condition is also sufficient.

7.17 Definition A complex function f, defined on an interval $I = [a, b]$, is said to be *absolutely continuous* on I (briefly, f is AC on I) if there corresponds to every $\epsilon > 0$ a $\delta > 0$ so that

$$\sum_{i=1}^{n} |f(\beta_i) - f(\alpha_i)| < \epsilon \tag{1}$$

for any n and any disjoint collection of segments $(\alpha_1, \beta_1), \ldots, (\alpha_n, \beta_n)$ in I whose lengths satisfy

$$\sum_{i=1}^{n} (\beta_i - \alpha_i) < \delta. \tag{2}$$

Such an f is obviously continuous: simply take $n = 1$.

In the following theorem, the implication $(b) \to (c)$ is probably the most interesting. That $(a) \to (c)$ without assuming monotonicity of f is the content of Theorem 7.20.

7.18 Theorem *Let $I = [a, b]$, let $f: I \to R^1$ be continuous and nondecreasing. Each of the following three statements about f implies the other two:*

(a) f is AC *on I.*
(b) f maps sets of measure 0 *to sets of measure* 0.
(c) f is differentiable a.e. *on I, $f' \in L^1$, and*

$$f(x) - f(a) = \int_a^x f'(t)\, dt \qquad (\alpha \le x \le b). \tag{1}$$

Note that the functions constructed in Example 7.16(*b*) map certain compact sets of measure 0 onto the whole unit interval!

Exercise 12 complements this theorem.

PROOF We will show that $(a) \to (b) \to (c) \to (a)$.

Let $\mathfrak{M}$ denote the σ-algebra of all Lebesgue measurable subsets of R^1.

Assume f is AC on I, pick $E \subset I$ so that $E \in \mathfrak{M}$ and $m(E) = 0$. We have to show that $f(E) \in \mathfrak{M}$ and $m(f(E)) = 0$. Without loss of generality, assume that neither a nor b lie in E.

Choose $\epsilon > 0$. Associate $\delta > 0$ to f and ϵ, as in Definition 7.17. There is then an open set V with $m(V) < \delta$, so that $E \subset V \subset I$. Let (α_i, β_i) be the disjoint segments whose union is V. Then $\sum (\beta_i - \alpha_i) < \delta$, and our choice of δ shows that therefore

$$\sum_i (f(\beta_i) - f(\alpha_i)) \le \epsilon. \tag{2}$$

[Definition 7.17 was stated in terms of finite sums; thus (2) holds for every partial sum of the (possibly) infinite series, hence (2) holds also for the sum of the whole series, as stated.]

Since $E \subset V$, $f(E) \subset \bigcup [f(\alpha_i), f(\beta_i)]$. The Lebesgue measure of this union is the left side of (2). This says that $f(E)$ is a subset of Borel sets of arbitrarily small measure. Since Lebesgue measure is complete, it follows that $f(E) \in \mathfrak{M}$ and $m(f(E)) = 0$.

We have now proved that (*a*) implies (*b*).

Assume next that (*b*) holds. Define

$$g(x) = x + f(x) \qquad (a \leq x \leq b). \tag{3}$$

If the f-image of some segment of length η has length η', then the g-image of that same segment has length $\eta + \eta'$. From this it follows easily that g satisfies (*b*), since f does.

Now suppose $E \subset I$, $E \in \mathfrak{M}$. Then $E = E_1 \cup E_0$ where $m(E_0) = 0$ and E_1 is an F_σ (Theorem 2.20). Thus E_1 is a countable union of compact sets, and so is $g(E_1)$, because g is continuous. Since g satisfies (*b*), $m(g(E_0)) = 0$. Since $g(E) = g(E_1) \cup g(E_0)$, we conclude: $g(E) \in \mathfrak{M}$.

Therefore we can define

$$\mu(E) = m(g(E)) \qquad (E \subset I,\ E \in \mathfrak{M}). \tag{4}$$

Since g is one-to-one (this is our reason for working with g rather than f), disjoint sets in I have disjoint g-images. The countable additivity of m shows therefore that μ is a (positive, bounded) measure on $\mathfrak{M}$. Also, $\mu \ll m$, because g satisfies (*b*). Thus

$$d\mu = h\, dm \tag{5}$$

for some $h \in L^1(m)$, by the Radon-Nikodym theorem.

If $E = [a, x]$, then $g(E) = [g(a), g(x)]$, and (5) gives

$$g(x) - g(a) = m(g(E)) = \mu(E) = \int_E h\, dm = \int_a^x h(t)\, dt.$$

If we now use (3), we conclude that

$$f(x) - f(a) = \int_a^x [h(t) - 1]\, dt \qquad (\alpha \leq x \leq b). \tag{6}$$

Thus $f'(x) = h(x) - 1$ a.e. $[m]$, by Theorem 7.11.

We have now proved that (*b*) implies (*c*).

The discussion that preceded Definition 7.17 showed that (*c*) implies (*a*).

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7.19 Theorem *Suppose $f: I \to R^1$ is* AC, *$I = [a, b]$. Define*

$$F(x) = \sup \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})| \qquad (a \leq x \leq b) \tag{1}$$

where the supremum is taken over all N and over all choices of $\{t_i\}$ such that

$$a = t_0 < t_1 < \cdots < t_N = x. \tag{2}$$

The functions F, $F + f$, $F - f$ are then nondecreasing and AC *on I.*

[F is called the *total variation* function of f. If f is any (complex) function on I, AC or not, and $F(b) < \infty$, then f is said to have *bounded variation* on I, and $F(b)$ is the *total variation* of f on I. Exercise 13 is relevant to this.]

PROOF If (2) holds and $x < y \leq b$, then

$$F(y) \geq |f(y) - f(x)| + \sum_{i=1}^{N} |f(t_i) - f(t_{i-1})|. \tag{3}$$

Hence $F(y) \geq |f(y) - f(x)| + F(x)$. In particular

$$F(y) \geq f(y) - f(x) + F(x) \quad \text{and} \quad F(y) \geq f(x) - f(y) + F(x). \tag{4}$$

This proves that F, $F + f$, $F - f$ are nondecreasing.

Since sums of two AC functions are obviously AC, it only remains to be proved that F is AC on I.

If $(\alpha, \beta) \subset I$ then

$$F(\beta) - F(\alpha) = \sup \sum_{1}^{n} |f(t_i) - f(t_{i-1})|, \tag{5}$$

the supremum being taken over all $\{t_i\}$ that satisfy $\alpha = t_0 < \cdots < t_n = \beta$.

Note that $\sum (t_i - t_{i-1}) = \beta - \alpha$.

Now pick $\epsilon > 0$, associate $\delta > 0$ to f and ϵ as in Definition 7.17, choose disjoint segments $(\alpha_j, \beta_j) \subset I$ with $\sum (\beta_j - \alpha_j) < \delta$, and apply (5) to each (α_j, β_j). It follows that

$$\sum_j (F(\beta_j) - F(\alpha_j)) \leq \epsilon, \tag{6}$$

by our choice of δ. Thus F is AC on I. ////

We have now reached our main objective:

7.20 Theorem *If f is a complex function that is* AC *on $I = [a, b]$, then f is differentiable at almost all points of I, $f' \in L^1(m)$, and*

$$f(x) - f(a) = \int_a^x f'(t)\, dt \qquad (a \leq x \leq b). \tag{1}$$

PROOF It is of course enough to prove this for real f. Let F be its total variation function, as in Theorem 7.19, define

$$f_1 = \tfrac{1}{2}(F + f), \qquad f_2 = \tfrac{1}{2}(F - f), \tag{2}$$

and apply the implication (*a*) → (*c*) of Theorem 7.18 to f_1 and f_2. Since

$$f = f_1 - f_2 \tag{3}$$

this yields (1). ////

The next theorem derives (1) from a different set of hypotheses, by an entirely different method of proof.

7.21 Theorem *If f: $[a, b] \to R^1$ is differentiable at every point of $[a, b]$ and $f' \in L^1$ on $[a, b]$, then*

$$f(x) - f(a) = \int_a^x f'(t)\, dt \qquad (a \le x \le b). \tag{1}$$

Note that differentiability is assumed to hold at *every* point of $[a, b]$.

PROOF It is clear that it is enough to prove this for $x = b$. Fix $\epsilon > 0$. Theorem 2.25 ensures the existence of a lower semicontinuous function g on $[a, b]$ such that $g > f'$ and

$$\int_a^b g(t)\, dt < \int_a^b f'(t)\, dt + \epsilon. \tag{2}$$

Actually, Theorem 2.25 only gives $g \ge f'$, but since $m([a, b]) < \infty$, we can add a small constant to g without affecting (2). For any $\eta > 0$, define

$$F_\eta(x) = \int_a^x g(t)\, dt - f(x) + f(a) + \eta(x - a) \qquad (a \le x \le b). \tag{3}$$

Keep η fixed for the moment. To each $x \in [a, b)$ there corresponds a $\delta_x > 0$ such that

$$g(t) > f'(x) \quad \text{and} \quad \frac{f(t) - f(x)}{t - x} < f'(x) + \eta \tag{4}$$

for all $t \in (x, x + \delta_x)$, since g is lower semicontinuous and $g(x) > f'(x)$. For any such t we therefore have

$$\begin{aligned} F_\eta(t) - F_\eta(x) &= \int_x^t g(s)\, ds - [f(t) - f(x)] + \eta(t - x) \\ &> (t - x)f'(x) - (t - x)[f'(x) + \eta] + \eta(t - x) = 0. \end{aligned}$$

Since $F_\eta(a) = 0$ and F_η is *continuous*, there is a last point $x \in [a, b]$ at which $F_\eta(x) = 0$. If $x < b$, the preceding computation implies that $F_\eta(t) > 0$ for $t \in (x, b]$. In any case, $F_\eta(b) \ge 0$. Since this holds for every $\eta > 0$, (2) and (3) now give

$$f(b) - f(a) \le \int_a^b g(t)\, dt < \int_a^b f'(t)\, dt + \epsilon, \tag{5}$$

and since ϵ was arbitrary, we conclude that

$$f(b) - f(a) \le \int_a^b f'(t)\, dt. \tag{6}$$

If f satisfies the hypotheses of the theorem, so does $-f$; therefore (6) holds with $-f$ in place of f, and these two inequalities together give (1). ////

Differentiable Transformations

7.22 Definitions Suppose V is an open set in R^k, T maps V into R^k, and $x \in V$. If there exists a linear operator A on R^k (i.e., a linear mapping of R^k into R^k, as in Definition 2.1) such that

$$\lim_{h \to 0} \frac{|T(x+h) - T(x) - Ah|}{|h|} = 0 \tag{1}$$

(where, of course, $h \in R^k$), then we say that T is *differentiable* at x, and define

$$T'(x) = A. \tag{2}$$

The linear operator $T'(x)$ is called the *derivative* of T at x. (One shows easily that there is at most one linear A that satisfies the preceding requirements; thus it is legitimate to talk about *the* derivative of T.) The term *differential* is also often used for $T'(x)$.

The point of (1) is of course that the difference $T(x+h) - T(x)$ is approximated by $T'(x)h$, a *linear* function of h.

Since every real number α gives rise to a linear operator on R^1 (mapping h to αh), our definition of $T'(x)$ coincides with the usual one when $k = 1$.

When $A: R^k \to R^k$ is linear, Theorem 2.20(*e*) shows that there is a number $\Delta(A)$ such that

$$m(A(E)) = \Delta(A)m(E) \tag{3}$$

for all measurable sets $E \subset R^k$. Since

$$A'(x) = A \qquad (x \in R^k) \tag{4}$$

and since every differentiable transformation T can be locally approximated by a constant plus a linear transformation, one may conjecture that

$$\frac{m(T(E))}{m(E)} \sim \Delta(T'(x)) \tag{5}$$

for suitable sets E that are close to x. This will be proved in Theorem 7.24, and furnishes the motivation for Theorem 7.26.

Recall that $\Delta(A) = |\det A|$ was proved in Sec. 2.23. When T is differentiable at x, the determinant of $T'(x)$ is called the *Jacobian* of T at x, and is denoted by $J_T(x)$. Thus

$$\Delta(T'(x)) = |J_T(x)|. \tag{6}$$

The following lemma seems geometrically obvious. Its proof depends on the Brouwer fixed point theorem. One can avoid the use of this theorem by imposing

stronger hypotheses on F, for example, by assuming that F is an open mapping. But this would lead to unnecessarily strong assumptions in Theorem 7.26.

7.23 Lemma *Let $S = \{x: |x| = 1\}$ be the sphere in R^k that is the boundary of the open unit ball $B = B(0, 1)$.*

If $F: \bar{B} \to R^k$ is continuous, $0 < \epsilon < 1$, and

$$|F(x) - x| < \epsilon \tag{1}$$

for all $x \in S$, then $F(B) \supset B(0, 1 - \epsilon)$.

PROOF Assume, to reach a contradiction, that some point $a \in B(0, 1 - \epsilon)$ is not in $F(B)$. By (1), $|F(x)| > 1 - \epsilon$ if $x \in S$. Thus a is not in $F(S)$, and therefore $a \neq F(x)$, for every $x \in \bar{B}$. This enables us to define a continuous map $G: \bar{B} \to \bar{B}$ by

$$G(x) = \frac{a - F(x)}{|a - F(x)|}. \tag{2}$$

If $x \in S$, then $x \cdot x = |x|^2 = 1$, so that

$$x \cdot (a - F(x)) = x \cdot a + x \cdot (x - F(x)) - 1 < |a| + \epsilon - 1 < 0. \tag{3}$$

This shows that $x \cdot G(x) < 0$, hence $x \neq G(x)$.

If $x \in B$, then obviously $x \neq G(x)$, simply because $G(x) \in S$.

Thus G fixes no point of $\bar{B}$, contrary to Brouwer's theorem which states that every continuous map of $\bar{B}$ into $\bar{B}$ has at least one fixed point. ////

A proof of Brouwer's theorem that is both elementary and simple may be found on pp. 38–40 of "Dimension Theory" by Hurewicz and Wallman, Princeton University Press, 1948.

7.24 Theorem *If*

(*a*) *V is open in R^k,*
(*b*) *$T: V \to R^k$ is continuous, and*
(*c*) *T is differentiable at some point $x \in V$, then*

$$\lim_{r \to 0} \frac{m(T(B(x, r)))}{m(B(x, r))} = \Delta(T'(x)). \tag{1}$$

Note that $T(B(x, r))$ is Lebesgue measurable; in fact, it is σ-compact, because $B(x, r)$ is σ-compact and T is continuous.

PROOF Assume, without loss of generality, that $x = 0$ and $T(x) = 0$. Put $A = T'(0)$.

The following elementary fact about linear operators on finite-dimensional vector spaces will be used: *A linear operator A on R^k is one-to-*

one if and only if the range of A is all of R^k. In that case, the inverse A^{-1} of A is also linear.

Accordingly, we split the proof into two cases.

CASE I *A is one-to-one.* Define

$$F(x) = A^{-1}T(x) \qquad (x \in V). \tag{2}$$

Then $F'(0) = A^{-1}T'(0) = A^{-1}A = I$, the identity operator. We shall prove that

$$\lim_{r \to 0} \frac{m(F(B(0, r)))}{m(B(0, r))} = 1. \tag{3}$$

Since $T(x) = AF(x)$, we have

$$m(T(B)) = m(A(F(B))) = \Delta(A)m(F(B)) \tag{4}$$

for every ball B, by 7.22(3). Hence (3) will give the desired result.

Choose $\epsilon > 0$. Since $F(0) = 0$ and $F'(0) = I$, there exists a $\delta > 0$ such that $0 < |x| < \delta$ implies

$$|F(x) - x| < \epsilon |x|. \tag{5}$$

We claim that the inclusions

$$B(0, (1 - \epsilon)r) \subset F(B(0, r)) \subset B(0, (1 + \epsilon)r) \tag{6}$$

hold if $0 < r < \delta$. The first of these follows from Lemma 7.23, applied to $B(0, r)$ in place of $B(0, 1)$, because $|F(x) - x| < \epsilon r$ for all x with $|x| = r$. The second follows directly from (5), since $|F(x)| < (1 + \epsilon)|x|$. It is clear that (6) implies

$$(1 - \epsilon)^k \leq \frac{m(F(B(0, r)))}{m(B(0, r))} \leq (1 + \epsilon)^k \tag{7}$$

and this proves (3).

CASE II *A is not one-to-one.* In this case, A maps R^k into a subspace of lower dimension, i.e., into a set of measure 0. Given $\epsilon > 0$, there is therefore an $\eta > 0$ such that $m(E_\eta) < \epsilon$ if E_η is the set of all points in R^k whose distance from $A(B(0, 1))$ is less than η. Since $A = T'(0)$, there is a $\delta > 0$ such that $|x| < \delta$ implies

$$|T(x) - Ax| \leq \eta |x|. \tag{8}$$

If $r < \delta$, then $T(B(0, r))$ lies therefore in the set E that consists of the points whose distance from $A(B(0, r))$ is less than ηr. Our choice of η shows that $m(E) < \epsilon r^k$. Hence

$$m(T(B(0, r))) < \epsilon r^k \qquad (0 < r < \delta). \tag{9}$$

Since $r^k = m(B(0, r))/m(B(0, 1))$, (9) implies that

$$\lim_{r \to 0} \frac{m(T(B(0, r)))}{m(B(0, r))} = 0. \tag{10}$$

This proves (1), since $\Delta(T'(0)) = \Delta(A) = 0$. ////

7.25 Lemma *Suppose $E \subset R^k$, $m(E) = 0$, T maps E into R^k, and*

$$\limsup \frac{|T(y) - T(x)|}{|y - x|} < \infty$$

for every $x \in E$, as y tends to x within E.

Then $m(T(E)) = 0$.

PROOF Fix positive integers n and p, let $F = F_{n,p}$ be the set of all $x \in E$ such that

$$|T(y) - T(x)| \leq n|y - x|$$

for all $y \in B(x, 1/p) \cap E$, and choose $\epsilon > 0$. Since $m(F) = 0$, F can be covered by balls $B_i = B(x_i, r_i)$, where $x_i \in F$, $r_i < 1/p$, in such a way that $\sum m(B_i) < \epsilon$. (To do this, cover F by an open set W of small measure, decompose W into disjoint boxes of small diameter, as in Sec. 2.19, and cover each of those that intersect F by a ball whose center lies in the box and in F.)

If $x \in F \cap B_i$ then $|x_i - x| < r_i < 1/p$ and $x_i \in F$. Hence

$$|T(x_i) - T(x)| \leq n|x_i - x| < nr_i$$

so that $T(F \cap B_i) \subset B(T(x_i), nr_i)$. Therefore

$$T(F) \subset \bigcup_i B(T(x_i), nr_i).$$

The measure of this union is at most

$$\sum_i m(B(T(x_i), nr_i) = n^k \sum_i m(B_i) < n^k \epsilon.$$

Since Lebesgue measure is complete and ϵ was arbitrary, it follows that $T(F)$ is measurable and $m(T(F)) = 0$.

To complete the proof, note that E is the union of the countable collection $\{F_{n,p}\}$. ////

Here is a special case of the lemma:

If V is open in R^k and $T: V \to R^k$ is differentiable at every point of V, then T maps sets of measure 0 to sets of measure 0.

We now come to the change-of-variables theorem.

7.26 Theorem *Suppose that*

(i) *$X \subset V \subset R^k$, V is open, $T: V \to R^k$ is continuous;*

(ii) *X is Lebesgue measurable, T is one-to-one on X, and T is differentiable at every point of X;*

(iii) $m(T(V - X)) = 0$.

Then, setting $Y = T(X)$,

$$\int_Y f\,dm = \int_X (f \circ T)|J_T|\,dm \tag{1}$$

for every measurable f: $R^k \to [0, \infty]$.

The case $X = V$ is perhaps the most interesting one. As regards condition (iii), it holds, for instance, when $m(V - X) = 0$ and T satisfies the hypotheses of Lemma 7.25 on $V - X$.

The proof has some elements in common with that of the implication $(b) \to (c)$ in Theorem 7.18.

It will be important in this proof to distinguish between Borel sets and Lebesgue measurable sets. The σ-algebra consisting of the Lebesgue measurable subsets of R^k will be denoted by $\mathfrak{M}$.

PROOF We break the proof into the following three steps:

(I) *If $E \in \mathfrak{M}$ and $E \subset V$, then $T(E) \in \mathfrak{M}$.*

(II) *For every $E \in \mathfrak{M}$,*

$$m(T(E \cap X)) = \int_X \chi_E |J_T|\,dm.$$

(III) *For every $A \in \mathfrak{M}$,*

$$\int_Y \chi_A\,dm = \int_X (\chi_A \circ T)|J_T|\,dm.$$

If $E_0 \in \mathfrak{M}$, $E_0 \subset V$, and $m(E_0) = 0$, then $m(T(E_0 - X)) = 0$ by (iii), and $m(T(E_0 \cap X)) = 0$ by Lemma 7.25. Thus $m(T(E_0)) = 0$.

If $E_1 \subset V$ is an F_σ, then E_1 is σ-compact, hence $T(E_1)$ is σ-compact, because T is continuous. Thus $T(E_1) \in \mathfrak{M}$.

Since every $E \in \mathfrak{M}$ is the union of an F_σ and a set of measure 0 (Theorem 2.20), (I) is proved.

To prove (II), let n be a positive integer, and put

$$V_n = \{x \in V: |T(x)| < n\}, \qquad X_n = X \cap V_n. \tag{2}$$

Because of (I), we can define

$$\mu_n(E) = m(T(E \cap X_n)) \qquad (E \in \mathfrak{M}). \tag{3}$$

Since T is one-to-one on X_n, the countable additivity of m shows that μ_n is a measure on $\mathfrak{M}$. Also, μ_n is bounded (this was the reason for replacing X temporarily by X_n), and $\mu_n \ll m$, by another application of Lemma 7.25.

Theorem 7.8 tells us therefore that $(D\mu_n)(x)$ exists a.e. $[m]$, that $D\mu_n \in L^1(m)$, and that

$$\mu_n(E) = \int_E (D\mu_n)\, dm \qquad (E \in \mathfrak{M}). \tag{4}$$

We claim next that

$$(D\mu_n)(x) = |J_T(x)| \qquad (x \in X_n). \tag{5}$$

To see this, fix $x \in X_n$, and note that $B(x, r) \subset V_n$ for all sufficiently small $r > 0$, because V_n is open. Since $V_n - X_n \subset V - X$, hypothesis (iii) enables us to replace X_n by V_n in (3) without changing $\mu_n(E)$. Hence, for small $r > 0$,

$$\mu_n(B(x, r)) = m(T(B(x, r))). \tag{6}$$

If we divide both sides of (6) by $m(B(x, r))$ and refer to Theorem 7.24 and formula 7.22(6), we obtain (5).

Since (3) implies that $\mu_n(E) = \mu_n(E \cap X_n)$, it follows from (3), (4), and (5) that

$$m(T(E \cap X_n)) = \int_{X_n} \chi_E |J_T|\, dm \qquad (E \in \mathfrak{M}). \tag{7}$$

If we apply the monotone convergence theorem to (7), letting $n \to \infty$, we obtain (II).

We begin the proof of (III) by letting A be a Borel set in R^k. Put

$$E = T^{-1}(A) = \{x \in V: T(x) \in A\}. \tag{8}$$

Then $\chi_E = \chi_A \circ T$. Since χ_A is a Borel function and T is continuous, χ_E is a Borel function (Theorem 1.12), hence $E \in \mathfrak{M}$. Also

$$T(E \cap X) = A \cap Y \tag{9}$$

which implies, by (II), that

$$\int_Y \chi_A\, dm = m(T(E \cap X)) = \int_X (\chi_A \circ T)|J_T|\, dm. \tag{10}$$

Finally, if $N \in \mathfrak{M}$ and $m(N) = 0$, there is a Borel set $A \supset N$ with $m(A) = 0$. For this A, (10) shows that $(\chi_A \circ T)|J_T| = 0$ a.e. $[m]$. Since $0 \leq \chi_N \leq \chi_A$, it follows that both integrals in (10) are 0 if A is replaced by N. Since every Lebesgue measurable set is the disjoint union of a Borel set and a set of measure 0, (10) holds for every $A \in \mathfrak{M}$. This proves (III).

Once we have (III), it is clear that (1) holds for every nonnegative Lebesgue measurable *simple* function f. Another application of the monotone convergence theorem completes the proof. ////

Note that we did not prove that $f \circ T$ is Lebesgue measurable for all Lebesgue measurable f. It need not be; see Exercise 8. What the proof does establish is the Lebesgue measurability of the product $(f \circ T)|J_T|$.

Here is a special case of the theorem:

Suppose φ: $[a, b] \to [\alpha, \beta]$ *is* AC, *monotonic,* $\varphi(a) = \alpha$, $\varphi(b) = \beta$, *and* $f \geq 0$ *is Lebesgue measurable. Then*

$$\int_\alpha^\beta f(t)\, dt = \int_a^b f(\varphi(x))\varphi'(x)\, dx. \tag{15}$$

To derive this from Theorem 7.26, put $V = (a, b)$, $T = \varphi$, let Ω be the union of the maximal segments on which φ is constant (if there are any) and let X be the set of all $x \in V - \Omega$ where $\varphi'(x)$ exists (and is finite).

Exercises

1 Show that $|f(x)| \leq (Mf)(x)$ at every Lebesgue point of f if $f \in L^1(R^k)$.

2 For $\delta > 0$, let $I(\delta)$ be the segment $(-\delta, \delta) \subset R^1$. Given α and β, $0 \leq \alpha \leq \beta \leq 1$, construct a measurable set $E \subset R^1$ so that the upper and lower limits of

$$\frac{m(E \cap I(\delta))}{2\delta}$$

are β and α, respectively, as $\delta \to 0$.

(Compare this with Section 7.12.)

3 Suppose that E is a measurable set of real numbers with arbitrarily small periods. Explicitly, this means that there are positive numbers p_i, converging to 0 as $i \to \infty$, so that

$$E + p_i = E \qquad (i = 1, 2, 3, \ldots).$$

Prove that then either E or its complement has measure 0.

Hint: Pick $\alpha \in R^1$, put $F(x) = m(E \cap [\alpha, x])$ for $x > \alpha$, show that

$$F(x + p_i) - F(x - p_i) = F(y + p_i) - F(y - p_i)$$

if $\alpha + p_i < x < y$. What does this imply about $F'(x)$ if $m(E) > 0$?

4 Call t a *period* of the function f on R^1 if $f(x + t) = f(x)$ for all $x \in R^1$. Suppose f is a real Lebesgue measurable function with periods s and t whose quotient s/t is irrational. Prove that there is a constant c such that $f(x) = c$ a.e., but that f need not be constant.

Hint: Apply Exercise 3 to the sets $\{f > \lambda\}$.

5 If $A \subset R^1$ and $B \subset R^1$, define $A + B = \{a + b: a \in A, b \in B\}$. Suppose $m(A) > 0$, $m(B) > 0$. Prove that $A + B$ contains a segment, by completing the following outline.

There are points a_0 and b_0 where A and B have metric density 1. Choose a small $\delta > 0$. Put $c_0 = a_0 + b_0$. For each ϵ, positive or negative, define B_ϵ to be the set of all $c_0 + \epsilon - b$ for which $b \in B$ and $|b - b_0| < \delta$. Then $B_\epsilon \subset (a_0 + \epsilon - \delta, a_0 + \epsilon + \delta)$. If δ was well chosen and $|\epsilon|$ is sufficiently small, it follows that A intersects B_ϵ, so that $A + B \supset (c_0 - \epsilon_0, c_0 + \epsilon_0)$ for some $\epsilon_0 > 0$.

Let C be Cantor's "middle thirds" set and show that $C + C$ is an interval, although $m(C) = 0$. (See also Exercise 19, Chap. 9.)

6 Suppose G is a subgroup of R^1 (relative to addition), $G \neq R^1$, and G is Lebesgue measurable. Prove that then $m(G) = 0$.

Hint: Use Exercise 5.

7 Construct a continuous monotonic function f on R^1 so that f is not constant on any segment although $f'(x) = 0$ a.e.

8 Let $V = (a, b)$ be a bounded segment in R^1. Choose segments $W_n \subset V$ in such a way that their union W is dense in V and the set $K = V - W$ has $m(K) > 0$. Choose continuous functions φ_n so that $\varphi_n(x) = 0$ outside W_n, $0 < \varphi_n(x) < 2^{-n}$ in W_n. Put $\varphi = \sum \varphi_n$ and define

$$T(x) = \int_a^x \varphi(t)\, dt \qquad (a < x < b).$$

Prove the following statements:

(*a*) T satisfies the hypotheses of Theorem 7.26, with $X = V$.

(*b*) T' is continuous, $T'(x) = 0$ on K, $m(T(K)) = 0$.

(*c*) If E is a nonmeasurable subset of K (see Theorem 2.22) and $A = T(E)$, then χ_A is Lebesgue measurable but $\chi_A \circ T$ is not.

(*d*) The functions φ_n can be so chosen that the resulting T is an *infinitely differentiable* homeomorphism of V onto some segment in R^1 and (*c*) still holds.

9 Suppose $0 < \alpha < 1$. Pick t so that $t^\alpha = 2$. Then $t > 2$, and the construction of Example (*b*) in Sec. 7.16 can be carried out with $\delta_n = (2/t)^n$. Show that the resulting function f belongs to Lip α on $[0, 1]$.

10 If $f \in$ Lip 1 on $[a, b]$, prove that f is absolutely continuous and that $f' \in L^\infty$.

11 Assume that $1 < p < \infty$, f is absolutely continuous on $[a, b]$, $f' \in L^p$, and $\alpha = 1/q$, where q is the exponent conjugate to p. Prove that $f \in$ Lip α.

12 Suppose $\varphi: [a, b] \to R^1$ is nondecreasing.

(*a*) Show that there is a left-continuous nondecreasing f on $[a, b]$ so that $\{f \neq \varphi\}$ is at most countable. [Left-continuous means: if $a < x \leq b$ and $\epsilon > 0$, then there is a $\delta > 0$ so that $|f(x) - f(x - t)| < \epsilon$ whenever $0 < t < \delta$.]

(*b*) Imitate the proof of Theorem 7.18 to show that there is a positive Borel measure μ on $[a, b]$ for which

$$f(x) - f(a) = \mu([a, x)) \qquad (a \leq x \leq b).$$

(*c*) Deduce from (*b*) that $f'(x)$ exists a.e. $[m]$, that $f' \in L^1(m)$, and that

$$f(x) - f(a) = \int_a^x f'(t)\, dt + s(x) \qquad (a \leq x \leq b)$$

where s is nondecreasing and $s'(x) = 0$ a.e. $[m]$.

(*d*) Show that $\mu \perp m$ if and only if $f'(x) = 0$ a.e. $[m]$, and that $\mu \ll m$ if and only if f is AC on $[a, b]$.

(*e*) Prove that $\varphi'(x) = f'(x)$ a.e. $[m]$.

13 Let BV be the class of all f on $[a, b]$ that have bounded variation on $[a, b]$, as defined after Theorem 7.19. Prove the following statements.

(*a*) Every monotonic bounded function on $[a, b]$ is in BV.

(*b*) If $f \in BV$ is real, there exist bounded monotonic functions f_1 and f_2 so that $f = f_1 - f_2$.

Hint: Imitate the proof of Theorem 7.19.

(*c*) If $f \in BV$ is left-continuous then f_1 and f_2 can be chosen in (*b*) so as to be also left-continuous.

(*d*) If $f \in BV$ is left-continuous then there is a Borel measure μ on $[a, b]$ that satisfies

$$f(x) - f(a) = \mu([a, x)) \qquad (a \leq x \leq b);$$

$\mu \ll m$ if and only if f is AC on $[a, b]$.

(*e*) Every $f \in BV$ is differentiable a.e. $[m]$, and $f' \in L^1(m)$.

14 Show that the product of two absolutely continuous functions on $[a, b]$ is absolutely continuous. Use this to derive a theorem about integration by parts.

15 Construct a monotonic function f on R^1 so that $f'(x)$ exists (finitely) for every $x \in R^1$, but f' is not a continuous function.

16 Suppose $E \subset [a, b]$, $m(E) = 0$. Construct an absolutely continuous monotonic function f on $[a, b]$ so that $f'(x) = \infty$ at every $x \in E$.

Hint: $E \subset \bigcap V_n$, V_n open, $m(V_n) < 2^{-n}$. Consider the sum of the characteristic functions of these sets.

17 Suppose $\{\mu_n\}$ is a sequence of positive Borel measures on R^k and

$$\mu(E) = \sum_{n=1}^{\infty} \mu_n(E).$$

Assume $\mu(R^k) < \infty$. Show that μ is a Borel measure. What is the relation between the Lebesgue decompositions of the μ_n and that of μ?

Prove that

$$(D\mu)(x) = \sum_{n=1}^{\infty} (D\mu_n)(x) \quad \text{a.e. } [m].$$

Derive corresponding theorems for sequences $\{f_n\}$ of positive nondecreasing functions on R^1 and their sums $f = \sum f_n$.

18 Let $\varphi_0(t) = 1$ on $[0, 1)$, $\varphi_0(t) = -1$ on $[1, 2)$, extend φ_0 to R^1 so as to have period 2, and define $\varphi_n(t) = \varphi_0(2^n t)$, $n = 1, 2, 3, \ldots$.

Assume that $\sum |c_n|^2 < \infty$ *and prove that the series*

$$\sum_{n=1}^{\infty} c_n \varphi_n(t) \tag{*}$$

converges then for almost every t.

Probabilistic interpretation: The series $\sum (\pm c_n)$ converges with probability 1.

Suggestion: $\{\varphi_n\}$ is orthonormal on $[0, 1]$, hence (*) is the Fourier series of some $f \in L^2$. If $a = j \cdot 2^{-N}$, $b = (j + 1) \cdot 2^{-N}$, $a < t < b$, and $s_N = c_1\varphi_1 + \cdots + c_N\varphi_N$, then, for $n > N$,

$$s_N(t) = \frac{1}{b-a}\int_a^b s_N \, dm = \frac{1}{b-a}\int_a^b s_n \, dm,$$

and the last integral converges to $\int_a^b f \, dm$, as $n \to \infty$. Show that (*) converges to $f(t)$ at almost every Lebesgue point of f.

19 Suppose f is continuous on R^1, $f(x) > 0$ if $0 < x < 1$, $f(x) = 0$ otherwise. Define

$$h_c(x) = \sup \{n^c f(nx): n = 1, 2, 3, \ldots\}.$$

Prove that

(*a*) h_c is in $L^1(R^1)$ if $0 < c < 1$,

(*b*) h_1 is in weak L^1 but not in $L^1(R^1)$,

(*c*) h_c is not in weak L^1 if $c > 1$.

20 (*a*) For any set $E \subset R^2$, the boundary ∂E of E is, by definition, the closure of E minus the interior of E. Show that E is Lebesgue measurable whenever $m(\partial E) = 0$.

(*b*) Suppose that E is the union of a (*possibly uncountable*) collection of *closed* discs in R^2 whose radii are at least 1. Use (*a*) to show that E is Lebesgue measurable.

(*c*) Show that the conclusion of (*b*) is true even when the radii are unrestricted.

(*d*) Show that some unions of closed discs of radius 1 are not Borel sets. (See Sec. 2.21.)

(*e*) Can discs be replaced by triangles, rectangles, arbitrary polygons, etc., in all this? What is the relevant geometric property?

21 If f is a real function on [0, 1] and

$$\gamma(t) = t + if(t),$$

the length of the graph of f is, by definition, the total variation of γ on [0, 1]. Show that this length is finite if and only if $f \in BV$. (See Exercise 13.) Show that it is equal to

$$\int_0^1 \sqrt{1 + [f'(t)]^2}\, dt$$

if f is absolutely continuous.

22 (*a*) Assume that both f and its maximal function Mf are in $L^1(R^k)$. Prove that then $f(x) = 0$ a.e. $[m]$. *Hint*: To every other $f \in L^1(R^k)$ corresponds a constant $c = c(f) > 0$ such that

$$(Mf)(x) \geq c\,|x|^{-k}$$

whenever $|x|$ is sufficiently large.

(*b*) Define $f(x) = x^{-1}(\log x)^{-2}$ if $0 < x < \frac{1}{2}$, $f(x) = 0$ on the rest of R^1. Then $f \in L^1(R^1)$. Show that

$$(Mf)(x) \geq |2x \log (2x)|^{-1} \qquad (0 < x < 1/4)$$

so that $\int_0^1 (Mf)(x)\, dx = \infty$.

23 The definition of Lebesgue points, as made in Sec. 7.6, applies to individual integrable functions, not to the equivalence classes discussed in Sec. 3.10. However, if $F \in L^1(R^k)$ is one of these equivalence classes, one may call a point $x \in R^k$ a *Lebesgue point of* F if there is a complex number, let us call it $(SF)(x)$, such that

$$\lim_{r \to 0} \frac{1}{m(B_r)} \int_{B(x,\, r)} |f - (SF)(x)|\, dm = 0$$

for one (hence for every) $f \in F$.

Define $(SF)(x)$ to be 0 at those points $x \in R^k$ that are not Lebesgue points of F.

Prove the following statement: If $f \in F$, and x is a Lebesgue point of f, then x is also a Lebesgue point of F, and $f(x) = (SF)(x)$. Hence $SF \in F$.

Thus S "selects" a member of F that has a *maximal* set of Lebesgue points.

CHAPTER

EIGHT

INTEGRATION ON PRODUCT SPACES

This chapter is devoted to the proof and discussion of the theorem of Fubini concerning integration of functions of two variables. We first present the theorem in its abstract form.

Measurability on Cartesian Products

8.1 Definitions If X and Y are two sets, their *cartesian product* $X \times Y$ is the set of all ordered pairs (x, y), with $x \in X$ and $y \in Y$. If $A \subset X$ and $B \subset Y$, it follows that $A \times B \subset X \times Y$. We call any set of the form $A \times B$ a *rectangle* in $X \times Y$.

Suppose now that $(X, \mathscr{S})$ and $(Y, \mathscr{T})$ are measurable spaces. Recall that this simply means that $\mathscr{S}$ is a σ-algebra in X and $\mathscr{T}$ is a σ-algebra in Y.

A *measurable rectangle* is any set of the form $A \times B$, where $A \in \mathscr{S}$ and $B \in \mathscr{T}$.

If $Q = R_1 \cup \cdots \cup R_n$, where each R_i is a measurable rectangle and $R_i \cap R_j = \varnothing$ for $i \neq j$, we say that $Q \in \mathscr{E}$, the class of all *elementary sets.*

$\mathscr{S} \times \mathscr{T}$ is defined to be the smallest σ-algebra in $X \times Y$ which contains every measurable rectangle.

A *monotone class* $\mathfrak{M}$ is a collection of sets with the following properties: If $A_i \in \mathfrak{M}$, $B_i \in \mathfrak{M}$, $A_i \subset A_{i+1}$, $B_i \supset B_{i+1}$, for $i = 1, 2, 3, \ldots$, and if

$$A = \bigcup_{i=1}^{\infty} A_i, \qquad B = \bigcap_{i=1}^{\infty} B_i, \tag{1}$$

then $A \in \mathfrak{M}$ and $B \in \mathfrak{M}$.

If $E \subset X \times Y$, $x \in X$, $y \in Y$, we define

$$E_x = \{y: (x, y) \in E\}, \qquad E^y = \{x: (x, y) \in E\}. \tag{2}$$

We call E_x and E^y the *x-section* and *y-section*, respectively, of E. Note that $E_x \subset Y$, $E^y \subset X$.

8.2 Theorem *If $E \in \mathscr{S} \times \mathscr{T}$, then $E_x \in \mathscr{T}$ and $E^y \in \mathscr{S}$, for every $x \in X$ and $y \in Y$.*

PROOF Let Ω be the class of all $E \in \mathscr{S} \times \mathscr{T}$ such that $E_x \in \mathscr{T}$ for every $x \in X$. If $E = A \times B$, then $E_x = B$ if $x \in A$, $E_x = \varnothing$ if $x \notin A$. Therefore every measurable rectangle belongs to Ω. Since $\mathscr{T}$ is a σ-algebra, the following three statements are true. They prove that Ω is a σ-algebra and hence that $\Omega = \mathscr{S} \times \mathscr{T}$:

(*a*) $X \times Y \in \Omega$.
(*b*) If $E \in \Omega$, then $(E^c)_x = (E_x)^c$, hence $E^c \in \Omega$.
(*c*) If $E_i \in \Omega$ $(i = 1, 2, 3, \ldots)$ and $E = \bigcup E_i$, then $E_x = \bigcup (E_i)_x$, hence $E \in \Omega$.

The proof is the same for E^y. ////

8.3 Theorem *$\mathscr{S} \times \mathscr{T}$ is the smallest monotone class which contains all elementary sets.*

PROOF Let $\mathfrak{M}$ be the smallest monotone class which contains $\mathscr{E}$; the proof that this class exists is exactly like that of Theorem 1.10. Since $\mathscr{S} \times \mathscr{T}$ is a monotone class, we have $\mathfrak{M} \subset \mathscr{S} \times \mathscr{T}$.

The identities

$$(A_1 \times B_1) \cap (A_2 \times B_2) = (A_1 \cap A_2) \times (B_1 \cap B_2),$$

$$(A_1 \times B_1) - (A_2 \times B_2) = [(A_1 - A_2) \times B_1] \cup [(A_1 \cap A_2) \times (B_1 - B_2)]$$

show that the intersection of two measurable rectangles is a measurable rectangle and that their difference is the union of two disjoint measurable rectangles, hence is an elementary set. If $P \in \mathscr{E}$ and $Q \in \mathscr{E}$, it follows easily that $P \cap Q \in \mathscr{E}$ and $P - Q \in \mathscr{E}$. Since

$$P \cup Q = (P - Q) \cup Q$$

and $(P - Q) \cap Q = \varnothing$, we also have $P \cup Q \in \mathscr{E}$.

For any set $P \subset X \times Y$, define $\Omega(P)$ to be the class of all $Q \subset X \times Y$ such that $P - Q \in \mathfrak{M}$, $Q - P \in \mathfrak{M}$, and $P \cup Q \in \mathfrak{M}$. The following properties are obvious:

(*a*) $Q \in \Omega(P)$ if and only if $P \in \Omega(Q)$.
(*b*) Since $\mathfrak{M}$ is a monotone class, so is each $\Omega(P)$.

Fix $P \in \mathscr{E}$. Our preceding remarks about $\mathscr{E}$ show that $Q \in \Omega(P)$ for all $Q \in \mathscr{E}$, hence $\mathscr{E} \subset \Omega(P)$, and now (*b*) implies that $\mathfrak{M} \subset \Omega(P)$.

Next, fix $Q \in \mathfrak{M}$. We just saw that $Q \in \Omega(P)$ if $P \in \mathscr{E}$. By (*a*), $P \in \Omega(Q)$, hence $\mathscr{E} \subset \Omega(Q)$, and if we refer to (*b*) once more we obtain $\mathfrak{M} \subset \Omega(Q)$.

Summing up: *If* P and $Q \in \mathfrak{M}$, then $P - Q \in \mathfrak{M}$ and $P \cup Q \in \mathfrak{M}$.

It now follows that $\mathfrak{M}$ is a σ-algebra in $X \times Y$:

(i) $X \times Y \in \mathscr{E}$. Hence $X \times Y \in \mathfrak{M}$.

(ii) If $Q \in \mathfrak{M}$, then $Q^c \in \mathfrak{M}$, since the difference of any two members of $\mathfrak{M}$ is in $\mathfrak{M}$.

(iii) If $P_i \in \mathfrak{M}$ for $i = 1, 2, 3, \ldots$, and $P = \bigcup P_i$, put

$$Q_n = P_1 \cup \cdots \cup P_n.$$

Since $\mathfrak{M}$ is closed under the formation of finite unions, $Q_n \in \mathfrak{M}$.

Since $Q_n \subset Q_{n+1}$ and $P = \bigcup Q_n$, the monotonicity of $\mathfrak{M}$ shows that $P \in \mathfrak{M}$.

Thus $\mathfrak{M}$ is a σ-algebra, $\mathscr{E} \subset \mathfrak{M} \subset \mathscr{S} \times \mathscr{T}$, and (by definition) $\mathscr{S} \times \mathscr{T}$ is the smallest σ-algebra which contains $\mathscr{E}$. Hence $\mathfrak{M} = \mathscr{S} \times \mathscr{T}$. ////

8.4 Definition With each function f on $X \times Y$ and with each $x \in X$ we associate a function f_x defined on Y by $f_x(y) = f(x, y)$.

Similarly, if $y \in Y$, f^y is the function defined on X by $f^y(x) = f(x, y)$.

Since we are now dealing with three σ-algebras, $\mathscr{S}$, $\mathscr{T}$, and $\mathscr{S} \times \mathscr{T}$, we shall, for the sake of clarity, indicate in the sequel to which of these three σ-algebras the word "measurable" refers.

8.5 Theorem *Let f be an $(\mathscr{S} \times \mathscr{T})$-measurable function on $X \times Y$. Then*

(*a*) *For each $x \in X$, f_x is a $\mathscr{T}$-measurable function.*

(*b*) *For each $y \in Y$, f^y is an $\mathscr{S}$-measurable function.*

PROOF For any open set V, put

$$Q = \{(x, y) : f(x, y) \in V\}.$$

Then $Q \in \mathscr{S} \times \mathscr{T}$, and

$$Q_x = \{y : f_x(y) \in V\}.$$

Theorem 8.2 shows that $Q_x \in \mathscr{T}$. This proves (*a*); the proof of (*b*) is similar. ////

Product Measures

8.6 Theorem *Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \lambda)$ be σ-finite measure spaces. Suppose $Q \in \mathscr{S} \times \mathscr{T}$. If*

$$\varphi(x) = \lambda(Q_x), \qquad \psi(y) = \mu(Q^y) \tag{1}$$

for every $x \in X$ and $y \in Y$, then φ is $\mathscr{S}$-measurable, ψ is $\mathscr{T}$-measurable, and

$$\int_X \varphi \, d\mu = \int_Y \psi \, d\lambda, \tag{2}$$

Notes: The assumptions on the measure spaces are, more explicitly, that μ and λ are positive measures on $\mathscr{S}$ and $\mathscr{T}$, respectively, that X is the union of countably many disjoint sets X_n with $\mu(X_n) < \infty$, and that Y is the union of countably many disjoint sets Y_m with $\lambda(Y_m) < \infty$.

Theorem 8.2 shows that the definitions (1) make sense. Since

$$\lambda(Q_x) = \int_Y \chi_Q(x, y) \, d\lambda(y) \qquad (x \in X), \tag{3}$$

with a similar statement for $\mu(Q^y)$, the conclusion (2) can be written in the form

$$\int_X d\mu(x) \int_Y \chi_Q(x, y) \, d\lambda(y) = \int_Y d\lambda(y) \int_X \chi_Q(x, y) \, d\mu(x). \tag{4}$$

PROOF Let Ω be the class of all $Q \in \mathscr{S} \times \mathscr{T}$ for which the conclusion of the theorem holds. We claim that Ω has the following four properties:

(*a*) Every measurable rectangle belongs to Ω.
(*b*) If $Q_1 \subset Q_2 \subset Q_3 \subset \cdots$, if each $Q_i \in \Omega$, and if $Q = \bigcup Q_i$, then $Q \in \Omega$.
(*c*) If $\{Q_i\}$ is a disjoint countable collection of members of Ω, and if $Q = \bigcup Q_i$, then $Q \in \Omega$.
(*d*) If $\mu(A) < \infty$ and $\lambda(B) < \infty$, if $A \times B \supset Q_1 \supset Q_2 \supset Q_3 \supset \cdots$, if $Q = \bigcap Q_i$ and $Q_i \in \Omega$ for $i = 1, 2, 3, \ldots$, then $Q \in \Omega$.

If $Q = A \times B$, where $A \in \mathscr{S}$, $B \in \mathscr{T}$, then

$$\lambda(Q_x) = \lambda(B)\chi_A(x) \quad \text{and} \quad \mu(Q^y) = \mu(A)\chi_B(y), \tag{5}$$

and therefore each of the integrals in (2) is equal to $\mu(A)\lambda(B)$. This gives (*a*).

To prove (*b*), let φ_i and ψ_i be associated with Q_i in the way in which (1) associates φ and ψ with Q. The countable additivity of μ and λ shows that

$$\varphi_i(x) \to \varphi(x), \qquad \psi_i(y) \to \psi(y) \qquad (i \to \infty), \tag{6}$$

the convergence being monotone increasing at every point. Since φ_i and ψ_i are assumed to satisfy the conclusion of the theorem, (*b*) follows from the monotone convergence theorem.

For finite unions of disjoint sets, (*c*) is clear, because the characteristic function of a union of *disjoint* sets is the sum of their characteristic functions. The general case of (*c*) now follows from (*b*).

The proof of (*d*) is like that of (*b*), except that we use the dominated convergence theorem in place of the monotone convergence theorem. This is legitimate, since $\mu(A) < \infty$ and $\lambda(B) < \infty$.

Now define

$$Q_{mn} = Q \cap (X_n \times Y_m) \qquad (m, n = 1, 2, 3, \ldots) \tag{7}$$

and let $\mathfrak{M}$ be the class of all $Q \in \mathscr{S} \times \mathscr{T}$ such that $Q_{mn} \in \Omega$ for all choices of m and n. Then (*b*) and (*d*) show that $\mathfrak{M}$ is a monotone class; (*a*) and (*c*) show that $\mathscr{E} \subset \mathfrak{M}$; and since $\mathfrak{M} \subset \mathscr{S} \times \mathscr{T}$, Theorem 8.3 implies that $\mathfrak{M} = \mathscr{S} \times \mathscr{T}$.

Thus $Q_{mn} \in \Omega$ for every $Q \in \mathscr{S} \times \mathscr{T}$ and for all choices of m and n. Since Q is the union of the sets Q_{mn} and since these sets are disjoint, we conclude from (*c*) that $Q \in \Omega$. This completes the proof. ////

8.7 Definition If $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \lambda)$ are as in Theorem 8.6, and if $Q \in \mathscr{S} \times \mathscr{T}$, we define

$$(\mu \times \lambda)(Q) = \int_X \lambda(Q_x)\, d\mu(x) = \int_Y \mu(Q^y)\, d\lambda(y). \tag{1}$$

The equality of the integrals in (1) is the content of Theorem 8.6. We call $\mu \times \lambda$ the *product* of the measures μ and λ. That $\mu \times \lambda$ is really a measure (i.e., that $\mu \times \lambda$ is countably additive on $\mathscr{S} \times \mathscr{T}$) follows immediately from Theorem 1.27.

Observe also that $\mu \times \lambda$ is σ-finite.

The Fubini Theorem

8.8 Theorem *Let $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \lambda)$ be σ-finite measure spaces, and let f be an $(\mathscr{S} \times \mathscr{T})$-measurable function on $X \times Y$.*

(*a*) *If $0 \leq f \leq \infty$, and if*

$$\varphi(x) = \int_Y f_x\, d\lambda, \qquad \psi(y) = \int_X f^y\, d\mu \qquad (x \in X,\ y \in Y), \tag{1}$$

then φ is $\mathscr{S}$-measurable, ψ is $\mathscr{T}$-measurable, and

$$\int_X \varphi\, d\mu = \int_{X \times Y} f\, d(\mu \times \lambda) = \int_Y \psi\, d\lambda. \tag{2}$$

(*b*) *If f is complex and if*

$$\varphi^*(x) = \int_Y |f|_x\, d\lambda \quad \text{and} \quad \int_X \varphi^*\, d\mu < \infty, \tag{3}$$

then $f \in L^1(\mu \times \lambda)$.

(*c*) *If* $f \in L^1(\mu \times \lambda)$, *then* $f_x \in L^1(\lambda)$ *for almost all* $x \in X$, $f^y \in L^1(\mu)$ *for almost all* $y \in Y$; *the functions* φ *and* ψ, *defined by* (1) *a.e., are in* $L^1(\mu)$ *and* $L^1(\lambda)$, *respectively, and* (2) *holds.*

Notes: The first and last integrals in (2) can also be written in the more usual form

$$\int_X d\mu(x) \int_Y f(x, y)\, d\lambda(y) = \int_Y d\lambda(y) \int_X f(x, y)\, d\mu(x). \tag{4}$$

These are the so-called "iterated integrals" of f. The middle integral in (2) is often referred to as a *double integral.*

The combination of (*b*) and (*c*) gives the following useful result: *If* f *is* $(\mathcal{S} \times \mathcal{T})$*-measurable and if*

$$\int_X d\mu(x) \int_Y |f(x, y)|\, d\lambda(y) < \infty, \tag{5}$$

then the two iterated integrals (4) *are finite and equal.*

In other words, "the order of integration may be reversed" for $(\mathcal{S} \times \mathcal{T})$-measurable functions f whenever $f \geq 0$ and also whenever one of the iterated integrals of $|f|$ is finite.

PROOF We first consider (*a*). By Theorem 8.5, the definitions of φ and ψ make sense. Suppose $Q \in \mathcal{S} \times \mathcal{T}$ and $f = \chi_Q$. By Definition 8.7, (2) is then exactly the conclusion of Theorem 8.6. Hence (*a*) holds for all nonnegative simple $(\mathcal{S} \times \mathcal{T})$-measurable functions s. In the general case, there is a sequence of such functions s_n, such that $0 \leq s_1 \leq s_2 \leq \cdots$ and $s_n(x, y) \to f(x, y)$ at every point of $X \times Y$. If φ_n is associated with s_n in the same way in which φ was associated to f, we have

$$\int_X \varphi_n\, d\mu = \int_{X \times Y} s_n\, d(\mu \times \lambda) \qquad (n = 1, 2, 3, \ldots). \tag{6}$$

The monotone convergence theorem, applied on $(Y, \mathcal{T}, \lambda)$, shows that $\varphi_n(x)$ increases to $\varphi(x)$, for every $x \in X$, as $n \to \infty$. Hence the monotone convergence theorem applies again, to the two integrals in (6), and the first equality (2) is obtained. The second half of (2) follows by interchanging the roles of x and y. This completes (*a*).

If we apply (*a*) to $|f|$, we see that (*b*) is true.

Obviously, it is enough to prove (*c*) for *real* $L^1(\mu \times \lambda)$; the complex case then follows. If f is real, (*a*) applies to f^+ and to f^-. Let φ_1 and φ_2 correspond to f^+ and f^- as φ corresponds to f in (1). Since $f \in L^1(\mu \times \lambda)$ and

$f^+ \leq |f|$, and since (*a*) holds for f^+, we see that $\varphi_1 \in L^1(\mu)$. Similarly, $\varphi_2 \in L^1(\mu)$. Since

$$f_x = (f^+)_x - (f^-)_x \tag{7}$$

we have $f_x \in L^1(\lambda)$ for every x for which $\varphi_1(x) < \infty$ and $\varphi_2(x) < \infty$; since φ_1 and φ_2 are in $L^1(\mu)$, this happens for almost all x; and at any such x, we have $\varphi(x) = \varphi_1(x) - \varphi_2(x)$. Hence $\varphi \in L^1(\mu)$. Now (2) holds with φ_1 and f^+ and with φ_2 and f^-, in place of φ and f; if we subtract the resulting equations, we obtain one half of (*c*). The other half is proved in the same manner, with f_x and ψ in place of f_x and φ. ////

8.9 Counterexamples The following three examples will show that the various hypotheses in Theorems 8.6 and 8.8 cannot be dispensed with.

(*a*) Let $X = Y = [0, 1]$, $\mu = \lambda =$ Lebesgue measure on $[0, 1]$. Choose $\{\delta_n\}$ so that $0 = \delta_1 < \delta_2 < \delta_3 < \cdots$, $\delta_n \to 1$, and let g_n be a real continuous function with support in (δ_n, δ_{n+1}), such that $\int_0^1 g_n(t)\, dt = 1$, for $n = 1, 2, 3, \ldots$. Define

$$f(x, y) = \sum_{n=1}^{\infty} [g_n(x) - g_{n+1}(x)]g_n(y).$$

Note that at each point (x, y) at most one term in this sum is different from 0. Thus no convergence problem arises in the definition of f. An easy computation shows that

$$\int_0^1 dx \int_0^1 f(x, y)\, dy = 1 \neq 0 = \int_0^1 dy \int_0^1 f(x, y)\, dx,$$

so that the conclusion of the Fubini theorem fails, although both iterated integrals exist. Note that f is continuous in this example, except at the point $(1, 1)$, but that

$$\int_0^1 dx \int_0^1 |f(x, y)|\, dy = \infty.$$

(*b*) Let $X = Y = [0, 1]$, $\mu =$ Lebesgue measure on $[0, 1]$, $\lambda =$ counting measure on Y, and put $f(x, y) = 1$ if $x = y$, $f(x, y) = 0$ if $x \neq y$. Then

$$\int_X f(x, y)\, d\mu(x) = 0, \qquad \int_Y f(x, y)\, d\lambda(y) = 1$$

for all x and y in $[0, 1]$, so that

$$\int_Y d\lambda(y) \int_X f(x, y)\, d\mu(x) = 0 \neq 1 = \int_X d\mu(x) \int_Y f(x, y)\, d\lambda(y).$$

This time the failure is due to the fact that λ is not σ-finite.

Observe that our function f *is* $(\mathscr{S} \times \mathscr{T})$-measurable, if $\mathscr{S}$ is the class of all Lebesgue measurable sets in $[0, 1]$ and $\mathscr{T}$ consists of all subsets of $[0, 1]$.

To see this, note that $f = \chi_D$, where D is the diagonal of the unit square. Given n, put

$$I_j = \left[\frac{j-1}{n}, \frac{j}{n}\right]$$

and put

$$Q_n = (I_1 \times I_1) \cup (I_2 \times I_2) \cup \cdots \cup (I_n \times I_n).$$

Then Q_n is a finite union of measurable rectangles, and $D = \bigcap Q_n$.

(c) In examples (*a*) and (*b*), the failure of the Fubini theorem was due to the fact that either the function or the space was "too big." We now turn to the role played by the requirement that f be measurable *with respect to the* σ*-algebra* $\mathcal{S} \times \mathcal{T}$.

To pose the question more precisely, suppose $\mu(X) = \lambda(Y) = 1$, $0 \le f \le 1$ (so that "bigness" is certainly avoided); assume f_x is $\mathcal{T}$-measurable and f^y is $\mathcal{S}$-measurable, for all x and y; and assume φ is $\mathcal{S}$-measurable and ψ is $\mathcal{T}$-measurable, where φ and ψ are defined as in 8.8(1). Then $0 \le \varphi \le 1$, $0 \le \psi \le 1$, and both iterated integrals are finite. (Note that no reference to product measures is needed to *define* iterated integrals.) Does it follow that the two iterated integrals of f are equal?

The (perhaps surprising) answer is no.

In the following example (due to Sierpinski), we take

$$(X, \mathcal{S}, \mu) = (Y, \mathcal{T}, \lambda) = [0, 1]$$

with Lebesgue measure. The construction depends on the continuum hypothesis. It is a consequence of this hypothesis that there is a one-to-one mapping j of the unit interval $[0, 1]$ onto a well-ordered set W such that $j(x)$ has at most countably many predecessors in W, for each $x \in [0, 1]$. Taking this for granted, let Q be the set of all (x, y) in the unit square such that $j(x)$ precedes $j(y)$ in W. For each $x \in [0, 1]$, Q_x contains all but countably many points of $[0, 1]$; for each $y \in [0, 1]$, Q^y contains at most countably many points of $[0, 1]$. If $f = \chi_Q$, it follows that f_x and f^y are Borel measurable and that

$$\varphi(x) = \int_0^1 f(x, y)\, dy = 1, \qquad \psi(y) = \int_0^1 f(x, y)\, dx = 0$$

for all x and y. Hence

$$\int_0^1 dx \int_0^1 f(x, y)\, dy = 1 \neq 0 = \int_0^1 dy \int_0^1 f(x, y)\, dx.$$

Completion of Product Measures

8.10 If $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ are complete measure spaces, it need not be true that $(X \times Y, \mathcal{S} \times \mathcal{T}, \mu \times \lambda)$ is complete. There is nothing pathological about

this phenomenon: Suppose that there exists an $A \in \mathcal{S}$, $A \neq \varnothing$, with $\mu(A) = 0$; and suppose that there exists a $B \subset Y$ so that $B \notin \mathcal{T}$. Then $A \times B \subset A \times Y$, $(\mu \times \lambda)(A \times Y) = 0$, but $A \times B \notin \mathcal{S} \times \mathcal{T}$. (The last assertion follows from Theorem 8.2.)

For instance, if $\mu = \lambda = m_1$ (Lebesgue measure on R^1), let A consist of any one point, and let B be any nonmeasurable set in R^1. Thus $m_1 \times m_1$ is not a complete measure; in particular, $m_1 \times m_1$ is not m_2, since the latter is complete, by its construction. However, m_2 *is the completion of* $m_1 \times m_1$. This result generalizes to arbitrary dimensions:

8.11 Theorem *Let m_k denote Lebesgue measure on R^k. If $k = r + s$, $r \geq 1$, $s \geq 1$, then m_k is the completion of the product measure $m_r \times m_s$.*

PROOF Let $\mathcal{B}_k$ and $\mathfrak{M}_k$ be the σ-algebras of all Borel sets and of all Lebesgue measurable sets in R^k, respectively. We shall first show that

$$\mathcal{B}_k \subset \mathfrak{M}_r \times \mathfrak{M}_s \subset \mathfrak{M}_k. \tag{1}$$

Every k-cell belongs to $\mathfrak{M}_r \times \mathfrak{M}_s$. The σ-algebra generated by the k-cells is $\mathcal{B}_k$. Hence $\mathcal{B}_k \subset \mathfrak{M}_r \times \mathfrak{M}_s$. Next, suppose $E \in \mathfrak{M}_r$ and $F \in \mathfrak{M}_s$. It is easy to see, by Theorem 2.20(*b*), that both $E \times R^s$ and $R^r \times F$ belong to $\mathfrak{M}_k$. The same is true of their intersection $E \times F$. It follows that $\mathfrak{M}_r \times \mathfrak{M}_s \subset \mathfrak{M}_k$.

Choose $Q \in \mathfrak{M}_r \times \mathfrak{M}_s$. Then $Q \in \mathfrak{M}_k$, so there are sets P_1 and $P_2 \in \mathcal{B}_k$ such that $P_1 \subset Q \subset P_2$ and $m_k(P_2 - P_1) = 0$. Both m_k and $m_r \times m_s$ are translation invariant Borel measures on R^k. They assign the same value to each k-cell. Hence they agree on $\mathcal{B}_k$, by Theorem 2.20(*d*). In particular,

$$(m_r \times m_s)(Q - P_1) \leq (m_r \times m_s)(P_2 - P_1) = m_k(P_2 - P_1) = 0$$

and therefore

$$(m_r \times m_s)(Q) = (m_r \times m_s)(P_1) = m_k(P_1) = m_k(Q).$$

So $m_r \times m_s$ agrees with m_k on $\mathfrak{M}_r \times \mathfrak{M}_s$.

It now follows that $\mathfrak{M}_k$ is the $(m_r \times m_s)$-completion of $\mathfrak{M}_r \times \mathfrak{M}_s$, and this is what the theorem asserts. ////

We conclude this section with an alternative statement of Fubini's theorem which is of special interest in view of Theorem 8.11.

8.12 Theorem *Let $(X, \mathcal{S}, \mu)$ and $(Y, \mathcal{T}, \lambda)$ be complete σ-finite measure spaces. Let $(\mathcal{S} \times \mathcal{T})^*$ be the completion of $\mathcal{S} \times \mathcal{T}$, relative to the measure $\mu \times \lambda$. Let f be an $(\mathcal{S} \times \mathcal{T})^*$-measurable function on $X \times Y$. Then all conclusions of Theorem 8.8 hold, the only difference being as follows:*

The $\mathcal{T}$-measurability of f_x can be asserted only for almost all $x \in X$, so that $\varphi(x)$ is only defined a.e. $[\mu]$ by 8.8(1); *a similar statement holds for f^y and ψ.*

The proof depends on the following two lemmas:

Lemma 1 *Suppose v is a positive measure on a σ-algebra $\mathfrak{M}$, $\mathfrak{M}^*$ is the completion of $\mathfrak{M}$ relative to v, and f is an $\mathfrak{M}^*$-measurable function. Then there exists an $\mathfrak{M}$-measurable function g such that $f = g$ a.e. $[v]$.*

(An interesting special case of this arises when v is Lebesgue measure on R^k and $\mathfrak{M}$ is the class of all Borel sets in R^k.)

Lemma 2 *Let h be an $(\mathscr{S} \times \mathscr{T})^*$-measurable function on $X \times Y$ such that $h = 0$ a.e. $[\mu \times \lambda]$. Then for almost all $x \in X$ it is true that $h(x, y) = 0$ for almost all $y \in Y$; in particular, h_x is $\mathscr{T}$-measurable for almost all $x \in X$. A similar statement holds for h^y.*

If we assume the lemmas, the proof of the theorem is immediate: If f is as in the theorem, Lemma 1 (with $v = \mu \times \lambda$) shows that $f = g + h$, where $h = 0$ a.e. $[\mu \times \lambda]$ and g is $(\mathscr{S} \times \mathscr{T})$-measurable. Theorem 8.8 applies to g. Lemma 2 shows that $f_x = g_x$ a.e. $[\lambda]$ for almost all x and that $f^y = g^y$ a.e. $[\mu]$ for almost all y. Hence the two iterated integrals of f, as well as the double integral, are the same as those of g, and the theorem follows.

PROOF OF LEMMA 1 Suppose f is $\mathfrak{M}^*$-measurable and $f \geq 0$. There exist $\mathfrak{M}^*$-measurable simple functions $0 = s_0 \leq s_1 \leq s_2 \leq \cdots$ such that $s_n(x) \to f(x)$ for each $x \in X$, as $n \to \infty$. Hence $f = \sum (s_{n+1} - s_n)$. Since $s_{n+1} - s_n$ is a finite linear combination of characteristic functions, it follows that there are constants $c_i > 0$ and sets $E_i \in \mathfrak{M}^*$ such that

$$f(x) = \sum_{i=1}^{\infty} c_i \chi_{E_i}(x) \qquad (x \in X).$$

The definition of $\mathfrak{M}^*$ (see Theorem 1.36) shows that there are sets $A_i \in \mathfrak{M}$, $B_i \in \mathfrak{M}$, such that $A_i \subset E_i \subset B_i$ and $v(B_i - A_i) = 0$. Define

$$g(x) = \sum_{i=1}^{\infty} c_i \chi_{A_i}(x) \qquad (x \in X).$$

Then the function g is $\mathfrak{M}$-measurable, and $g(x) = f(x)$, except possibly when $x \in \bigcup (E_i - A_i) \subset \bigcup (B_i - A_i)$. Since $v(B_i - A_i) = 0$ for each i, we conclude that $g = f$ a.e. $[v]$. The general case (f real or complex) follows from this. ////

PROOF OF LEMMA 2 Let P be the set of all points in $X \times Y$ at which $h(x, y) \neq 0$. Then $P \in (\mathscr{S} \times \mathscr{T})^*$ and $(\mu \times \lambda)(P) = 0$. Hence there exists a $Q \in \mathscr{S} \times \mathscr{T}$ such that $P \subset Q$ and $(\mu \times \lambda)(Q) = 0$. By Theorem 8.6,

$$\int_X \lambda(Q_x)\, d\mu(x) = 0. \tag{1}$$

Let N be the set of all $x \in X$ at which $\lambda(Q_x) > 0$. It follows from (1) that $\mu(N) = 0$. For every $x \notin N$, $\lambda(Q_x) = 0$. Since $P_x \subset Q_x$ and $(Y, \mathscr{T}, \lambda)$ is a complete measure space, every subset of P_x belongs to $\mathscr{T}$ if $x \notin N$. If $y \notin P_x$, then $h_x(y) = 0$. Thus we see, for every $x \notin N$, that h_x is $\mathscr{T}$-measurable and that $h_x(y) = 0$ a.e. $[\lambda]$. ////

Convolutions

8.13 It happens occasionally that one can prove that a certain set is not empty by proving that it is actually large. The word "large" may of course refer to various properties. One of these (a rather crude one) is cardinality. An example is furnished by the familiar proof that there exist transcendental numbers: There are only countably many algebraic numbers but uncountably many real numbers, hence the set of transcendental real numbers is not empty. Applications of Baire's theorem are based on a topological notion of largeness: The dense G_δ's are "large" subsets of a complete metric space. A third type of largeness is measure-theoretic: One can try to show that a certain set in a measure space is not empty by showing that it has positive measure or, better still, by showing that its complement has measure zero. Fubini's theorem often occurs in this type of argument.

For example, let f and $g \in L^1(R^1)$, assume $f \geq 0$ and $g \geq 0$ for the moment, and consider the integral

$$h(x) = \int_{-\infty}^{\infty} f(x - t)g(t)\, dt \qquad (-\infty < x < \infty). \tag{1}$$

For any fixed x, the integrand in (1) is a measurable function with range in $[0, \infty]$, so that $h(x)$ is certainly well defined by (1), and $0 \leq h(x) \leq \infty$.

But is there any x for which $h(x) < \infty$? Note that the integrand in (1) is, for each fixed x, the product of two members of L^1, and such a product is not always in L^1. [Example: $f(x) = g(x) = 1/\sqrt{x}$ if $0 < x < 1$, 0 otherwise.] The Fubini theorem will give an affirmative answer. In fact, it will show that $h \in L^1(R^1)$, hence that $h(x) < \infty$ a.e.

8.14 Theorem *Suppose $f \in L^1(R^1)$, $g \in L^1(R^1)$. Then*

$$\int_{-\infty}^{\infty} |f(x - y)g(y)|\, dy < \infty \tag{1}$$

for almost all x. For these x, define

$$h(x) = \int_{-\infty}^{\infty} f(x - y)g(y)\, dy. \tag{2}$$

Then $h \in L^1(R^1)$, and

$$\|h\|_1 \leq \|f\|_1 \|g\|_1, \tag{3}$$

where

$$\|f\|_1 = \int_{-\infty}^{\infty} |f(x)|\, dx. \tag{4}$$

We call h the *convolution* of f and g, and write $h = f * g$.

PROOF There exist Borel functions f_0 and g_0 such that $f_0 = f$ a.e. and $g_0 = g$ a.e. The integrals (1) and (2) are unchanged, for every x, if we replace f by f_0 and g by g_0. Hence we may assume, to begin with, that f and g are Borel functions.

To apply Fubini's theorem, we shall first prove that the function F defined by

$$F(x, y) = f(x - y)g(y) \tag{5}$$

is a Borel function on R^2.

Define $\varphi: R^2 \to R^1$ and $\psi: R^2 \to R^1$ by

$$\varphi(x, y) = x - y, \qquad \psi(x, y) = y. \tag{6}$$

Then $f(x - y) = (f \circ \varphi)(x, y)$ and $g(y) = (g \circ \psi)(x, y)$. Since φ and ψ are Borel functions, Theorem 1.12(d) shows that $f \circ \varphi$ and $g \circ \psi$ are Borel functions on R^2. Hence so is their product.

Next we observe that

$$\int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |F(x, y)|\, dx = \int_{-\infty}^{\infty} |g(y)|\, dy \int_{-\infty}^{\infty} |f(x - y)|\, dx = \|f\|_1 \|g\|_1, \tag{7}$$

since

$$\int_{-\infty}^{\infty} |f(x - y)|\, dx = \|f\|_1 \tag{8}$$

for every $y \in R^1$, by the translation-invariance of Lebesgue measure.

Thus $F \in L^1(R^2)$, and Fubini's theorem implies that the integral (2) exists for almost all $x \in R^1$ and that $h \in L^1(R^1)$. Finally,

$$\begin{aligned} \|h\|_1 &= \int_{-\infty}^{\infty} |h(x)|\, dx \le \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} |F(x, y)|\, dy \\ &= \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} |F(x, y)|\, dx = \|f\|_1 \|g\|_1, \end{aligned}$$

by (7). This gives (3), and completes the proof. ////

Convolutions will play an important role in Chap. 9.

Distribution Functions

8.15 Definition Let μ be a σ-finite positive measure on some σ-algebra in some set X. Let $f: X \to [0, \infty]$ be measurable. The function that assigns to each $t \in [0, \infty)$ the number

$$\mu\{f > t\} = \mu(\{x \in X : f(x) > t\}) \tag{1}$$

is called the *distribution function* of f. It is clearly a monotonic (nonincreasing) function of t and is therefore Borel measurable.

One reason for introducing distribution functions is that they make it possible to replace integrals over X by integrals over $[0, \infty)$; the formula

$$\int_X f \, d\mu = \int_0^\infty \mu\{f > t\} \, dt \tag{2}$$

is the special case $\varphi(t) = t$ of our next theorem. This will then be used to derive an L^p-property of the maximal functions that were introduced in Chap. 7.

8.16 Theorem *Suppose that f and μ are as above, that $\varphi: [0, \infty] \to [0, \infty]$ is monotonic, absolutely continuous on $[0, T]$ for every $T < \infty$, and that $\varphi(0) = 0$ and $\varphi(t) \to \varphi(\infty)$ as $t \to \infty$. Then*

$$\int_X (\varphi \circ f) \, d\mu = \int_0^\infty \mu\{f > t\} \varphi'(t) \, dt. \tag{1}$$

PROOF Let E be the set of all $(x, t) \in X \times [0, \infty)$ where $f(x) > t$. When f is simple, then E is a union of finitely many measurable rectangles, and is therefore measurable. In the general case, the measurability of E follows via the standard approximation of f by simple functions (Theorem 1.17). As in Sec. 8.1, put

$$E^t = \{x \in X : (x, t) \in E\} \qquad (0 \le t < \infty). \tag{2}$$

The distribution function of f is then

$$\mu(E^t) = \int_X \chi_E(x, t) \, d\mu(x). \tag{3}$$

The right side of (1) is therefore

$$\int_0^\infty \mu(E^t) \varphi'(t) \, dt = \int_X d\mu(x) \int_0^\infty \chi_E(x, t) \varphi'(t) \, dt, \tag{4}$$

by Fubini's theorem.

For each $x \in X$, $\chi_E(x, t) = 1$ if $t < f(x)$ and is 0 if $t \geq f(x)$. The inner integral in (4) is therefore

$$\int_0^{f(x)} \varphi'(t)\, dt = \varphi(f(x)) \tag{5}$$

by Theorem 7.20. Now (1) follows from (4) and (5). ////

8.17 Recall now that the maximal function Mf lies in weak L^1 when $f \in L^1(R^k)$ (Theorem 7.4). We also have the trivial estimate

$$\|Mf\|_\infty \leq \|f\|_\infty \tag{1}$$

valid for all $f \in L^\infty(R^k)$. A technique invented by Marcinkiewicz makes it possible to "interpolate" between these two extremes and to prove the following theorem of Hardy and Littlewood (which fails when $p = 1$; see Exercise 22, Chap. 7).

8.18 Theorem *If $1 < p < \infty$ and $f \in L^p(R^k)$ then $Mf \in L^p(R^k)$.*

PROOF Since $Mf = M(|f|)$ we may assume, without loss of generality, that $f \geq 0$. Theorem 7.4 shows that there is a constant A, depending only on the dimension k, such that

$$m\{Mg > t\} \leq \frac{A}{t} \|g\|_1 \tag{1}$$

for every $g \in L^1(R^k)$. Here, and in the rest of this proof, $m = m_k$, the Lebesgue measure on R^k.

Pick a constant c, $0 < c < 1$, which will be specified later so as to minimize a certain upper bound. For each $t \in (0, \infty)$, split f into a sum

$$f = g_t + h_t \tag{2}$$

where

$$g_t(x) = \begin{cases} f(x) & \text{if } f(x) > ct \\ 0 & \text{if } f(x) \leq ct. \end{cases} \tag{3}$$

Then $0 \leq h_t(x) \leq ct$ for every $x \in R^k$. Hence $h_t \in L^\infty$, $Mh_t \leq ct$, and

$$Mf \leq Mg_t + Mh_t \leq Mg_t + ct. \tag{4}$$

If $(Mf)(x) > t$ for some x, it follows that

$$(Mg_t)(x) > (1 - c)t. \tag{5}$$

Setting $E_t = \{f > ct\}$, (5), (1), and (3) imply that

$$m\{Mf > t\} \leq m\{Mg_t > (1 - c)t\} \leq \frac{A}{(1 - c)t} \|g_t\|_1 = \frac{A}{(1 - c)t} \int_{E_t} f\, dm.$$

We now use Theorem 8.16, with $X = R^k$, $\mu = m$, $\varphi(t) = t^p$, to calculate

$$\int_{R^k} (Mf)^p \, dm = p \int_0^\infty m\{Mf > t\} t^{p-1} \, dt \leq \frac{Ap}{1-c} \int_0^\infty t^{p-2} \, dt \int_{E_t} f \, dm$$

$$= \frac{Ap}{1-c} \int_{R^k} f \, dm \int_0^{f/c} t^{p-2} \, dt = \frac{Apc^{1-p}}{(1-c)(p-1)} \int_{R^k} f^p \, dm.$$

This proves the theorem. However, to get a good constant, let us choose c so as to minimize that last expression. This happens when $c = (p-1)/p = 1/q$, where q is the exponent conjugate to p. For this c,

$$c^{1-p} = \left(1 + \frac{1}{p-1}\right)^{p-1} < e,$$

and the preceding computation yields

$$\|Mf\|_p \leq C_p \|f\|_p \tag{6}$$

where $C_p = (Aepq)^{1/p}$. ////

Note that $C_p \to 1$ as $p \to \infty$, which agrees with formula 8.17(1), and that $C_p \to \infty$ as $p \to 1$.

Exercises

1 Find a monotone class $\mathfrak{M}$ in R^1 which is not a σ-algebra, even though $R^1 \in \mathfrak{M}$ and $R^1 - A \in \mathfrak{M}$ for every $A \in \mathfrak{M}$.

2 Suppose f is a Lebesgue measurable nonnegative real function on R^1 and $A(f)$ is the *ordinate set* of f. This is the set of all points $(x, y) \in R^2$ for which $0 < y < f(x)$.

(*a*) Is it true that $A(f)$ is Lebesgue measurable, in the two-dimensional sense?
(*b*) If the answer to (*a*) is affirmative, is the integral of f over R^1 equal to the measure of $A(f)$?
(*c*) Is the graph of f a measurable subset of R^2?
(*d*) If the answer to (*c*) is affirmative, is the measure of the graph equal to zero?

3 Find an example of a positive continuous function f in the open unit square in R^2, whose integral (relative to Lebesgue measure) is finite but such that $\varphi(x)$ (in the notation of Theorem 8.8) is infinite for some $x \in (0, 1)$.

4 Suppose $1 \leq p \leq \infty$, $f \in L^1(R^1)$, and $g \in L^p(R^1)$.

(*a*) Imitate the proof of Theorem 8.14 to show that the integral defining $(f * g)(x)$ exists for almost all x, that $f * g \in L^p(R^1)$, and that

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

(*b*) Show that equality can hold in (*a*) if $p = 1$ and if $p = \infty$, and find the conditions under which this happens.

(*c*) Assume $1 < p < \infty$, and equality holds in (*a*). Show that then either $f = 0$ a.e. or $g = 0$ a.e.

(*d*) Assume $1 \leq p \leq \infty$, $\epsilon > 0$, and show that there exist $f \in L^1(R^1)$ and $g \in L^p(R^1)$ such that

$$\|f * g\|_p > (1 - \epsilon)\|f\|_1 \|g\|_p.$$

5 Let M be the Banach space of all complex Borel measures on R^1. The norm in M is $\|\mu\| = |\mu|(R^1)$. Associate to each Borel set $E \subset R^1$ the set

$$E_2 = \{(x, y): x + y \in E\} \subset R^2.$$

If μ and $\lambda \in M$, define their convolution $\mu * \lambda$ to be the set function given by

$$(\mu * \lambda)(E) = (\mu \times \lambda)(E_2)$$

for every Borel set $E \subset R^1$; $\mu \times \lambda$ is as in Definition 8.7.

(*a*) Prove that $\mu * \lambda \in M$ and that $\|\mu * \lambda\| \leq \|\mu\|\ \|\lambda\|$.

(*b*) Prove that $\mu * \lambda$ is the unique $v \in M$ such that

$$\int f\, dv = \iint f(x + y)\, d\mu(x)\, d\lambda(y)$$

for every $f \in C_0(R^1)$. (All integrals extend over R^1.)

(*c*) Prove that convolution in M is commutative, associative, and distributive with respect to addition.

(*d*) Prove the formula

$$(\mu * \lambda)(E) = \int \mu(E - t)\, d\lambda(t)$$

for every μ and $\lambda \in M$ and every Borel set E. Here

$$E - t = \{x - t: x \in E\}.$$

(*e*) Define μ to be *discrete* if μ is concentrated on a countable set; define μ to be *continuous* if $\mu(\{x\}) = 0$ for every point $x \in R^1$; let m be Lebesgue measure on R^1 (note that $m \notin M$). Prove that $\mu * \lambda$ is discrete if both μ and λ are discrete, that $\mu * \lambda$ is continuous if μ is continuous and $\lambda \in M$, and that $\mu * \lambda \ll m$ if $\mu \ll m$.

(*f*) Assume $d\mu = f\, dm$, $d\lambda = g\, dm$, $f \in L^1(R^1)$, and $g \in L^1(R^1)$, and prove that $d(\mu * \lambda) = (f * g)\, dm$.

(*g*) Properties (*a*) and (*c*) show that the Banach space M is what one calls a *commutative Banach algebra*. Show that (*e*) and (*f*) imply that the set of all discrete measures in M is a subalgebra of M, that the continuous measures form an ideal in M, and that the absolutely continuous measures (relative to m) form an ideal in M which is isomorphic (as an algebra) to $L^1(R^1)$.

(*h*) Show that M has a unit, i.e., show that there exists a $\delta \in M$ such that $\delta * \mu = \mu$ for all $\mu \in M$.

(*i*) Only two properties of R^1 have been used in this discussion: R^1 is a commutative group (under addition), and there exists a translation invariant Borel measure m on R^1 which is not identically 0 and which is finite on all compact subsets of R^1. Show that the same results hold if R^1 is replaced by R^k or by T (the unit circle) or by T^k (the k-dimensional torus, the cartesian product of k copies of T), as soon as the definitions are properly formulated.

6 (Polar coordinates in R^k.) Let S_{k-1} be the unit sphere in R^k, i.e., the set of all $u \in R^k$ whose distance from the origin 0 is 1. Show that every $x \in R^k$, except for $x = 0$, has a unique representation of the form $x = ru$, where r is a positive real number and $u \in S_{k-1}$. Thus $R^k - \{0\}$ may be regarded as the cartesian product $(0, \infty) \times S_{k-1}$.

Let m_k be Lebesgue measure on R^k, and define a measure σ_{k-1} on S_{k-1} as follows: If $A \subset S_{k-1}$ and A is a Borel set, let $\tilde{A}$ be the set of all points ru, where $0 < r < 1$ and $u \in A$, and define

$$\sigma_{k-1}(A) = k \cdot m_k(\tilde{A}).$$

Prove that the formula

$$\int_{R^k} f\,dm_k = \int_0^\infty r^{k-1}\,dr \int_{S_{k-1}} f(ru)\,d\sigma_{k-1}(u)$$

is valid for every nonnegative Borel function f on R^k. Check that this coincides with familiar results when $k = 2$ and when $k = 3$.

Suggestion: If $0 < r_1 < r_2$ and if A is an open subset of S_{k-1}, let E be the set of all ru with $r_1 < r < r_2$, $u \in A$, and verify that the formula holds for the characteristic function of E. Pass from these to characteristic functions of Borel sets in R^k.

7 Suppose $(X, \mathscr{S}, \mu)$ and $(Y, \mathscr{T}, \lambda)$ are σ-finite measure spaces, and suppose ψ is a measure on $\mathscr{S} \times \mathscr{T}$ such that

$$\psi(A \times B) = \mu(A)\lambda(B)$$

whenever $A \in \mathscr{S}$ and $B \in \mathscr{T}$. Prove that then $\psi(E) = (\mu \times \lambda)(E)$ for every $E \in \mathscr{S} \times \mathscr{T}$.

8 (*a*) Suppose f is a real function on R^2 such that each section f_x is Borel measurable and each section f^y is continuous. Prove that f is Borel measurable on R^2. Note the contrast between this and Example 8.9(*c*).

(*b*) Suppose g is a real function on R^k which is continuous in each of the k variables separately. More explicitly, for every choice of $x_2, \ldots, x_k$, the mapping $x_1 \to g(x_1, x_2, \ldots, x_k)$ is continuous, etc. Prove that g is a Borel function.

Hint: If $(i-1)/n = a_{i-1} \le x \le a_i = i/n$, put

$$f_n(x, y) = \frac{a_i - x}{a_i - a_{i-1}} f(a_{i-1}, y) + \frac{x - a_{i-1}}{a_i - a_{i-1}} f(a_i, y).$$

9 Suppose E is a dense set in R^1 and f is a real function on R^2 such that (*a*) f_x is Lebesgue measurable for each $x \in E$ and (*b*) f^y is continuous for almost all $y \in R^1$. Prove that f is Lebesgue measurable on R^2.

10 Suppose f is a real function on R^2, f_x is Lebesgue measurable for each x, and f^y is continuous for each y. Suppose $g: R^1 \to R^1$ is Lebesgue measurable, and put $h(y) = f(g(y), y)$. Prove that h is Lebesgue measurable on R^1.

Hint: Define f_n as in Exercise 8, put $h_n(y) = f_n(g(y), y)$, show that each h_n is measurable, and that $h_n(y) \to h(y)$.

11 Let $\mathscr{B}_k$ be the σ-algebra of all Borel sets in R^k. Prove that $\mathscr{B}_{m+n} = \mathscr{B}_m \times \mathscr{B}_n$. This is relevant in Theorem 8.14.

12 Use Fubini's theorem and the relation

$$\frac{1}{x} = \int_0^\infty e^{-xt}\,dt \qquad (x > 0)$$

to prove that

$$\lim_{A \to \infty} \int_0^A \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

13. If μ is a complex measure on a σ-algebra $\mathfrak{M}$, show that every set $E \in \mathfrak{M}$ has a subset A for which

$$|\mu(A)| \ge \frac{1}{\pi}\,|\mu|(E).$$

Suggestion: There is a measurable real function θ so that $d\mu = e^{i\theta}\, d|\mu|$. Let A_α be the subset of E where $\cos(\theta - \alpha) > 0$, show that

$$\operatorname{Re}\,[e^{-i\alpha}\mu(A_\alpha)] = \int_E \cos^+(\theta - \alpha)\, d|\mu|,$$

and integrate with respect to α (as in Lemma 6.3).

Show, by an example, that $1/\pi$ is the best constant in this inequality.

14 Complete the following proof of Hardy's inequality (Chap. 3, Exercise 14): Suppose $f \geq 0$ on $(0, \infty)$, $f \in L^p$, $1 < p < \infty$, and

$$F(x) = \frac{1}{x}\int_0^x f(t)\, dt.$$

Write $xF(x) = \int_0^x f(t)t^\alpha t^{-\alpha}\, dt$, where $0 < \alpha < 1/q$, use Hölder's inequality to get an upper bound for $F(x)^p$, and integrate to obtain

$$\int_0^\infty F^p(x)\, dx \leq (1 - \alpha q)^{1-p}(\alpha p)^{-1}\int_0^\infty f^p(t)\, dt.$$

Show that the best choice of α yields

$$\int_0^\infty F^p(x)\, dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f^p(t)\, dt.$$

15 Put $\varphi(t) = 1 - \cos t$ if $0 \leq t \leq 2\pi$, $\varphi(t) = 0$ for all other real t. For $-\infty < x < \infty$, define

$$f(x) = 1, \qquad g(x) = \varphi'(x), \qquad h(x) = \int_{-\infty}^x \varphi(t)\, dt.$$

Verify the following statements about convolutions of these functions:

(i) $(f * g)(x) = 0$ for all x.

(ii) $(g * h)(x) = (\varphi * \varphi)(x) > 0$ on $(0, 4\pi)$.

(iii) Therefore $(f * g) * h = 0$, whereas $f * (g * h)$ is a positive constant.

But convolution is supposedly associative, by Fubini's theorem (Exercise 5(*c*)). What went wrong?

16 Prove the following analogue of Minkowski's inequality, for $f \geq 0$:

$$\left\{\int\left[\int f(x, y)\, d\lambda(y)\right]^p d\mu(x)\right\}^{1/p} \leq \int\left[\int f^p(x, y)\, d\mu(x)\right]^{1/p} d\lambda(y).$$

Supply the required hypotheses. (Many further developments of this theme may be found in [9].)

CHAPTER

NINE

FOURIER TRANSFORMS

Formal Properties

9.1 Definitions In this chapter we shall depart from the previous notation and use the letter m not for Lebesgue measure on R^1 but for Lebesgue measure divided by $\sqrt{2\pi}$. This convention simplifies the appearance of results such as the inversion theorem and the Plancherel theorem. Accordingly, we shall use the notation

$$\int_{-\infty}^{\infty} f(x)\, dm(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, dx, \tag{1}$$

where dx refers to ordinary Lebesgue measure, and we define

$$\|f\|_p = \left\{ \int_{-\infty}^{\infty} |f(x)|^p\, dm(x) \right\}^{1/p} \qquad (1 \le p < \infty), \tag{2}$$

$$(f * g)(x) = \int_{-\infty}^{\infty} f(x - y) g(y)\, dm(y) \qquad (x \in R^1), \tag{3}$$

and

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-ixt}\, dm(x) \qquad (t \in R^1). \tag{4}$$

Throughout this chapter, we shall write L^p in place of $L^p(R^1)$, and C_0 will denote the space of all continuous functions on R^1 which vanish at infinity.

If $f \in L^1$, the integral (4) is well defined for every real t. The function $\hat{f}$ is called the *Fourier transform* of f. Note that the term "Fourier transform" is also applied to the *mapping* which takes f to $\hat{f}$.

The formal properties which are listed in Theorem 9.2 depend intimately on the translation-invariance of m and on the fact that for each real α the mapping $x \to e^{i\alpha x}$ is a *character* of the additive group R^1. By definition, a function φ is a character of R^1 if $|\varphi(t)| = 1$ and if

$$\varphi(s + t) = \varphi(s)\varphi(t) \tag{5}$$

for all real s and t. In other words, φ is to be a homomorphism of the additive group R^1 into the multiplicative group of the complex numbers of absolute value 1. We shall see later (in the proof of Theorem 9.23) that every continuous character of R^1 is given by an exponential.

9.2 Theorem *Suppose $f \in L^1$, and α and λ are real numbers.*

(*a*) *If $g(x) = f(x)e^{i\alpha x}$, then $\hat{g}(t) = \hat{f}(t - \alpha)$.*
(*b*) *If $g(x) = f(x - \alpha)$, then $\hat{g}(t) = \hat{f}(t)e^{-i\alpha t}$.*
(*c*) *If $g \in L^1$ and $h = f * g$, then $\hat{h}(t) = \hat{f}(t)\hat{g}(t)$.*

Thus the Fourier transform converts multiplication by a character into translation, and vice versa, and it converts convolutions to pointwise products.

(*d*) *If $g(x) = \overline{f(-x)}$, then $\hat{g}(t) = \overline{\hat{f}(t)}$.*
(*e*) *If $g(x) = f(x/\lambda)$ and $\lambda > 0$, then $\hat{g}(t) = \lambda\hat{f}(\lambda t)$.*
(*f*) *If $g(x) = -ixf(x)$ and $g \in L^1$, then $\hat{f}$ is differentiable and $\hat{f}'(t) = \hat{g}(t)$.*

PROOF (*a*), (*b*), (*d*), and (*e*) are proved by direct substitution into formula 9.1(4). The proof of (*c*) is an application of Fubini's theorem (see Theorem 8.14 for the required measurability proof):

$$\begin{aligned}
\hat{h}(t) &= \int_{-\infty}^{\infty} e^{-itx}\, dm(x) \int_{-\infty}^{\infty} f(x - y)g(y)\, dm(y) \\
&= \int_{-\infty}^{\infty} g(y)e^{-ity}\, dm(y) \int_{-\infty}^{\infty} f(x - y)e^{-it(x-y)}\, dm(x) \\
&= \int_{-\infty}^{\infty} g(y)e^{-ity}\, dm(y) \int_{-\infty}^{\infty} f(x)e^{-itx}\, dm(x) \\
&= \hat{g}(t)\hat{f}(t).
\end{aligned}$$

Note how the translation-invariance of m was used.

To prove (*f*), note that

$$\frac{\hat{f}(s) - \hat{f}(t)}{s - t} = \int_{-\infty}^{\infty} f(x)e^{-ixt}\varphi(x, s - t)\, dm(x) \qquad (s \neq t), \tag{1}$$

where $\varphi(x, u) = (e^{-ixu} - 1)/u$. Since $|\varphi(x, u)| \leq |x|$ for all real $u \neq 0$ and since $\varphi(x, u) \to -ix$ as $u \to 0$, the dominated convergence theorem applies to (1), if s tends to t, and we conclude that

$$\hat{f}'(t) = -i \int_{-\infty}^{\infty} xf(x)e^{-ixt}\, dm(x). \tag{2}$$

////

9.3 Remarks

(*a*) In the preceding proof, the appeal to the dominated convergence theorem may seem to be illegitimate since the dominated convergence theorem deals only with *countable* sequences of functions. However, it does enable us to conclude that

$$\lim_{n \to \infty} \frac{\hat{f}(s_n) - \hat{f}(t)}{s_n - t} = -i \int_{-\infty}^{\infty} xf(x)e^{-ixt}\, dm(t)$$

for every sequence $\{s_n\}$ which converges to t, and this says exactly that

$$\lim_{s \to t} \frac{\hat{f}(s) - \hat{f}(t)}{s - t} = -i \int_{-\infty}^{\infty} xf(x)e^{-ixt}\, dm(t).$$

We shall encounter similar situations again, and shall apply convergence theorems to them without further comment.

(*b*) Theorem 9.2(*b*) shows that the Fourier transform of

$$[f(x + \alpha) - f(x)]/\alpha$$

is

$$\hat{f}(t)\, \frac{e^{i\alpha t} - 1}{\alpha}.$$

This suggests that an analogue of Theorem 9.2(*f*) should be true under certain conditions, namely, that the Fourier transform of f' is $it\hat{f}(t)$. If $f \in L^1$, $f' \in L^1$, and if f is the indefinite integral of f', the result is easily established by an integration by parts. We leave this, and some related results, as exercises. The fact that the Fourier transform converts differentiation to multiplication by ti makes the Fourier transform a useful tool in the study of differential equations.

The Inversion Theorem

9.4 We have just seen that certain operations on functions correspond nicely to operations on their Fourier transforms. The usefulness and interest of this correspondence will of course be enhanced if there is a way of returning from the transforms to the functions, that is to say, if there is an inversion formula.

Let us see what such a formula might look like, by analogy with Fourier series. If

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx \qquad (n \in Z), \tag{1}$$

then the inversion formula is

$$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}. \tag{2}$$

We know that (2) holds, in the sense of L^2-convergence, if $f \in L^2(T)$. We also know that (2) does not necessarily hold in the sense of pointwise convergence, even if f is continuous. Suppose now that $f \in L^1(T)$, that $\{c_n\}$ is given by (1), and that

$$\sum_{-\infty}^{\infty} |c_n| < \infty. \tag{3}$$

Put

$$g(x) = \sum_{-\infty}^{\infty} c_n e^{inx}. \tag{4}$$

By (3), the series in (4) converges uniformly (hence g is continuous), and the Fourier coefficients of g are easily computed:

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} g(x)e^{-ikx}\,dx &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\sum_{n=-\infty}^{\infty} c_n e^{inx}\right\}e^{-ikx}\,dx \\ &= \sum_{n=-\infty}^{\infty} c_n \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-k)x}\,dx \\ &= c_k. \end{aligned} \tag{5}$$

Thus f and g have the same Fourier coefficients. This implies $f = g$ a.e., so the Fourier series of f converges to $f(x)$ a.e.

The analogous assumptions in the context of Fourier transforms are that $f \in L^1$ and $\hat{f} \in L^1$, and we might then expect that a formula like

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t)e^{itx}\,dm(t) \tag{6}$$

is valid. Certainly, if $\hat{f} \in L^1$, the right side of (6) is well defined; call it $g(x)$; but if we want to argue as in (5), we run into the integral

$$\int_{-\infty}^{\infty} e^{i(t-s)x}\,dx, \tag{7}$$

which is meaningless as it stands. Thus even under the strong assumption that $\hat{f} \in L^1$, a proof of (6) (which *is* true) has to proceed over a more devious route.

[It should be mentioned that (6) may hold even if $\hat{f} \notin L^1$, if the integral over $(-\infty, \infty)$ is interpreted as the limit, as $A \to \infty$, of integrals over $(-A, A)$. (Analogue: a series may converge without converging absolutely.) We shall not go into this.]

9.5 Theorem *For any function f on R^1 and every $y \in R^1$, let f_y be the translate of f defined by*

$$f_y(x) = f(x - y) \qquad (x \in R^1). \tag{1}$$

If $1 \leq p < \infty$ and if $f \in L^p$, the mapping

$$y \to f_y \tag{2}$$

is a uniformly continuous mapping of R^1 into $L^p(R^1)$.

PROOF Fix $\epsilon > 0$. Since $f \in L^p$, there exists a continuous function g whose support lies in a bounded interval $[-A, A]$, such that

$$\|f - g\|_p < \epsilon$$

(Theorem 3.14). The uniform continuity of g shows that there exists a $\delta \in (0, A)$ such that $|s - t| < \delta$ implies

$$|g(s) - g(t)| < (3A)^{-1/p}\epsilon.$$

If $|s - t| < \delta$, it follows that

$$\int_{-\infty}^{\infty} |g(x - s) - g(x - t)|^p \, dx < (3A)^{-1}\epsilon^p(2A + \delta) < \epsilon^p,$$

so that $\|g_s - g_t\|_p < \epsilon$.

Note that L^p-norms (relative to Lebesgue measure) are translation-invariant: $\|f\|_p = \|f_s\|_p$. Thus

$$\begin{aligned} \|f_s - f_t\|_p &\leq \|f_s - g_s\|_p + \|g_s - g_t\|_p + \|g_t - f_t\|_p \\ &= \|(f - g)_s\|_p + \|g_s - g_t\|_p + \|(g - f)_t\|_p < 3\epsilon \end{aligned}$$

whenever $|s - t| < \delta$. This completes the proof. ////

9.6 Theorem *If $f \in L^1$, then $\hat{f} \in C_0$ and*

$$\|\hat{f}\|_\infty \leq \|f\|_1. \tag{1}$$

PROOF The inequality (1) is obvious from 9.1(4). If $t_n \to t$, then

$$|\hat{f}(t_n) - \hat{f}(t)| \leq \int_{-\infty}^{\infty} |f(x)| \, |e^{-it_n x} - e^{-itx}| \, dm(x). \tag{2}$$

The integrand is bounded by $2|f(x)|$ and tends to 0 for every x, as $n \to \infty$. Hence $\hat{f}(t_n) \to \hat{f}(t)$, by the dominated convergence theorem. Thus $\hat{f}$ is continuous.

Since $e^{\pi i} = -1$, 9.1(4) gives

$$\hat{f}(t) = -\int_{-\infty}^{\infty} f(x)e^{-it(x+\pi/t)}\, dm(x) = -\int_{-\infty}^{\infty} f(x - \pi/t)e^{-itx}\, dm(x). \tag{3}$$

Hence

$$2\hat{f}(t) = \int_{-\infty}^{\infty} \left\{f(x) - f\left(x - \frac{\pi}{t}\right)\right\} e^{-itx}\, dm(x), \tag{4}$$

so that

$$2|\hat{f}(t)| \leq \|f - f_{\pi/t}\|_1, \tag{5}$$

which tends to 0 as $t \to \pm\infty$, by Theorem 9.5. ////

9.7 A Pair of Auxiliary Functions In the proof of the inversion theorem it will be convenient to know a positive function H which has a positive Fourier transform whose integral is easily calculated. Among the many possibilities we choose one which is of interest in connection with harmonic functions in a half plane. (See Exercise 25, Chap. 11.)

Put

$$H(t) = e^{-|t|} \tag{1}$$

and define

$$h_\lambda(x) = \int_{-\infty}^{\infty} H(\lambda t)e^{itx}\, dm(t) \qquad (\lambda > 0). \tag{2}$$

A simple computation gives

$$h_\lambda(x) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{\lambda^2 + x^2} \tag{3}$$

and hence

$$\int_{-\infty}^{\infty} h_\lambda(x)\, dm(x) = 1. \tag{4}$$

Note also that $0 < H(t) \leq 1$ and that $H(\lambda t) \to 1$ as $\lambda \to 0$.

9.8 Proposition *If $f \in L^1$, then*

$$(f * h_\lambda)(x) = \int_{-\infty}^{\infty} H(\lambda t)\hat{f}(t)e^{ixt}\, dm(t).$$

PROOF This is a simple application of Fubini's theorem.

$$\begin{aligned}(f * h_\lambda)(x) &= \int_{-\infty}^{\infty} f(x-y)\,dm(y) \int_{-\infty}^{\infty} H(\lambda t)e^{ity}\,dm(t) \\ &= \int_{-\infty}^{\infty} H(\lambda t)\,dm(t) \int_{-\infty}^{\infty} f(x-y)e^{ity}\,dm(y) \\ &= \int_{-\infty}^{\infty} H(\lambda t)\,dm(t) \int_{-\infty}^{\infty} f(y)e^{it(x-y)}\,dm(y) \\ &= \int_{-\infty}^{\infty} H(\lambda t)\hat{f}(t)e^{itx}\,dm(t). \end{aligned}$$ ////

9.9 Theorem *If $g \in L^\infty$ and g is continuous at a point x, then*

$$\lim_{\lambda \to 0} (g * h_\lambda)(x) = g(x). \tag{1}$$

PROOF On account of 9.7(4), we have

$$\begin{aligned}(g * h_\lambda)(x) - g(x) &= \int_{-\infty}^{\infty} [g(x-y) - g(x)]h_\lambda(y)\,dm(y) \\ &= \int_{-\infty}^{\infty} [g(x-y) - g(x)]\lambda^{-1}h_1\left(\frac{y}{\lambda}\right) dm(y) \\ &= \int_{-\infty}^{\infty} [g(x-\lambda s) - g(x)]h_1(s)\,dm(s). \end{aligned}$$

The last integrand is dominated by $2\|g\|_\infty h_1(s)$ and converges to 0 pointwise for every s, as $\lambda \to 0$. Hence (1) follows from the dominated convergence theorem. ////

9.10 Theorem *If $1 \le p < \infty$ and $f \in L^p$, then*

$$\lim_{\lambda \to 0} \|f * h_\lambda - f\|_p = 0. \tag{1}$$

The cases $p = 1$ and $p = 2$ will be the ones of interest to us, but the general case is no harder to prove.

PROOF Since $h_\lambda \in L^q$, where q is the exponent conjugate to p, $(f * h_\lambda)(x)$ is defined for every x. (In fact, $f * h_\lambda$ is continuous; see Exercise 8.) Because of 9.7(4) we have

$$(f * h_\lambda)(x) - f(x) = \int_{-\infty}^{\infty} [f(x-y) - f(x)]h_\lambda(y)\,dm(y) \tag{2}$$

and Theorem 3.3 gives

$$|(f * h_\lambda)(x) - f(x)|^p \leq \int_{-\infty}^{\infty} |f(x-y) - f(x)|^p h_\lambda(y)\, dm(y). \tag{3}$$

Integrate (3) with respect to x and apply Fubini's theorem:

$$\|f * h_\lambda - f\|_p^p \leq \int_{-\infty}^{\infty} \|f_y - f\|_p^p h_\lambda(y)\, dm(y). \tag{4}$$

If $g(y) = \|f_y - f\|_p^p$, then g is bounded and continuous, by Theorem 9.5, and $g(0) = 0$. Hence the right side of (4) tends to 0 as $\lambda \to 0$, by Theorem 9.9. ////

9.11 The Inversion Theorem *If $f \in L^1$ and $\hat{f} \in L^1$, and if*

$$g(x) = \int_{-\infty}^{\infty} \hat{f}(t) e^{ixt}\, dm(t) \qquad (x \in R^1), \tag{1}$$

then $g \in C_0$ and $f(x) = g(x)$ a.e.

PROOF By Proposition 9.8,

$$(f * h_\lambda)(x) = \int_{-\infty}^{\infty} H(\lambda t) \hat{f}(t) e^{ixt}\, dm(t). \tag{2}$$

The integrands on the right side of (2) are bounded by $|\hat{f}(t)|$, and since $H(\lambda t) \to 1$ as $\lambda \to 0$, the right side of (2) converges to $g(x)$, for every $x \in R^1$, by the dominated convergence theorem.

If we combine Theorems 9.10 and 3.12, we see that there is a sequence $\{\lambda_n\}$ such that $\lambda_n \to 0$ and

$$\lim_{n \to \infty} (f * h_{\lambda_n})(x) = f(x) \text{ a.e.} \tag{3}$$

Hence $f(x) = g(x)$ a.e. That $g \in C_0$ follows from Theorem 9.6. ////

9.12 The Uniqueness Theorem *If $f \in L^1$ and $\hat{f}(t) = 0$ for all $t \in R^1$, then $f(x) = 0$ a.e.*

PROOF Since $\hat{f} = 0$ we have $\hat{f} \in L^1$, and the result follows from the inversion theorem. ////

The Plancherel Theorem

Since the Lebesgue measure of R^1 is infinite, L^2 is not a subset of L^1, and the definition of the Fourier transform by formula 9.1(4) is therefore not directly applicable to every $f \in L^2$. The definition does apply, however, if $f \in L^1 \cap L^2$, and it turns out that then $\hat{f} \in L^2$. In fact, $\|\hat{f}\|_2 = \|f\|_2$! This isometry of $L^1 \cap L^2$ into L^2 extends to an isometry of L^2 onto L^2, and this extension defines the Fourier

transform (sometimes called the *Plancherel transform*) of every $f \in L^2$. The resulting L^2-theory has in fact a great deal more symmetry than is the case in L^1. In L^2, f and $\hat{f}$ play exactly the same role.

9.13 Theorem *One can associate to each $f \in L^2$ a function $\hat{f} \in L^2$ so that the following properties hold:*

(*a*) *If $f \in L^1 \cap L^2$, then $\hat{f}$ is the previously defined Fourier transform of f.*
(*b*) *For every $f \in L^2$, $\|\hat{f}\|_2 = \|f\|_2$.*
(*c*) *The mapping $f \to \hat{f}$ is a Hilbert space isomorphism of L^2 onto L^2.*
(*d*) *The following symmetric relation exists between f and $\hat{f}$: If*

$$\varphi_A(t) = \int_{-A}^{A} f(x)e^{-ixt}\, dm(x) \quad \text{and} \quad \psi_A(x) = \int_{-A}^{A} \hat{f}(t)e^{ixt}\, dm(t),$$

then $\|\varphi_A - \hat{f}\|_2 \to 0$ and $\|\psi_A - f\|_2 \to 0$ as $A \to \infty$.

Note: Since $L^1 \cap L^2$ is dense in L^2, properties (*a*) and (*b*) determine the mapping $f \to \hat{f}$ uniquely. Property (*d*) may be called the *L^2 inversion theorem.*

PROOF Our first objective is the relation

$$\|\hat{f}\|_2 = \|f\|_2 \qquad (f \in L^1 \cap L^2). \tag{1}$$

We fix $f \in L^1 \cap L^2$, put $\tilde{f}(x) = \overline{f(-x)}$, and define $g = f * \tilde{f}$. Then

$$g(x) = \int_{-\infty}^{\infty} f(x-y)\overline{f(-y)}\, dm(y) = \int_{-\infty}^{\infty} f(x+y)\overline{f(y)}\, dm(y), \tag{2}$$

or

$$g(x) = (f_{-x}, f), \tag{3}$$

where the inner product is taken in the Hilbert space L^2 and f_{-x} denotes a translate of f, as in Theorem 9.5. By that theorem, $x \to f_{-x}$ is a continuous mapping of R^1 into L^2, and the continuity of the inner product (Theorem 4.6) therefore implies that g is a continuous function. The Schwarz inequality shows that

$$|g(x)| \leq \|f_{-x}\|_2 \|f\|_2 = \|f\|_2^2 \tag{4}$$

so that g is bounded. Also, $g \in L^1$ since $f \in L^1$ and $\tilde{f} \in L^1$.

Since $g \in L^1$, we may apply Proposition 9.8:

$$(g * h_\lambda)(0) = \int_{-\infty}^{\infty} H(\lambda t)\hat{g}(t)\, dm(t). \tag{5}$$

Since g is continuous and bounded, Theorem 9.9 shows that

$$\lim_{\lambda \to 0} (g * h_\lambda)(0) = g(0) = \|f\|_2^2. \tag{6}$$

Theorem 9.2(*d*) shows that $\hat{g} = |\hat{f}|^2 \geq 0$, and since $H(\lambda t)$ increases to 1 as $\lambda \to 0$, the monotone convergence theorem gives

$$\lim_{\lambda \to 0} \int_{-\infty}^{\infty} H(\lambda t)\hat{g}(t)\, dm(t) = \int_{-\infty}^{\infty} |\hat{f}(t)|^2\, dm(t). \tag{7}$$

Now (5), (6), and (7) shows that $f \in L^2$ and that (1) holds.

This was the crux of the proof.

Let Y be the space of all Fourier transforms $\hat{f}$ of functions $f \in L^1 \cap L^2$. By (1), $Y \subset L^2$. We claim that Y is dense in L^2, i.e., that $Y^\perp = \{0\}$.

The functions $x \to e^{i\alpha x}H(\lambda x)$ are in $L^1 \cap L^2$, for all real α and all $\lambda > 0$. Their Fourier transforms

$$h_\lambda(\alpha - t) = \int_{-\infty}^{\infty} e^{i\alpha x}H(\lambda x)e^{-ixt}\, dm(x) \tag{8}$$

are therefore in Y. If $w \in L^2$, $w \in Y^\perp$, it follows that

$$(h_\lambda * \bar{w})(\alpha) = \int_{-\infty}^{\infty} h_\lambda(\alpha - t)\bar{w}(t)\, dm(t) = 0 \tag{9}$$

for all α. Hence $w = 0$, by Theorem 9.10, and therefore Y is dense in L^2.

Let us introduce the temporary notation Φf for $\hat{f}$. From what has been proved so far, we see that Φ is an L^2-isometry from one dense subspace of L^2, namely $L^1 \cap L^2$, onto another, namely Y. Elementary Cauchy sequence arguments (compare with Lemma 4.16) imply therefore that Φ extends to an isometry $\tilde{\Phi}$ of L^2 onto L^2. If we write $\hat{f}$ for $\tilde{\Phi}f$, we obtain properties (*a*) and (*b*).

Property (*c*) follows from (*b*), as in the proof of Theorem 4.18. The *Parseval formula*

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dm(x) = \int_{-\infty}^{\infty} \hat{f}(t)\overline{\hat{g}(t)}\, dm(t) \tag{10}$$

holds therefore for all $f \in L^2$ and $g \in L^2$.

To prove (*d*), let k_A be the characteristic function of $[-A, A]$. Then $k_A f \in L^1 \cap L^2$ if $f \in L^2$, and

$$\varphi_A = (k_A f)\hat{}. \tag{11}$$

Since $\|f - k_A f\|_2 \to 0$ as $A \to \infty$, it follows from (*b*) that

$$\|\hat{f} - \varphi_A\|_2 = \|(f - k_A f)\hat{}\,\|_2 \to 0 \tag{12}$$

as $A \to \infty$.

The other half of (*d*) is proved the same way. ////

9.14 Theorem *If $f \in L^2$ and $\hat{f} \in L^1$, then*

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(t)e^{ixt}\, dm(t) \qquad a.e.$$

PROOF This is corollary of Theorem 9.13(*d*). ////

9.15 Remark If $f \in L^1$, formula 9.1(4) defines $\hat{f}(t)$ unambiguously for every t. If $f \in L^2$, the Plancherel theorem defines $\hat{f}$ uniquely as an element of the Hilbert space L^2, but as a point function $\hat{f}(t)$ is only determined almost everywhere. This is an important difference between the theory of Fourier transforms in L^1 and in L^2. The indeterminacy of $\hat{f}(t)$ as a point function will cause some difficulties in the problem to which we now turn.

9.16 Translation-Invariant Subspaces of L^2 A subspace M of L^2 is said to be *translation-invariant* if $f \in M$ implies that $f_\alpha \in M$ for every real α, where $f_\alpha(x) = f(x - \alpha)$. Translations have already played an important part in our study of Fourier transforms. We now pose a problem whose solution will afford an illustration of how the Plancherel theorem can be used. (Other applications will occur in Chap. 19.) The problem is:

Describe the closed translation-invariant subspaces of L^2.

Let M be a closed translation-invariant subspace of L^2, and let $\hat{M}$ be the image of M under the Fourier transform. Then $\hat{M}$ is closed (since the Fourier transform is an L^2-isometry). If f_α is a translate of f, the Fourier transform of f_α is $\hat{f}e_\alpha$, where $e_\alpha(t) = e^{-i\alpha t}$; we proved this for $f \in L^1$ in Theorem 9.2; the result extends to L^2, as can be seen from Theorem 9.13(*d*). *It follows that $\hat{M}$ is invariant under multiplication by e_α, for all $\alpha \in R^1$.*

Let E be any measurable set in R^1. If $\hat{M}$ is the set of all $\varphi \in L^2$ which vanish a.e. on E, then $\hat{M}$ certainly is a subspace of L^2, which is invariant under multiplication by all e_α (note that $|e_\alpha| = 1$, so $\varphi e_\alpha \in L^2$ if $\varphi \in L^2$), and $\hat{M}$ is also closed. *Proof*: $\varphi \in \hat{M}$ if and only if φ is orthogonal to every $\psi \in L^2$ which vanishes a.e. on the complement of E.

If M is the inverse image of this $\hat{M}$, under the Fourier transform, then M is a space with the desired properties.

One may now conjecture that every one of our spaces M is obtained in this manner, from a set $E \subset R^1$. To prove this, we have to show that to every closed translation-invariant $M \subset L^2$ there corresponds a set $E \subset R^1$ such that $f \in M$ if and only if $\hat{f}(t) = 0$ a.e. on E. The obvious way of constructing E from M is to associate with each $f \in M$ the set E_f consisting of all points at which $\hat{f}(t) = 0$, and to define E as the intersection of these sets E_f. But this obvious attack runs into a serious difficulty: Each E_f is defined only up to sets of measure 0. If $\{A_i\}$ is a countable collection of sets, each determined up to sets of measure 0, then $\bigcap A_i$ is also determined up to sets of measure 0. But there are uncountably many $f \in M$, so we lose all control over $\bigcap E_f$.

This difficulty disappears entirely if we think of our functions as elements of the Hilbert space L^2, and not primarily as point functions.

We shall now prove the conjecture. Let $\hat{M}$ be the image of a closed translation-invariant subspace $M \subset L^2$, under the Fourier transform. Let P be the

orthogonal projection of L^2 onto $\hat{M}$ (Theorem 4.11): To each $f \in L^2$ there corresponds a unique $Pf \in \hat{M}$ such that $f - Pf$ is orthogonal to $\hat{M}$. Hence

$$f - Pf \perp Pg \qquad (f \text{ and } g \in L^2) \tag{1}$$

and since $\hat{M}$ is invariant under multiplication by e_α, we also have

$$f - Pf \perp (Pg)e_\alpha \qquad (f \text{ and } g \in L^2, \alpha \in R^1). \tag{2}$$

If we recall how the inner product is defined in L^2, we see that (2) is equivalent to

$$\int_{-\infty}^{\infty} (f - Pf) \cdot \overline{Pg} \cdot e_{-\alpha}\, dm = 0 \qquad (f \text{ and } g \in L^2, \alpha \in R^1) \tag{3}$$

and this says that the Fourier transform of

$$(f - Pf) \cdot \overline{Pg} \tag{4}$$

is 0. The function (4) is the product of two L^2-functions, hence is in L^1, and the uniqueness theorem for Fourier transforms shows now that the function (4) is 0 a.e. This remains true if $\overline{Pg}$ is replaced by Pg. Hence

$$f \cdot Pg = (Pf) \cdot (Pg) \qquad (f \text{ and } g \in L^2). \tag{5}$$

Interchanging the roles of f and g leads from (5) to

$$f \cdot Pg = g \cdot Pf \qquad (f \text{ and } g \in L^2). \tag{6}$$

Now let g be a fixed positive function in L^2; for instance, put $g(t) = e^{-|t|}$. Define

$$\varphi(t) = \frac{(Pg)(t)}{g(t)}. \tag{7}$$

$(Pg)(t)$ may only be defined a.e.; choose any one determination in (7). Now (6) becomes

$$Pf = \varphi \cdot f \qquad (f \in L^2). \tag{8}$$

If $f \in \hat{M}$, then $Pf = f$. This says that $P^2 = P$, and it fcllows that $\varphi^2 = \varphi$, because

$$\varphi^2 \cdot g = \varphi \cdot Pg = P^2 g = Pg = \varphi \cdot g. \tag{9}$$

Since $\varphi^2 = \varphi$, we have $\varphi = 0$ or 1 a.e., and if we let E be the set of all t where $\varphi(t) = 0$, then $\hat{M}$ consists precisely of those $f \in L^2$ which are 0 a.e. on E, since $f \in \hat{M}$ if and only if $f = Pf = \varphi \cdot f$.

We therefore obtain the following solution to our problem.

9.17 Theorem *Associate to each measurable set $E \subset R^1$ the space M_E of all $f \in L^2$ such that $\hat{f} = 0$ a.e. on E. Then M_E is a closed translation-invariant subspace of L^2. Every closed translation-invariant subspace of L^2 is M_E for some E, and $M_A = M_B$ if and only if*

$$m((A - B) \cup (B - A)) = 0.$$

The uniqueness statement is easily proved; we leave the details to the reader.

The above problem can of course be posed in other function spaces. It has been studied in great detail in L^1. The known results show that the situation is infinitely more complicated there than in L^2.

The Banach Algebra L^1

9.18 Definition A Banach space A is said to be a *Banach algebra* if there is a multiplication defined in A which satisfies the inequality

$$\|xy\| \leq \|x\| \, \|y\| \qquad (x \text{ and } y \in A), \tag{1}$$

the associative law $x(yz) = (xy)z$, the distributive laws

$$x(y + z) = xy + xz, \quad (y + z)x = yx + zx \quad (x, y, \text{ and } z \in A), \tag{2}$$

and the relation

$$(\alpha x)y = x(\alpha y) = \alpha(xy) \tag{3}$$

where α is any scalar.

9.19 Examples

(*a*) Let $A = C(X)$, where X is a compact Hausdorff space, with the supremum norm and the usual pointwise multiplication of functions: $(fg)(x) = f(x)g(x)$. This is a commutative Banach algebra ($fg = gf$) with unit (the constant function 1).

(*b*) $C_0(R^1)$ is a commutative Banach algebra without unit, i.e., without an element u such that $uf = f$ for all $f \in C_0(R^1)$.

(*c*) The set of all linear operators on R^k (or on any Banach space), with the operator norm as in Definition 5.3, and with addition and multiplication defined by

$$(A + B)(x) = Ax + Bx, \quad (AB)x = A(Bx),$$

is a Banach algebra with unit which is not commutative when $k > 1$.

(*d*) L^1 is a Banach algebra if we define multiplication by convolution; since

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1,$$

the norm inequality is satisfied. The associative law could be verified directly (an application of Fubini's theorem), but we can proceed as

follows: We know that the Fourier transform of $f * g$ is $\hat{f} \cdot \hat{g}$, and we know that the mapping $f \to \hat{f}$ is one-to-one. For every $t \in R^1$,

$$\hat{f}(t)[\hat{g}(t)\hat{h}(t)] = [\hat{f}(t)\hat{g}(t)]\hat{h}(t),$$

by the associative law for complex numbers. It follows that

$$f * (g * h) = (f * g) * h.$$

In the same way we see immediately that $f * g = g * f$. The remaining requirements of Definition 9.18 are also easily seen to hold in L^1.

Thus L^1 is a commutative Banach algebra. The Fourier transform is an algebra isomorphism of L^1 into C_0. Hence there is no $f \in L^1$ with $\hat{f} \equiv 1$, and therefore L^1 has no unit.

9.20 Complex Homomorphisms The most important complex functions on a Banach algebra A are the homomorphisms of A into the complex field. These are precisely the linear functionals which also preserve multiplication, i.e., the functions φ such that

$$\varphi(\alpha x + \beta y) = \alpha\varphi(x) + \beta\varphi(y), \qquad \varphi(xy) = \varphi(x)\varphi(y)$$

for all x and $y \in A$ and all scalars α and β. Note that no boundedness assumption is made in this definition. It is a very interesting fact that this would be redundant:

9.21 Theorem *If φ is a complex homomorphism on a Banach algebra A, then the norm of φ, as a linear functional, is at most* 1.

PROOF Assume, to get a contradiction, that $|\varphi(x_0)| > \|x_0\|$ for some $x_0 \in A$. Put $\lambda = \varphi(x_0)$, and put $x = x_0/\lambda$. Then $\|x\| < 1$ and $\varphi(x) = 1$.

Since $\|x^n\| \leq \|x\|^n$ and $\|x\| < 1$, the elements

$$s_n = -x - x^2 - \cdots - x^n \tag{1}$$

form a Cauchy sequence in A. Since A is complete, being a Banach space, there exists a $y \in A$ such that $\|y - s_n\| \to 0$, and it is easily seen that $x + s_n = xs_{n-1}$, so that

$$x + y = xy. \tag{2}$$

Hence $\varphi(x) + \varphi(y) = \varphi(x)\varphi(y)$, which is impossible since $\varphi(x) = 1$. ////

9.22 The Complex Homomorphisms of L^1 Suppose φ is a complex homomorphism of L^1, i.e., a linear functional (of norm at most 1, by Theorem 9.21) which also satisfies the relation

$$\varphi(f * g) = \varphi(f)\varphi(g) \qquad (f \text{ and } g \in L^1). \tag{1}$$

By Theorem 6.16, there exists a $\beta \in L^\infty$ such that

$$\varphi(f) = \int_{-\infty}^{\infty} f(x)\beta(x)\, dm(x) \qquad (f \in L^1). \tag{2}$$

We now exploit the relation (1) to see what else we can say about β. On the one hand,

$$\begin{aligned}
\varphi(f * g) &= \int_{-\infty}^{\infty} (f * g)(x)\beta(x)\, dm(x) \\
&= \int_{-\infty}^{\infty} \beta(x)\, dm(x) \int_{-\infty}^{\infty} f(x-y)g(y)\, dm(y) \\
&= \int_{-\infty}^{\infty} g(y)\, dm(y) \int_{-\infty}^{\infty} f_y(x)\beta(x)\, dm(x) \\
&= \int_{-\infty}^{\infty} g(y)\varphi(f_y)\, dm(y).
\end{aligned} \tag{3}$$

On the other hand,

$$\varphi(f)\varphi(g) = \varphi(f) \int_{-\infty}^{\infty} g(y)\beta(y)\, dm(y). \tag{4}$$

Let us now assume that φ is not identically 0. Fix $f \in L^1$ so that $\varphi(f) \neq 0$. Since the last integral in (3) is equal to the right side of (4) for every $g \in L^1$, the uniqueness assertion of Theorem 6.16 shows that

$$\varphi(f)\beta(y) = \varphi(f_y) \tag{5}$$

for almost all y. But $y \to f_y$ is a continuous mapping of R^1 into L^1 (Theorem 9.5) and φ is continuous on L^1. Hence the right side of (5) is a continuous function of y, and we may assume [by changing $\beta(y)$ on a set of measure 0 if necessary, which does not affect (2)] that β is continuous. If we replace y by $x + y$ and then f by f_x in (5), we obtain

$$\varphi(f)\beta(x+y) = \varphi(f_{x+y}) = \varphi((f_x)_y) = \varphi(f_x)\beta(y) = \varphi(f)\beta(x)\beta(y),$$

so that

$$\beta(x+y) = \beta(x)\beta(y) \qquad (x \text{ and } y \in R^1). \tag{6}$$

Since β is not identically 0, (6) implies that $\beta(0) = 1$, and the continuity of β shows that there is a $\delta > 0$ such that

$$\int_0^{\delta} \beta(y)\, dy = c \neq 0. \tag{7}$$

Then

$$c\beta(x) = \int_0^\delta \beta(y)\beta(x)\,dy = \int_0^\delta \beta(y+x)\,dy = \int_x^{x+\delta} \beta(y)\,dy. \tag{8}$$

Since β is continuous, the last integral is a differentiable function of x; hence (8) shows that β is differentiable. Differentiate (6) with respect to y, then put $y = 0$; the result is

$$\beta'(x) = A\beta(x), \qquad A = \beta'(0). \tag{9}$$

Hence the derivative of $\beta(x)e^{-Ax}$ is 0, and since $\beta(0) = 1$, we obtain

$$\beta(x) = e^{Ax}. \tag{10}$$

But β is bounded on R^1. Therefore A must be pure imaginary, and we conclude: There exists a $t \in R^1$ such that

$$\beta(x) = e^{-itx}. \tag{11}$$

We have thus arrived at the Fourier transform.

9.23 Theorem *To every complex homomorphism φ on L^1 (except to $\varphi = 0$) there corresponds a unique $t \in R^1$ such that $\varphi(f) = \hat{f}(t)$.*

The existence of t was proved above. The uniqueness follows from the observation that if $t \neq s$ then there exists an $f \in L^1$ such that $\hat{f}(t) \neq \hat{f}(s)$; take for $f(x)$ a suitable translate of $e^{-|x|}$.

Exercises

1 Suppose $f \in L^1, f > 0$. Prove that $|\hat{f}(y)| < \hat{f}(0)$ for every $y \neq 0$.

2 Compute the Fourier transform of the characteristic function of an interval. For $n = 1, 2, 3, \ldots$, let g_n be the characteristic function of $[-n, n]$, let h be the characteristic function of $[-1, 1]$, and compute $g_n * h$ explicitly. (The graph is piecewise linear.) Show that $g_n * h$ is the Fourier transform of a function $f_n \in L^1$; except for a multiplicative constant,

$$f_n(x) = \frac{\sin x \sin nx}{x^2}$$

Show that $\|f_n\|_1 \to \infty$ and conclude that the mapping $f \to \hat{f}$ maps L^1 into a *proper* subset of C_0.

Show, however, that the range of this mapping is dense in C_0.

3 Find

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{\sin \lambda t}{t} e^{itx}\,dt \qquad (-\infty < x < \infty)$$

where λ is a positive constant.

4 Give examples of $f \in L^2$ such that $f \notin L^1$ but $\hat{f} \in L^1$. Under what circumstances can this happen?

5 If $f \in L^1$ and $\int |t\hat{f}(t)|\, dm(t) < \infty$, prove that f coincides a.e. with a differentiable function whose derivative is

$$i \int_{-\infty}^{\infty} t\hat{f}(t) e^{ixt}\, dm(t).$$

6 Suppose $f \in L^1$, f is differentiable almost everywhere, and $f' \in L^1$. Does it follow that the Fourier transform of f' is $ti\hat{f}(t)$?

7 Let S be the class of all functions f on R^1 which have the following property: f is infinitely differentiable, and there are numbers $A_{mn}(f) < \infty$, for m and $n = 0, 1, 2, \ldots$, such that

$$|x^n D^m f(x)| \leq A_{mn}(f) \qquad (x \in R^1).$$

Here D is the ordinary differentiation operator.

Prove that the Fourier transform maps S onto S. Find examples of members of S.

8 If p and q are conjugate exponents, $f \in L^p$, $g \in L^q$, and $h = f * g$, prove that h is uniformly continuous. If also $1 < p < \infty$, then $h \in C_0$; show that this fails for some $f \in L^1$, $g \in L^\infty$.

9 Suppose $1 \leq p < \infty$, $f \in L^p$, and

$$g(x) = \int_x^{x+1} f(t)\, dt.$$

Prove that $g \in C_0$. What can you say about g if $f \in L^\infty$?

10 Let C^∞ be the class of all infinitely differentiable complex functions on R^1, and let C_c^∞ consist of all $g \in C^\infty$ whose supports are compact. Show that C_c^∞ does not consist of 0 alone.

Let L^1_{loc} be the class of all f which belong to L^1 locally; that is, $f \in L^1_{\text{loc}}$ provided that f is measurable and $\int_I |f| < \infty$ for every bounded interval I.

If $f \in L^1_{\text{loc}}$ and $g \in C_c^\infty$, prove that $f * g \in C^\infty$.

Prove that there are sequences $\{g_n\}$ in C_c^∞ such that

$$\|f * g_n - f\|_1 \to 0$$

as $n \to \infty$, for every $f \in L^1$. (Compare Theorem 9.10.) Prove that $\{g_n\}$ can also be so chosen that $(f * g_n)(x) \to f(x)$ a.e., for every $f \in L^1_{\text{loc}}$; in fact, for suitable $\{g_n\}$ the convergence occurs at every point x at which f is the derivative of its indefinite integral.

Prove that $(f * h_\lambda)(x) \to f(x)$ a.e. if $f \in L^1$, as $\lambda \to 0$, and that $f * h_\lambda \in C^\infty$, although h_λ does not have compact support. (h_λ is defined in Sec. 9.7.)

11 Find conditions on f and/or $\hat{f}$ which ensure the correctness of the following formal argument: If

$$\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-itx}\, dx$$

and

$$F(x) = \sum_{k=-\infty}^{\infty} f(x + 2k\pi)$$

then F is periodic, with period 2π, the nth Fourier coefficient of F is $\varphi(n)$, hence $F(x) = \sum \varphi(n) e^{inx}$. In particular,

$$\sum_{k=-\infty}^{\infty} f(2k\pi) = \sum_{n=-\infty}^{\infty} \varphi(n).$$

More generally,

$$\sum_{k=-\infty}^{\infty} f(k\beta) = \alpha \sum_{n=-\infty}^{\infty} \varphi(n\alpha) \qquad \text{if } \alpha > 0,\ \beta > 0,\ \alpha\beta = 2\pi. \tag{*}$$

What does (*) say about the limit, as $\alpha \to 0$, of the right-hand side (for "nice" functions, of course)? Is this in agreement with the inversion theorem?

[(*) is known as the Poisson summation formula.]

12 Take $f(x) = e^{-|x|}$ in Exercise 11 and derive the identity

$$\frac{e^{2\pi\alpha} + 1}{e^{2\pi\alpha} - 1} = \frac{1}{\pi} \sum_{n=-\infty}^{\infty} \frac{\alpha}{\alpha^2 + n^2}.$$

13 If $0 < c < \infty$, define $f_c(x) = \exp(-cx^2)$.

(a) Compute $\hat{f}_c$. *Hint*: If $\varphi = \hat{f}_c$, an integration by parts gives $2c\varphi'(t) + t\varphi(t) = 0$.

(b) Show that there is one (and only one) c for which $\hat{f}_c = f_c$.

(c) Show that $f_a * f_b = \gamma f_c$; find γ and c explicitly in terms of a and b.

(d) Take $f = f_c$ in Exercise 11. What is the resulting identity?

14 The Fourier transform can be defined for $f \in L^1(R^k)$ by

$$\hat{f}(y) = \int_{R^k} f(x) e^{-ix \cdot y}\, dm_k(x) \qquad (y \in R^k),$$

where $x \cdot y = \sum \xi_i \eta_i$ if $x = (\xi_1, \ldots, \xi_k)$, $y = (\eta_1, \ldots, \eta_k)$, and m_k is Lebesgue measure on R^k, divided by $(2\pi)^{k/2}$ for convenience. Prove the inversion theorem and the Plancherel theorem in this context, as well as the analogue of Theorem 9.23.

15 If $f \in L^1(R^k)$, A is a linear operator on R^k, and $g(x) = f(Ax)$, how is $\hat{g}$ related to $\hat{f}$? If f is invariant under rotations, i.e., if $f(x)$ depends only on the euclidean distance of x from the origin, prove that the same is true of $\hat{f}$.

16 The *Laplacian* of a function f on R^k is

$$\Delta f = \sum_{j=1}^{k} \frac{\partial^2 f}{\partial x_j^2},$$

provided the partial derivatives exist. What is the relation between $\hat{f}$ and $\hat{g}$ if $g = \Delta f$ and all necessary integrability conditions are satisfied? It is clear that the Laplacian commutes with translations. Prove that it also commutes with rotations, i.e., that

$$\Delta(f \circ A) = (\Delta f) \circ A$$

whenever f has continuous second derivatives and A is a rotation of R^k. (Show that it is enough to do this under the additional assumption that f has compact support.)

17 Show that every Lebesgue measurable character of R^1 is continuous. Do the same for R^k. (Adapt part of the proof of Theorem 9.23.) Compare with Exercise 18.

18 Show (with the aid of the Hausdorff maximality theorem) that there exist real *discontinuous* functions f on R^1 such that

$$f(x + y) = f(x) + f(y) \tag{1}$$

for all x and $y \in R^1$.

Show that if (1) holds and f is Lebesgue measurable, then f is continuous.

Show that if (1) holds and the graph of f is not dense in the plane, then f is continuous.

Find all continuous functions which satisfy (1).

19 Suppose A and B are measurable subsets of R^1, having finite positive measure. Show that the convolution $\chi_A * \chi_B$ is continuous and not identically 0. Use this to prove that $A + B$ contains a segment.

(A different proof was suggested in Exercise 5, Chap. 7.)

CHAPTER

TEN

ELEMENTARY PROPERTIES OF HOLOMORPHIC FUNCTIONS

Complex Differentiation

We shall now study complex functions defined in subsets of the complex plane. It will be convenient to adopt some standard notations which will be used throughout the rest of this book.

10.1 Definitions If $r > 0$ and a is a complex number,

$$D(a; r) = \{z: |z - a| < r\} \tag{1}$$

is the open circular disc with center at a and radius r. $\bar{D}(a; r)$ is the closure of $D(a; r)$, and

$$D'(a; r) = \{z: 0 < |z - a| < r\} \tag{2}$$

is the punctured disc with center at a and radius r.

A set E in a topological space X is said to be *not connected* if E is the union of two nonempty sets A and B such that

$$\bar{A} \cap B = \varnothing = A \cap \bar{B}. \tag{3}$$

If A and B are as above, and if V and W are the complements of $\bar{A}$ and $\bar{B}$, respectively, it follows that $A \subset W$ and $B \subset V$. Hence

$$E \subset V \cup W, \qquad E \cap V \neq \varnothing, \qquad E \cap W \neq \varnothing, \qquad E \cap V \cap W = \varnothing. \tag{4}$$

Conversely, if open sets V and W exist such that (4) holds, it is easy to see that E is not connected, by taking $A = E \cap W$, $B = E \cap V$.

If E is closed and not connected, then (3) shows that E is the union of two disjoint nonempty closed sets; for if $\bar{A} \subset A \cup B$ and $\bar{A} \cap B = \varnothing$, then $\bar{A} = A$.

If E is open and not connected, then (4) shows that E is the union of two disjoint nonempty open sets, namely $E \cap V$ and $E \cap W$.

Each set consisting of a single point is obviously connected. If $x \in E$, the family Φ_x of all connected subsets of E that contain x is therefore not empty. The union of all members of Φ_x is easily seen to be connected, and to be a *maximal connected subset* of E. These sets are called the *components* of E. Any two components of E are thus disjoint, and E is the union of its components.

By a *region* we shall mean a nonempty connected open subset of the complex plane. Since each open set Ω in the plane is a union of discs, and since all discs are connected, each component of Ω is open. Every plane open set is thus a union of disjoint regions. The letter Ω will from now on denote a plane open set.

10.2 Definition Suppose f is a complex function defined in Ω. If $z_0 \in \Omega$ and if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \tag{1}$$

exists, we denote this limit by $f'(z_0)$ and call it the *derivative* of f at z_0. If $f'(z_0)$ exists for every $z_0 \in \Omega$, we say that f is *holomorphic* (or *analytic*) in Ω. The class of all holomorphic functions in Ω will be denoted by $H(\Omega)$.

To be quite explicit, $f'(z_0)$ exists if to every $\epsilon > 0$ there corresponds a $\delta > 0$ such that

$$\left| \frac{f(z) - f(z_0)}{z - z_0} - f'(z_0) \right| < \epsilon \qquad \text{for all } z \in D'(z_0; \delta). \tag{2}$$

Thus $f'(z_0)$ is a complex number, obtained as a limit of quotients of complex numbers. Note that f is a mapping of Ω into R^2 and that Definition 7.22 associates with such mappings another kind of derivative, namely, a linear operator on R^2. In our present situation, if (2) is satisfied, this linear operator turns out to be multiplication by $f'(z_0)$ (regarding R^2 as the complex field). We leave it to the reader to verify this.

10.3 Remarks If $f \in H(\Omega)$ and $g \in H(\Omega)$, then also $f + g \in H(\Omega)$ and $fg \in H(\Omega)$, so that $H(\Omega)$ is a ring; the usual differentiation rules apply.

More interesting is the fact that superpositions of holomorphic functions are holomorphic: *If* $f \in H(\Omega)$, *if* $f(\Omega) \subset \Omega_1$, *if* $g \in H(\Omega_1)$, *and if* $h = g \circ f$, *then* $h \in H(\Omega)$, *and* h' *can be computed by the chain rule*

$$h'(z_0) = g'(f(z_0))f'(z_0) \qquad (z_0 \in \Omega). \tag{1}$$

To prove this, fix $z_0 \in \Omega$, and put $w_0 = f(z_0)$. Then

$$f(z) - f(z_0) = [f'(z_0) + \epsilon(z)](z - z_0), \tag{2}$$

$$g(w) - g(w_0) = [g'(w_0) + \eta(w)](w - w_0), \tag{3}$$

where $\epsilon(z) \to 0$ as $z \to z_0$ and $\eta(w) \to 0$ as $w \to w_0$. Put $w = f(z)$, and substitute (2) into (3): If $z \neq z_0$,

$$\frac{h(z) - h(z_0)}{z - z_0} = [g'(f(z_0)) + \eta(f(z))][f'(z_0) + \epsilon(z)]. \tag{4}$$

The differentiability of f forces f to be continuous at z_0. Hence (1) follows from (4).

10.4 Examples For $n = 0, 1, 2, \ldots$, z^n is holomorphic in the whole plane, and the same is true of every polynomial in z. One easily verifies directly that $1/z$ is holomorphic in $\{z: z \neq 0\}$. Hence, taking $g(w) = 1/w$ in the chain rule, we see that if f_1 and f_2 are in $H(\Omega)$ and Ω_0 is an open subset of Ω in which f_2 has no zero, then $f_1/f_2 \in H(\Omega_0)$.

Another example of a function which is holomorphic in the whole plane (such functions are called *entire*) is the exponential function defined in the Prologue. In fact, we saw there that exp is differentiable everywhere, in the sense of Definition 10.2, and that $\exp'(z) = \exp(z)$ for every complex z.

10.5 Power Series From the theory of power series we shall assume only one fact as known, namely, that to each power series

$$\sum_{n=0}^{\infty} c_n(z - a)^n \tag{1}$$

there corresponds a number $R \in [0, \infty]$ such that the series converges absolutely and uniformly in $\bar{D}(a; r)$, for every $r < R$, and diverges if $z \notin \bar{D}(a; R)$. The "radius of convergence" R is given by the root test:

$$\frac{1}{R} = \limsup_{n \to \infty} |c_n|^{1/n}. \tag{2}$$

Let us say that a function f defined in Ω is *representable by power series in* Ω if to every disc $D(a; r) \subset \Omega$ there corresponds a series (1) which converges to $f(z)$ for all $z \in D(a; r)$.

10.6 Theorem *If f is representable by power series in Ω, then $f \in H(\Omega)$ and f' is also representable by power series in Ω. In fact, if*

$$f(z) = \sum_{n=0}^{\infty} c_n(z - a)^n \tag{1}$$

for $z \in D(a; r)$, then for these z we also have

$$f'(z) = \sum_{n=1}^{\infty} nc_n(z - a)^{n-1}. \tag{2}$$

PROOF If the series (1) converges in $D(a; r)$, the root test shows that the series (2) also converges there. Take $a = 0$, without loss of generality, denote the sum of the series (2) by $g(z)$, fix $w \in D(a; r)$, and choose ρ so that $|w| < \rho < r$. If $z \neq w$, we have

$$\frac{f(z) - f(w)}{z - w} - g(w) = \sum_{n=1}^{\infty} c_n \left[\frac{z^n - w^n}{z - w} - n w^{n-1} \right]. \tag{3}$$

The expression in brackets is 0 if $n = 1$, and is

$$(z - w) \sum_{k=1}^{n-1} k w^{k-1} z^{n-k-1} \tag{4}$$

if $n \geq 2$. If $|z| < \rho$, the absolute value of the sum in (4) is less than

$$\frac{n(n-1)}{2} \rho^{n-2} \tag{5}$$

so

$$\left| \frac{f(z) - f(w)}{z - w} - g(w) \right| \leq |z - w| \sum_{n=2}^{\infty} n^2 |c_n| \rho^{n-2}. \tag{6}$$

Since $\rho < r$, the last series converges. Hence the left side of (6) tends to 0 as $z \to w$. This says that $f'(w) = g(w)$, and completes the proof. ////

Corollary *Since f' is seen to satisfy the same hypothesis as f does, the theorem can be applied to f'. It follows that f has derivatives of all orders, that each derivative is representable by power series in Ω, and that*

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) c_n (z - a)^{n-k} \tag{7}$$

if (1) *holds. Hence* (1) *implies that*

$$k! c_k = f^{(k)}(a) \qquad (k = 0, 1, 2, \ldots), \tag{8}$$

so that for each $a \in \Omega$ there is a unique sequence $\{c_n\}$ for which (1) *holds.*

We now describe a process which manufactures functions that are representable by power series. Special cases will be of importance later.

10.7 Theorem *Suppose μ is a complex (finite) measure on a measurable space X, φ is a complex measurable function on X, Ω is an open set in the plane which does not intersect $\varphi(X)$, and*

$$f(z) = \int_X \frac{d\mu(\zeta)}{\varphi(\zeta) - z} \qquad (z \in \Omega). \tag{1}$$

Then f is representable by power series in Ω.

PROOF Suppose $D(a; r) \subset \Omega$. Since

$$\left| \frac{z-a}{\varphi(\zeta)-a} \right| \leq \frac{|z-a|}{r} < 1 \tag{2}$$

for every $z \in D(a; r)$ and every $\zeta \in X$, the geometric series

$$\sum_{n=0}^{\infty} \frac{(z-a)^n}{(\varphi(\zeta)-a)^{n+1}} = \frac{1}{\varphi(\zeta)-z} \tag{3}$$

converges uniformly on X, for every fixed $z \in D(a; r)$. Hence the series (3) may be substituted into (1), and $f(z)$ may be computed by interchanging summation and integration. It follows that

$$f(z) = \sum_{0}^{\infty} c_n (z-a)^n \qquad (z \in D(a; r)) \tag{4}$$

where

$$c_n = \int_X \frac{d\mu(\zeta)}{(\varphi(\zeta)-a)^{n+1}} \qquad (n = 0, 1, 2, \ldots). \tag{5}$$

////

Note: The convergence of the series (4) in $D(a; r)$ is a consequence of the proof. We can also derive it from (5), since (5) shows that

$$|c_n| \leq \frac{|\mu|(X)}{r^{n+1}} \qquad (n = 0, 1, 2, \ldots). \tag{6}$$

Integration over Paths

Our first major objective in this chapter is the converse of Theorem 10.6: *Every $f \in H(\Omega)$ is representable by power series in Ω.* The quickest route to this is via Cauchy's theorem which leads to an important integral representation of holomorphic functions. In this section the required integration theory will be developed; we shall keep it as simple as possible, and shall regard it merely as a useful tool in the investigation of properties of holomorphic functions.

10.8 Definitions If X is a topological space, a *curve in X* is a continuous mapping γ of a compact interval $[\alpha, \beta] \subset R^1$ into X; here $\alpha < \beta$. We call $[\alpha, \beta]$ the *parameter interval* of γ and denote the range of γ by γ^*. Thus γ is a mapping, and γ^* is the set of all points $\gamma(t)$, for $\alpha \leq t \leq \beta$.

If the *initial point* $\gamma(\alpha)$ of γ coincides with its *end point* $\gamma(\beta)$, we call γ a *closed curve.*

A *path* is a piecewise continuously differentiable curve in the plane. More explicitly, a path with parameter interval $[\alpha, \beta]$ is a continuous complex function γ on $[\alpha, \beta]$, such that the following holds: There are finitely many

points s_j, $\alpha = s_0 < s_1 < \cdots < s_n = \beta$, and the restriction of γ to each interval $[s_{j-1}, s_j]$ has a continuous derivative on $[s_{j-1}, s_j]$; however, at the points $s_1, \ldots, s_{n-1}$ the left- and right-hand derivatives of γ may differ.

A *closed path* is a closed curve which is also a path.

Now suppose γ is a path, and f is a continuous function on γ^*. The integral of f over γ is defined as an integral over the parameter interval $[\alpha, \beta]$ of γ:

$$\int_\gamma f(z)\,dz = \int_\alpha^\beta f(\gamma(t))\gamma'(t)\,dt. \tag{1}$$

Let φ be a continuously differentiable one-to-one mapping of an interval $[\alpha_1, \beta_1]$ onto $[\alpha, \beta]$, such that $\varphi(\alpha_1) = \alpha$, $\varphi(\beta_1) = \beta$, and put $\gamma_1 = \gamma \circ \varphi$. Then γ_1 is a path with parameter interval $[\alpha_1, \beta_1]$; the integral of f over γ_1 is

$$\int_{\alpha_1}^{\beta_1} f(\gamma_1(t))\gamma_1'(t)\,dt = \int_{\alpha_1}^{\beta_1} f(\gamma(\varphi(t)))\gamma'(\varphi(t))\varphi'(t)\,dt = \int_\alpha^\beta f(\gamma(s))\gamma'(s)\,ds,$$

so that our "reparametrization" has not changed the integral:

$$\int_{\gamma_1} f(z)\,dz = \int_\gamma f(z)\,dz. \tag{2}$$

Whenever (2) holds for a pair of paths γ and γ_1 (and for all f), we shall regard γ and γ_1 as equivalent.

It is convenient to be able to replace a path by an equivalent one, i.e., to choose parameter intervals at will. For instance, if the end point of γ_1 coincides with the initial point of γ_2, we may locate their parameter intervals so that γ_1 and γ_2 join to form one path γ, with the property that

$$\int_\gamma f = \int_{\gamma_1} f + \int_{\gamma_2} f \tag{3}$$

for every continuous f on $\gamma^* = \gamma_1^* \cup \gamma_2^*$.

However, suppose that $[0, 1]$ is the parameter interval of a path γ, and $\gamma_1(t) = \gamma(1 - t)$, $0 \le t \le 1$. We call γ_1 the path *opposite* to γ, for the following reason: For any f continuous on $\gamma_1^* = \gamma^*$, we have

$$\int_0^1 f(\gamma_1(t))\gamma_1'(t)\,dt = -\int_0^1 f(\gamma(1-t))\gamma'(1-t)\,dt = -\int_0^1 f(\gamma(s))\gamma'(s)\,ds,$$

so that

$$\int_{\gamma_1} f = -\int_\gamma f. \tag{4}$$

From (1) we obtain the inequality

$$\left| \int_\gamma f(z)\, dz \right| \leq \|f\|_\infty \int_\alpha^\beta |\gamma'(t)|\, dt, \tag{5}$$

where $\|f\|_\infty$ is the maximum of $|f|$ on γ^* and the last integral in (5) is (by definition) the *length* of γ.

10.9 Special Cases

(*a*) If a is a complex number and $r > 0$, the path defined by

$$\gamma(t) = a + re^{it} \qquad (0 \leq t \leq 2\pi) \tag{1}$$

is called the *positively oriented circle* with center at a and radius r; we have

$$\int_\gamma f(z)\, dz = ir \int_0^{2\pi} f(a + re^{i\theta}) e^{i\theta}\, d\theta, \tag{2}$$

and the length of γ is $2\pi r$, as expected.

(*b*) If a and b are complex numbers, the path γ given by

$$\gamma(t) = a + (b - a)t \qquad (0 \leq t \leq 1) \tag{3}$$

is the *oriented interval* $[a, b]$; its length is $|b - a|$, and

$$\int_{[a,\, b]} f(z)\, dz = (b - a) \int_0^1 f[a + (b - a)t]\, dt. \tag{4}$$

If

$$\gamma_1(t) = \frac{a(\beta - t) + b(t - \alpha)}{\beta - \alpha} \qquad (\alpha \leq t \leq \beta), \tag{5}$$

we obtain an equivalent path, which we still denote by $[a, b]$. The path opposite to $[a, b]$ is $[b, a]$.

(*c*) Let $\{a, b, c\}$ be an ordered triple of complex numbers, let

$$\Delta = \Delta(a, b, c)\,.$$

be the triangle with vertices at a, b, and c (Δ is the smallest convex set which contains a, b, and c), and define

$$\int_{\partial\Delta} f = \int_{[a,\, b]} f + \int_{[b,\, c]} f + \int_{[c,\, a]} f, \tag{6}$$

for any f continuous on the boundary of Δ. We may regard (6) as the definition of its left side. Or we may regard $\partial\Delta$ as a path obtained by joining $[a, b]$ to $[b, c]$ to $[c, a]$, as outlined in Definition 10.8, in which case (6) is easily proved to be true.

If $\{a, b, c\}$ is permuted cyclically, we see from (6) that the left side of (6) is unaffected. If $\{a, b, c\}$ is replaced by $\{a, c, b\}$, then the left side of (6) changes sign.

We now come to a theorem which plays a very important role in function theory.

10.10 Theorem *Let γ be a closed path, let Ω be the complement of γ^* (relative to the plane), and define*

$$\operatorname{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\gamma \frac{d\zeta}{\zeta - z} \qquad (z \in \Omega). \tag{1}$$

Then $\operatorname{Ind}_\gamma$ is an integer-valued function on Ω which is constant in each component of Ω and which is 0 in the unbounded component of Ω.

We call $\operatorname{Ind}_\gamma(z)$ the *index* of z with respect to γ. Note that γ^* is compact, hence γ^* lies in a bounded disc D whose complement D^c is connected; thus D^c lies in some component of Ω. This shows that Ω has precisely one unbounded component.

PROOF Let $[\alpha, \beta]$ be the parameter interval of γ, fix $z \in \Omega$, then

$$\operatorname{Ind}_\gamma(z) = \frac{1}{2\pi i}\int_\alpha^\beta \frac{\gamma'(s)}{\gamma(s) - z}\, ds. \tag{2}$$

Since $w/2\pi i$ is an integer if and only if $e^w = 1$, the first assertion of the theorem, namely, that $\operatorname{Ind}_\gamma(z)$ is an integer, is equivalent to the assertion that $\varphi(\beta) = 1$, where

$$\varphi(t) = \exp\left\{\int_\alpha^t \frac{\gamma'(s)}{\gamma(s) - z}\, ds\right\} \qquad (\alpha \le t \le \beta). \tag{3}$$

Differentiation of (3) shows that

$$\frac{\varphi'(t)}{\varphi(t)} = \frac{\gamma'(t)}{\gamma(t) - z} \tag{4}$$

except possibly on a finite set S where γ is not differentiable. Therefore $\varphi/(\gamma - z)$ is a continuous function on $[\alpha, \beta]$ whose derivative is zero in $[\alpha, \beta] - S$. Since S is finite, $\varphi/(\gamma - z)$ is constant on $[\alpha, \beta]$; and since $\varphi(\alpha) = 1$, we obtain

$$\varphi(t) = \frac{\gamma(t) - z}{\gamma(\alpha) - z} \qquad (\alpha \le t \le \beta). \tag{5}$$

We now use the assumption that γ is a closed path, i.e., that $\gamma(\beta) = \gamma(\alpha)$; (5) shows that $\varphi(\beta) = 1$, and this, as we observed above, implies that $\operatorname{Ind}_\gamma(z)$ is an integer.

By Theorem 10.7, (1) shows that $\text{Ind}_\gamma \in H(\Omega)$. The image of a connected set under a continuous mapping is connected ([26], Theorem 4.22), and since Ind_γ is an integer-valued function, Ind_γ must be constant on each component of Ω.

Finally, (2) shows that $|\text{Ind}_\gamma(z)| < 1$ if $|z|$ is sufficiently large. This implies that $\text{Ind}_\gamma(z) = 0$ in the unbounded component of Ω. ////

Remark: If $\lambda(t)$ denotes the integral in (3), the preceding proof shows that $2\pi\,\text{Ind}_\gamma(z)$ is the net increase in the imaginary part of $\lambda(t)$, as t runs from α to β, and this is the same as the net increase of the argument of $\gamma(t) - z$. (We have not defined "argument" and will have no need for it.) If we divide this increase by 2π, we obtain "the number of times that γ winds around z," and this explains why the term "winding number" is frequently used for the index. One virtue of the preceding proof is that it establishes the main properties of the index without any reference to the (multiple-valued) argument of a complex number.

10.11 Theorem *If γ is the positively oriented circle with center at a and radius r, then*

$$\text{Ind}_\gamma(z) = \begin{cases} 1 & \text{if } |z-a| < r, \\ 0 & \text{if } |z-a| > r. \end{cases}$$

PROOF We take γ as in Sec. 10.9(a). By Theorem 10.10, it is enough to compute $\text{Ind}_\gamma(a)$, and 10.9(2) shows that this equals

$$\frac{1}{2\pi i}\int_\gamma \frac{dz}{z-a} = \frac{r}{2\pi}\int_0^{2\pi} (re^{it})^{-1} e^{it}\,dt = 1. \qquad ////$$

The Local Cauchy Theorem

There are several forms of Cauchy's theorem. They all assert that if γ is a closed path or cycle in Ω, and if γ and Ω satisfy certain topological conditions, then the integral of every $f \in H(\Omega)$ over γ is 0. We shall first derive a simple local version of this (Theorem 10.14) which is quite sufficient for many applications. A more general global form will be established later.

10.12 Theorem *Suppose $F \in H(\Omega)$ and F' is continuous in Ω. Then*

$$\int_\gamma F'(z)\,dz = 0$$

for every closed path γ in Ω.

PROOF If $[\alpha, \beta]$ is the parameter interval of γ, the fundamental theorem of calculus shows that

$$\int_\gamma F'(z)\,dz = \int_\alpha^\beta F'(\gamma(t))\gamma'(t)\,dt = F(\gamma(\beta)) - F(\gamma(\alpha)) = 0,$$

since $\gamma(\beta) = \gamma(\alpha)$. ////

Corollary *Since z^n is the derivative of $z^{n+1}/(n+1)$ for all integers $n \neq -1$, we have*

$$\int_\gamma z^n\,dz = 0$$

for every closed path γ if $n = 0, 1, 2, \ldots$, and for those closed paths γ for which $0 \notin \gamma^*$ if $n = -2, -3, -4, \ldots$.

The case $n = -1$ was dealt with in Theorem 10.10.

10.13 Cauchy's Theorem for a Triangle *Suppose Δ is a closed triangle in a plane open set Ω, $p \in \Omega$, f is continuous on Ω, and $f \in H(\Omega - \{p\})$. Then*

$$\int_{\partial\Delta} f(z)\,dz = 0. \tag{1}$$

For the definition of $\partial\Delta$ we refer to Sec. 10.9(*c*). We shall see later that our hypothesis actually implies that $f \in H(\Omega)$, i.e., that the exceptional point p is not really exceptional. However, the above formulation of the theorem will be useful in the proof of the Cauchy formula.

PROOF We assume first that $p \notin \Delta$. Let a, b, and c be the vertices of Δ, let a', b', and c' be the midpoints of $[b, c]$, $[c, a]$, and $[a, b]$, respectively, and consider the four triangles Δ^j formed by the ordered triples

$$\{a, c', b'\}, \qquad \{b, a', c'\}, \qquad \{c, b', a'\}, \qquad \{a', b', c'\}. \tag{2}$$

If J is the value of the integral (1), it follows from 10.9(6) that

$$J = \sum_{j=1}^{4} \int_{\partial\Delta^j} f(z)\,dz. \tag{3}$$

The absolute value of at least one of the integrals on the right of (3) is therefore at least $|J/4|$. Call the corresponding triangle Δ_1, repeat the argument with Δ_1 in place of Δ, and so forth. This generates a sequence of triangles Δ_n

such that $\Delta \supset \Delta_1 \supset \Delta_2 \supset \cdots$, such that the length of $\partial\Delta_n$ is $2^{-n}L$, if L is the length of $\partial\Delta$, and such that

$$|J| \leq 4^n \left| \int_{\partial\Delta_n} f(z)\, dz \right| \qquad (n = 1, 2, 3, \ldots). \tag{4}$$

There is a (unique) point z_0 which the triangles Δ_n have in common. Since Δ is compact, $z_0 \in \Delta$, so f is differentiable at z_0.

Let $\epsilon > 0$ be given. There exists an $r > 0$ such that

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \leq \epsilon |z - z_0| \tag{5}$$

whenever $|z - z_0| < r$, and there exists an n such that $|z - z_0| < r$ for all $z \in \Delta_n$. For this n we also have $|z - z_0| \leq 2^{-n}L$ for all $z \in \Delta_n$. By the Corollary to Theorem 10.12,

$$\int_{\partial\Delta_n} f(z)\, dz = \int_{\partial\Delta_n} [f(z) - f(z_0) - f'(z_0)(z - z_0)]\, dz, \tag{6}$$

so that (5) implies

$$\left| \int_{\partial\Delta_n} f(z)\, dz \right| \leq \epsilon(2^{-n}L)^2, \tag{7}$$

and now (4) shows that $|J| \leq \epsilon L^2$. Hence $J = 0$ if $p \notin \Delta$.

Assume next that p is a vertex of Δ, say $p = a$. If a, b, and c are collinear, then (1) is trivial, for any continuous f. If not, choose points $x \in [a, b]$ and $y \in [a, c]$, both close to a, and observe that the integral of f over $\partial\Delta$ is the sum of the integrals over the boundaries of the triangles $\{a, x, y\}$, $\{x, b, y\}$, and $\{b, c, y\}$. The last two of these are 0, since these triangles do not contain p. Hence the integral over $\partial\Delta$ is the sum of the integrals over $[a, x]$, $[x, y]$, and $[y, a]$, and since these intervals can be made arbitrarily short and f is bounded on Δ, we again obtain (1).

Finally, if p is an arbitrary point of Δ, apply the preceding result to $\{a, b, p\}$, $\{b, c, p\}$, and $\{c, a, p\}$ to complete the proof. ////

10.14 Cauchy's Theorem in a Convex Set *Suppose Ω is a convex open set, $p \in \Omega$, f is continuous on Ω, and $f \in H(\Omega - \{p\})$. Then $f = F'$ for some $F \in H(\Omega)$. Hence*

$$\int_\gamma f(z)\, dz = 0 \tag{1}$$

for every closed path γ in Ω.

PROOF Fix $a \in \Omega$. Since Ω is convex, Ω contains the straight line interval from a to z for every $z \in \Omega$, so we can define

$$F(z) = \int_{[a,\, z]} f(\xi)\, d\xi \qquad (z \in \Omega). \tag{2}$$

For any z and $z_0 \in \Omega$, the triangle with vertices at a, z_0, and z lies in Ω; hence $F(z) - F(z_0)$ is the integral of f over $[z_0, z]$, by Theorem 10.13. Fixing z_0, we thus obtain

$$\frac{F(z) - F(z_0)}{z - z_0} - f(z_0) = \frac{1}{z - z_0} \int_{[z_0,\, z]} [f(\xi) - f(z_0)]\, d\xi, \tag{3}$$

if $z \neq z_0$. Given $\epsilon > 0$, the continuity of f at z_0 shows that there is a $\delta > 0$ such that $|f(\xi) - f(z_0)| < \epsilon$ if $|\xi - z_0| < \delta$; hence the absolute value of the left side of (3) is less than ϵ as soon as $|z - z_0| < \delta$. This proves that $f = F'$. In particular, $F \in H(\Omega)$. Now (1) follows from Theorem 10.12. ////

10.15 Cauchy's Formula in a Convex Set *Suppose γ is a closed path in a convex open set Ω, and $f \in H(\Omega)$. If $z \in \Omega$ and $z \notin \gamma^*$, then*

$$f(z) \cdot \operatorname{Ind}_\gamma(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi. \tag{1}$$

The case of greatest interest is, of course, $\operatorname{Ind}_\gamma(z) = 1$.

PROOF Fix z so that the above conditions hold, and define

$$g(\xi) = \begin{cases} \dfrac{f(\xi) - f(z)}{\xi - z} & \text{if } \xi \in \Omega,\ \xi \neq z, \\ f'(z) & \text{if } \xi = z. \end{cases} \tag{2}$$

Then g satisfies the hypotheses of Theorem 10.14. Hence

$$\frac{1}{2\pi i} \int_\gamma g(\xi)\, d\xi = 0. \tag{3}$$

If we substitute (2) into (3) we obtain (1). ////

The theorem concerning the representability of holomorphic functions by power series is an easy consequence of Theorem 10.15, if we take a circle for γ:

10.16 Theorem *For every open set Ω in the plane, every $f \in H(\Omega)$ is representable by power series in Ω.*

PROOF Suppose $f \in H(\Omega)$ and $D(a; R) \subset \Omega$. If γ is a positively oriented circle with center at a and radius $r < R$, the convexity of $D(a; R)$ allows us to apply Theorem 10.15; by Theorem 10.11, we obtain

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)}{\xi - z}\, d\xi \qquad (z \in D(a; r)). \tag{1}$$

But now we can apply Theorem 10.7, with $X = [0, 2\pi]$, $\varphi = \gamma$, and $d\mu(t) = f(\gamma(t))\gamma'(t)\, dt$, and we conclude that there is a sequence $\{c_n\}$ such that

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \qquad (z \in D(a; r)). \tag{2}$$

The uniqueness of $\{c_n\}$ (see the Corollary to Theorem 10.6) shows that the same power series is obtained for every $r < R$ (as long as a is fixed). Hence the representation (2) is valid for every $z \in D(a; R)$, and the proof is complete. ////

Corollary *If $f \in H(\Omega)$, then $f' \in H(\Omega)$.*

PROOF Combine Theorems 10.6 and 10.16. ////

The Cauchy theorem has a useful converse:

10.17 Morera's Theorem *Suppose f is a continuous complex function in an open set Ω such that*

$$\int_{\partial\Delta} f(z)\, dz = 0$$

for every closed triangle $\Delta \subset \Omega$. Then $f \in H(\Omega)$.

PROOF Let V be a convex open set in Ω. As in the proof of Theorem 10.14, we can construct $F \in H(V)$ such that $F' = f$. Since derivatives of holomorphic functions are holomorphic (Theorem 10.16), we have $f \in H(V)$, for every convex open $V \subset \Omega$, hence $f \in H(\Omega)$. ////

The Power Series Representation

The fact that every holomorphic function is locally the sum of a convergent power series has a large number of interesting consequences. A few of these are developed in this section.

10.18 Theorem *Suppose Ω is a region, $f \in H(\Omega)$, and*

$$Z(f) = \{a \in \Omega : f(a) = 0\}. \tag{1}$$

Then either $Z(f) = \Omega$, or $Z(f)$ has no limit point in Ω. In the latter case there corresponds to each $a \in Z(f)$ a unique positive integer $m = m(a)$ such that

$$f(z) = (z - a)^m g(z) \qquad (z \in \Omega), \tag{2}$$

where $g \in H(\Omega)$ and $g(a) \neq 0$; furthermore, $Z(f)$ is at most countable.

(We recall that regions are *connected* open sets.)

The integer m is called the *order* of the zero which f has at the point a. Clearly, $Z(f) = \Omega$ if and only if f is identically 0 in Ω. We call $Z(f)$ the *zero set* of f. Analogous results hold of course for the set of α-points of f, i.e., the zero set of $f - \alpha$, where α is any complex number.

PROOF Let A be the set of all limit points of $Z(f)$ in Ω. Since f is continuous, $A \subset Z(f)$.

Fix $a \in Z(f)$, and choose $r > 0$ so that $D(a; r) \subset \Omega$. By Theorem 10.16,

$$f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n \qquad (z \in D(a; r)). \tag{3}$$

There are now two possibilities. Either all c_n are 0, in which case $D(a; r) \subset A$ and a is an interior point of A, or there is a smallest integer m [necessarily positive, since $f(a) = 0$] such that $c_m \neq 0$. In that case, define

$$g(z) = \begin{cases} (z - a)^{-m} f(z) & (z \in \Omega - \{a\}), \\ c_m & (z = a). \end{cases} \tag{4}$$

Then (2) holds. It is clear that $g \in H(\Omega - \{a\})$. But (3) implies

$$g(z) = \sum_{k=0}^{\infty} c_{m+k} (z - a)^k \qquad (z \in D(a; r)). \tag{5}$$

Hence $g \in H(D(a; r))$, so actually $g \in H(\Omega)$.

Moreover, $g(a) \neq 0$, and the continuity of g shows that there is a neighborhood of a in which g has no zero. Thus a is an isolated point of $Z(f)$, by (2).

If $a \in A$, the first case must therefore occur. So A is open. If $B = \Omega - A$, it is clear from the definition of A as a set of limit points that B is open. Thus Ω is the union of the disjoint open sets A and B. Since Ω is connected, we have either $A = \Omega$, in which case $Z(f) = \Omega$, or $A = \varnothing$. In the latter case, $Z(f)$ has at most finitely many points in each compact subset of Ω, and since Ω is σ-compact, $Z(f)$ is at most countable. ////

Corollary *If f and g are holomorphic functions in a region Ω and if $f(z) = g(z)$ for all z in some set which has a limit point in Ω, then $f(z) = g(z)$ for all $z \in \Omega$.*

In other words, a holomorphic function in a region Ω is determined by its values on any set which has a limit point in Ω. This is an important uniqueness theorem.

Note: The theorem fails if we drop the assumption that Ω is connected: If $\Omega = \Omega_0 \cup \Omega_1$, and Ω_0 and Ω_1 are disjoint open sets, put $f = 0$ in Ω_0 and $f = 1$ in Ω_1.

10.19 Definition If $a \in \Omega$ and $f \in H(\Omega - \{a\})$, then f is said to have an *isolated singularity* at the point a. If f can be so defined at a that the extended function is holomorphic in Ω, the singularity is said to be *removable.*

10.20 Theorem *Suppose $f \in H(\Omega - \{a\})$ and f is bounded in $D'(a; r)$, for some $r > 0$. Then f has a removable singularity at a.*

Recall that $D'(a; r) = \{z: 0 < |z - a| < r\}$.

PROOF Define $h(a) = 0$, and $h(z) = (z - a)^2 f(z)$ in $\Omega - \{a\}$. Our boundedness assumption shows that $h'(a) = 0$. Since h is evidently differentiable at every other point of Ω, we have $h \in H(\Omega)$, so

$$h(z) = \sum_{n=2}^{\infty} c_n (z - a)^n \qquad (z \in D(a; r)).$$

We obtain the desired holomorphic extension of f by setting $f(a) = c_2$, for then

$$f(z) = \sum_{n=0}^{\infty} c_{n+2} (z - a)^n \qquad (z \in D(a; r)). \qquad ////$$

10.21 Theorem *If $a \in \Omega$ and $f \in H(\Omega - \{a\})$, then one of the following three cases must occur:*

(*a*) *f has a removable singularity at a.*

(*b*) *There are complex numbers $c_1, \ldots, c_m$, where m is a positive integer and $c_m \neq 0$, such that*

$$f(z) - \sum_{k=1}^{m} \frac{c_k}{(z - a)^k}$$

has a removable singularity at a.

(*c*) *If $r > 0$ and $D(a; r) \subset \Omega$, then $f(D'(a; r))$ is dense in the plane.*

In case (*b*), f is said to have a *pole of order m* at a. The function

$$\sum_{k=1}^{m} c_k (z - a)^{-k},$$

a polynomial in $(z-a)^{-1}$, is called the *principal part* of f at a. It is clear in this situation that $|f(z)| \to \infty$ as $z \to a$.

In case (c), f is said to have an *essential singularity* at a. A statement equivalent to (c) is that to each complex number w there corresponds a sequence $\{z_n\}$ such that $z_n \to a$ and $f(z_n) \to w$ as $n \to \infty$.

PROOF Suppose (c) fails. Then there exist $r > 0$, $\delta > 0$, and a complex number w such that $|f(z) - w| > \delta$ in $D'(a; r)$. Let us write D for $D(a; r)$ and D' for $D'(a; r)$. Define

$$g(z) = \frac{1}{f(z) - w} \qquad (z \in D'). \tag{1}$$

Then $g \in H(D')$ and $|g| < 1/\delta$. By Theorem 10.20, g extends to a holomorphic function in D.

If $g(a) \neq 0$, (1) shows that f is bounded in $D'(a; \rho)$ for some $\rho > 0$. Hence (a) holds, by Theorem 10.20.

If g has a zero of order $m \geq 1$ at a, Theorem 10.18 shows that

$$g(z) = (z-a)^m g_1(z) \qquad (z \in D), \tag{2}$$

where $g_1 \in H(D)$ and $g_1(a) \neq 0$. Also, g_1 has no zero in D', by (1). Put $h = 1/g_1$ in D. Then $h \in H(D)$, h has no zero in D, and

$$f(z) - w = (z-a)^{-m} h(z) \qquad (z \in D'). \tag{3}$$

But h has an expansion of the form

$$h(z) = \sum_{n=0}^{\infty} b_n (z-a)^n \qquad (z \in D), \tag{4}$$

with $b_0 \neq 0$. Now (3) shows that (b) holds, with $c_k = b_{m-k}$, $k = 1, \ldots, m$.

This completes the proof. ////

We shall now exploit the fact that the restriction of a power series $\sum c_n(z-a)^n$ to a circle with center at a is a trigonometric series.

10.22 Theorem *If*

$$f(z) = \sum_{n=0}^{\infty} c_n (z-a)^n \qquad (z \in D(a; R)) \tag{1}$$

and if $0 < r < R$, *then*

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(a + re^{i\theta})|^2 \, d\theta. \tag{2}$$

PROOF We have

$$f(a + re^{i\theta}) = \sum_{n=0}^{\infty} c_n r^n e^{in\theta}. \tag{3}$$

For $r < R$, the series (3) converges uniformly on $[-\pi, \pi]$. Hence

$$c_n r^n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(a + re^{i\theta}) e^{-in\theta}\, d\theta \qquad (n = 0, 1, 2, \ldots), \tag{4}$$

and (2) is seen to be a special case of Parseval's formula. ////

Here are some consequences:

10.23 Liouville's Theorem *Every bounded entire function is constant.*

Recall that a function is *entire* if it is holomorphic in the whole plane.

PROOF Suppose f is entire, $|f(z)| < M$ for all z, and $f(z) = \sum c_n z^n$ for all z. By Theorem 10.22,

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} < M^2$$

for all r, which is possible only if $c_n = 0$ for all $n \geq 1$. ////

10.24 The Maximum Modulus Theorem *Suppose Ω is a region, $f \in H(\Omega)$, and $\bar{D}(a; r) \subset \Omega$. Then*

$$|f(a)| \leq \max_{\theta} |f(a + re^{i\theta})|. \tag{1}$$

Equality occurs in (1) *if and only if f is constant in Ω.*

Consequently, $|f|$ has no local maximum at any point of Ω, unless f is constant.

PROOF Assume that $|f(a + re^{i\theta})| \leq |f(a)|$ for all real θ. In the notation of Theorem 10.22 it follows then that

$$\sum_{n=0}^{\infty} |c_n|^2 r^{2n} \leq |f(a)|^2 = |c_0|^2.$$

Hence $c_1 = c_2 = c_3 = \cdots = 0$, which implies that $f(z) = f(a)$ in $D(a; r)$. Since Ω is connected, Theorem 10.18 shows that f is constant in Ω. ////

Corollary *Under the same hypotheses,*

$$|f(a)| \geq \min_{\theta} |f(a + re^{i\theta})| \tag{2}$$

if f has no zero in $D(a; r)$.

PROOF If $f(a + re^{i\theta}) = 0$ for some θ then (2) is obvious. In the other case, there is a region Ω_0 that contains $\bar{D}(a; r)$ and in which f has no zero; hence (1) can be applied to $1/f$, and (2) follows. ////

10.25 Theorem *If n is a positive integer and*

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0,$$

where $a_0, \ldots, a_{n-1}$ are complex numbers, then P has precisely n zeros in the plane.

Of course, these zeros are counted according to their multiplicities: A zero of order m, say, is counted as m zeros. This theorem contains the fact that the complex field is algebraically closed, i.e., that every nonconstant polynomial with complex coefficients has at least one complex zero.

PROOF Choose $r > 1 + 2|a_0| + |a_1| + \cdots + |a_{n-1}|$. Then

$$|P(re^{i\theta})| > |P(0)| \qquad (0 \le \theta \le 2\pi).$$

If P had no zeros, then the function $f = 1/P$ would be entire and would satisfy $|f(0)| > |f(re^{i\theta})|$ for all θ, which contradicts the maximum modulus theorem. Thus $P(z_1) = 0$ for some z_1. Consequently, there is a polynomial Q, of degree $n - 1$, such that $P(z) = (z - z_1)Q(z)$. The proof is completed by induction on n. ////

10.26 Theorem (Cauchy's Estimates) *If $f \in H(D(a; R))$ and $|f(z)| \le M$ for all $z \in D(a; R)$, then*

$$|f^{(n)}(a)| \le \frac{n!\,M}{R^n} \qquad (n = 1, 2, 3, \ldots). \tag{1}$$

PROOF For each $r < R$, each term of the series 10.22(2) is bounded above by M^2. ////

If we take $a = 0$, $R = 1$, and $f(z) = z^n$, then $M = 1$, $f^{(n)}(0) = n!$, and we see that (1) cannot be improved.

10.27 Definition A sequence $\{f_j\}$ of functions in Ω is said to *converge to f uniformly on compact subsets of Ω* if to every compact $K \subset \Omega$ and to every $\epsilon > 0$ there corresponds an $N = N(K, \epsilon)$ such that $|f_j(z) - f(z)| < \epsilon$ for all $z \in K$ if $j > N$.

For instance, the sequence $\{z^n\}$ converges to 0 uniformly on compact subsets of $D(0; 1)$, but the convergence is *not* uniform in $D(0; 1)$.

It is uniform convergence on compact subsets which arises most naturally in connection with limit operations on holomorphic functions. The term "almost uniform convergence" is sometimes used for this concept.

10.28 Theorem *Suppose $f_j \in H(\Omega)$, for $j = 1, 2, 3, \ldots$, and $f_j \to f$ uniformly on compact subsets of Ω. Then $f \in H(\Omega)$, and $f_j' \to f'$ uniformly on compact subsets of Ω.*

PROOF Since the convergence is uniform on each compact disc in Ω, f is continuous. Let Δ be a triangle in Ω. Then Δ is compact, so

$$\int_{\partial\Delta} f(z)\,dz = \lim_{j\to\infty} \int_{\partial\Delta} f_j(z)\,dz = 0,$$

by Cauchy's theorem. Hence Morera's theorem implies that $f \in H(\Omega)$.

Let K be compact, $K \subset \Omega$. There exists an $r > 0$ such that the union E of the closed discs $\bar{D}(z; r)$, for all $z \in K$, is a compact subset of Ω. Applying Theorem 10.26 to $f - f_j$, we have

$$|f'(z) - f_j'(z)| \le r^{-1}\|f - f_j\|_E \qquad (z \in K),$$

where $\|f\|_E$ denotes the supremum of $|f|$ on E. Since $f_j \to f$ uniformly on E, it follows that $f_j' \to f'$ uniformly on K. ////

Corollary *Under the same hypothesis, $f_j^{(n)} \to f^{(n)}$ uniformly, as $j \to \infty$, on every compact set $K \subset \Omega$, and for every positive integer n.*

Compare this with the situation on the real line, where sequences of infinitely differentiable functions can converge uniformly to nowhere differentiable functions!

The Open Mapping Theorem

If Ω is a region and $f \in H(\Omega)$, then $f(\Omega)$ is either a region or a point.

This important property of holomorphic functions will be proved, in more detailed form, in Theorem 10.32.

10.29 Lemma *If $f \in H(\Omega)$ and g is defined in $\Omega \times \Omega$ by*

$$g(z, w) = \begin{cases} \dfrac{f(z) - f(w)}{z - w} & \text{if } w \ne z, \\ f'(z) & \text{if } w = z, \end{cases}$$

then g is continuous in $\Omega \times \Omega$.

PROOF The only points $(z, w) \in \Omega \times \Omega$ at which the continuity of g is possibly in doubt have $z = w$.

Fix $a \in \Omega$. Fix $\epsilon > 0$. There exists $r > 0$ such that $D(a; r) \subset \Omega$ and $|f'(\zeta) - f'(a)| < \epsilon$ for all $\zeta \in D(a; r)$. If z and w are in $D(a; r)$ and if

$$\zeta(t) = (1 - t)z + tw,$$

then $\zeta(t) \in D(a; r)$ for $0 \leq t \leq 1$, and

$$g(z, w) - g(a, a) = \int_0^1 [f'(\zeta(t)) - f'(a)]\, dt.$$

The absolute value of the integrand is $< \epsilon$, for every t. Thus $|g(z, w) - g(a, a)| < \epsilon$. This proves that g is continuous at (a, a). ////

10.30 Theorem *Suppose $\varphi \in H(\Omega)$, $z_0 \in \Omega$, and $\varphi'(z_0) \neq 0$. Then Ω contains a neighborhood V of z_0 such that*

(a) φ is one-to-one in V,
(b) $W = \varphi(V)$ is an open set, and
(c) if $\psi: W \to V$ is defined by $\psi(\varphi(z)) = z$, then $\psi \in H(W)$.

Thus $\varphi: V \to W$ has a holomorphic inverse.

PROOF Lemma 10.29, applied to φ in place of f, shows that Ω contains a neighborhood V of z_0 such that

$$|\varphi(z_1) - \varphi(z_2)| \geq \tfrac{1}{2}|\varphi'(z_0)|\,|z_1 - z_2| \tag{1}$$

if $z_1 \in V$ and $z_2 \in V$. Thus (a) holds, and also

$$\varphi'(z) \neq 0 \qquad (z \in V). \tag{2}$$

To prove (b), pick $a \in V$ and choose $r > 0$ so that $\bar{D}(a, r) \subset V$. By (1) there exists $c > 0$ such that

$$|\varphi(a + re^{i\theta}) - \varphi(a)| > 2c \qquad (-\pi \leq \theta \leq \pi). \tag{3}$$

If $\lambda \in D(\varphi(a); c)$, then $|\lambda - \varphi(a)| < c$, hence (3) implies

$$\min_{\theta} |\lambda - \varphi(a + re^{i\theta})| > c. \tag{4}$$

By the corollary to Theorem 10.24, $\lambda - \varphi$ must therefore have a zero in $D(a; r)$. Thus $\lambda = \varphi(z)$ for some $z \in D(a; r) \subset V$.

This proves that $D(\varphi(a); c) \subset \varphi(V)$. Since a was an arbitrary point of V, $\varphi(V)$ is open.

To prove (c), fix $w_1 \in W$. Then $\varphi(z_1) = w_1$ for a unique $z_1 \in V$. If $w \in W$ and $\psi(w) = z \in V$, we have

$$\frac{\psi(w) - \psi(w_1)}{w - w_1} = \frac{z - z_1}{\varphi(z) - \varphi(z_1)}. \tag{5}$$

By (1), $z \to z_1$ when $w \to w_1$. Hence (2) implies that $\psi'(w_1) = 1/\varphi'(z_1)$. Thus $\psi \in H(W)$. ////

10.31 Definition For $m = 1, 2, 3, \ldots$, we denote the "m^{th} power function" $z \to z^m$ by π_m.

Each $w \neq 0$ is $\pi_m(z)$ for precisely m distinct values of z: If $w = re^{i\theta}$, $r > 0$, then $\pi_m(z) = w$ if and only if $z = r^{1/m}e^{i(\theta + 2k\pi)/m}$, $k = 1, \ldots, m$.

Note also that each π_m is an open mapping: If V is open and does not contain 0, then $\pi_m(V)$ is open by Theorem 10.30. On the other hand, $\pi_m(D(0; r)) = D(0; r^m)$.

Compositions of open mappings are clearly open. In particular, $\pi_m \circ \varphi$ is open, by Theorem 10.30, if φ' has no zero. The following theorem (which contains the more detailed version of the open mapping theorem that was mentioned prior to Lemma 10.29) states a converse: Every nonconstant holomorphic function in a region is locally of the form $\pi_m \circ \varphi$, except for an additive constant.

10.32 Theorem *Suppose Ω is a region, $f \in H(\Omega)$, f is not constant, $z_0 \in \Omega$, and $w_0 = f(z_0)$. Let m be the order of the zero which the function $f - w_0$ has at z_0.*

Then there exists a neighborhood V of z_0, $V \subset \Omega$, and there exists $\varphi \in H(V)$, such that

(a) $f(z) = w_0 + [\varphi(z)]^m$ for all $z \in V$,
(b) φ' has no zero in V and φ is an invertible mapping of V onto a disc $D(0; r)$.

Thus $f - w_0 = \pi_m \circ \varphi$ in V. It follows that f is an exactly m-to-1 mapping of $V - \{z_0\}$ onto $D'(w_0; r^m)$, and that each $w_0 \in f(\Omega)$ is an interior point of $f(\Omega)$. Hence $f(\Omega)$ is open.

PROOF Without loss of generality we may assume that Ω is a convex neighborhood of z_0 which is so small that $f(z) \neq w_0$ if $z \in \Omega - \{z_0\}$. Then

$$f(z) - w_0 = (z - z_0)^m g(z) \qquad (z \in \Omega) \tag{1}$$

for some $g \in H(\Omega)$ which has no zero in Ω. Hence $g'/g \in H(\Omega)$. By Theorem 10.14, $g'/g = h'$ for some $h \in H(\Omega)$. The derivative of $g \cdot \exp(-h)$ is 0 in Ω. If h is modified by the addition of a suitable constant, it follows that $g = \exp(h)$. Define

$$\varphi(z) = (z - z_0) \exp \frac{h(z)}{m} \qquad (z \in \Omega). \tag{2}$$

Then (*a*) holds, for all $z \in \Omega$.

Also, $\varphi(z_0) = 0$ and $\varphi'(z_0) \neq 0$. The existence of an open set V that satisfies (*b*) follows now from Theorem 10.30. This completes the proof. ////

The next theorem is really contained in the preceding results, but it seems advisable to state it explicitly.

10.33 Theorem *Suppose Ω is a region, $f \in H(\Omega)$, and f is one-to-one in Ω. Then $f'(z) \neq 0$ for every $z \in \Omega$, and the inverse of f is holomorphic.*

PROOF If $f'(z_0)$ were 0 for some $z_0 \in \Omega$, the hypotheses of Theorem 10.32 would hold with some $m > 1$, so that f would be m-to-1 in some deleted neighborhood of z_0. Now apply part (*c*) of Theorem 10.30. ////

Note that the converse of Theorem 10.33 is false: If $f(z) = e^z$, then $f'(z) \neq 0$ for every z, but f is not one-to-one in the whole complex plane.

The Global Cauchy Theorem

Before we state and prove this theorem, which will remove the restriction to convex regions that was imposed in Theorem 10.14, it will be convenient to add a little to the integration apparatus which was sufficient up to now. Essentially, it is a matter of no longer restricting ourselves to integrals over single paths, but to consider finite "sums" of paths instead. A simple instance of this occurred already in Sec. 10.9(*c*).

10.34 Chains and Cycles Suppose $\gamma_1, \ldots, \gamma_n$ are paths in the plane, and put $K = \gamma_1^* \cup \cdots \cup \gamma_n^*$. Each γ_i induces a linear functional $\tilde{\gamma}_i$ on the vector space $C(K)$, by the formula

$$\tilde{\gamma}_i(f) = \int_{\gamma_i} f(z)\, dz. \tag{1}$$

Define

$$\tilde{\Gamma} = \tilde{\gamma}_1 + \cdots + \tilde{\gamma}_n. \tag{2}$$

Explicitly, $\tilde{\Gamma}(f) = \tilde{\gamma}_1(f) + \cdots + \tilde{\gamma}_n(f)$ for all $f \in C(K)$. The relation (2) suggests that we introduce a "formal sum"

$$\Gamma = \gamma_1 \dot{+} \cdots \dot{+} \gamma_n \tag{3}$$

and define

$$\int_\Gamma f(z)\, dz = \tilde{\Gamma}(f). \tag{4}$$

Then (3) is merely an abbreviation for the statement

$$\int_\Gamma f(z)\, dz = \sum_{i=1}^{n} \int_{\gamma_i} f(z)\, dz \qquad (f \in C(K)). \tag{5}$$

Note that (5) serves as the definition of its left side.

The objects Γ so defined are called *chains*. If each γ_j in (3) is a *closed* path, then Γ is called a *cycle*. If each γ_j in (3) is a path in some open set Ω, we say that Γ is a *chain in* Ω.

If (3) holds, we define

$$\Gamma^* = \gamma_1^* \cup \cdots \cup \gamma_n^*. \tag{6}$$

If Γ is a cycle and $\alpha \notin \Gamma^*$, we define the *index* of α with respect to Γ by

$$\operatorname{Ind}_\Gamma(\alpha) = \frac{1}{2\pi i} \int_\Gamma \frac{dz}{z - \alpha}, \tag{7}$$

just as in Theorem 10.10. Obviously, (3) implies

$$\operatorname{Ind}_\Gamma(\alpha) = \sum_{i=1}^{n} \operatorname{Ind}_{\gamma_i}(\alpha). \tag{8}$$

If each γ_i in (3) is replaced by its opposite path (see Sec. 10.8), the resulting chain will be denoted by $-\Gamma$. Then

$$\int_{-\Gamma} f(z)\, dz = -\int_\Gamma f(z)\, dz \qquad (f \in C(\Gamma^*)). \tag{9}$$

In particular, $\operatorname{Ind}_{-\Gamma}(\alpha) = -\operatorname{Ind}_\Gamma(\alpha)$ if Γ is a cycle and $\alpha \notin \Gamma^*$.

Chains can be added and subtracted in the obvious way, by adding or subtracting the corresponding functionals: The statement $\Gamma = \Gamma_1 \dot{+} \Gamma_2$ means

$$\int_\Gamma f(z)\, dz = \int_{\Gamma_1} f(z)\, dz + \int_{\Gamma_2} f(z)\, dz \tag{10}$$

for every $f \in C(\Gamma_1^* \cup \Gamma_2^*)$.

Finally, note that a chain may be represented as a sum of paths in many ways. To say that

$$\gamma_1 \dot{+} \cdots \dot{+} \gamma_n = \delta_1 \dot{+} \cdots \dot{+} \delta_k$$

means simply that

$$\sum_i \int_{\gamma_i} f(z)\, dz = \sum_j \int_{\delta_j} f(z)\, dz$$

for every f that is continuous on $\gamma_1^* \cup \cdots \cup \gamma_n^* \cup \delta_1^* \cup \cdots \cup \delta_k^*$. In particular, a cycle may very well be represented as a sum of paths that are not closed.

10.35 Cauchy's Theorem *Suppose $f \in H(\Omega)$, where Ω is an arbitrary open set in the complex plane. If Γ is a cycle in Ω that satisfies*

$$\operatorname{Ind}_\Gamma(\alpha) = 0 \qquad \textit{for every } \alpha \textit{ not in } \Omega, \tag{1}$$

then

$$f(z) \cdot \operatorname{Ind}_\Gamma(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(w)}{w - z}\, dw \qquad \text{for } z \in \Omega - \Gamma^* \tag{2}$$

and

$$\int_\Gamma f(z)\, dz = 0. \tag{3}$$

If Γ_0 and Γ_1 are cycles in Ω such that

$$\operatorname{Ind}_{\Gamma_0}(\alpha) = \operatorname{Ind}_{\Gamma_1}(\alpha) \qquad \text{for every } \alpha \text{ not in } \Omega, \tag{4}$$

then

$$\int_{\Gamma_0} f(z)\, dz = \int_{\Gamma_1} f(z)\, dz. \tag{5}$$

PROOF The function g defined in $\Omega \times \Omega$ by

$$g(z, w) = \begin{cases} \dfrac{f(w) - f(z)}{w - z} & \text{if } w \neq z, \\ f'(z) & \text{if } w = z, \end{cases} \tag{6}$$

is continuous in $\Omega \times \Omega$ (Lemma 10.29). Hence we can define

$$h(z) = \frac{1}{2\pi i} \int_\Gamma g(z, w)\, dw \qquad (z \in \Omega). \tag{7}$$

For $z \in \Omega - \Gamma^*$, the Cauchy formula (2) is clearly equivalent to the assertion that

$$h(z) = 0. \tag{8}$$

To prove (8), let us first prove that $h \in H(\Omega)$. Note that g is uniformly continuous on every compact subset of $\Omega \times \Omega$. If $z \in \Omega$, $z_n \in \Omega$, and $z_n \to z$, it follows that $g(z_n, w) \to g(z, w)$ uniformly for $w \in \Gamma^*$ (a compact subset of Ω). Hence $h(z_n) \to h(z)$. This proves that h is continuous in Ω. Let Δ be a closed triangle in Ω. Then

$$\int_{\partial\Delta} h(z)\, dz = \frac{1}{2\pi i} \int_\Gamma \left(\int_{\partial\Delta} g(z, w)\, dz \right) dw. \tag{9}$$

For each $w \in \Omega$, $z \to g(z, w)$ is holomorphic in Ω. (The singularity at $z = w$ is removable.) The inner integral on the right side of (9) is therefore 0 for every $w \in \Gamma^*$. Morera's theorem shows now that $h \in H(\Omega)$.

Next, we let Ω_1 be the set of all complex numbers z for which $\mathrm{Ind}_\Gamma(z) = 0$, and we define

$$h_1(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w - z}\,dw \qquad (z \in \Omega_1). \tag{10}$$

If $z \in \Omega \cap \Omega_1$, the definition of Ω_1 makes it clear that $h_1(z) = h(z)$. Hence there is a function $\varphi \in H(\Omega \cup \Omega_1)$ whose restriction to Ω is h and whose restriction to Ω_1 is h_1.

Our hypothesis (1) shows that Ω_1 contains the complement of Ω. Thus φ is an entire function. Ω_1 also contains the unbounded component of the complement of Γ^*, since $\mathrm{Ind}_\Gamma(z)$ is 0 there. Hence

$$\lim_{|z|\to\infty} \varphi(z) = \lim_{|z|\to\infty} h_1(z) = 0. \tag{11}$$

Liouville's theorem implies now that $\varphi(z) = 0$ for every z. This proves (8), and hence (2).

To deduce (3) from (2), pick $a \in \Omega - \Gamma^*$ and define $F(z) = (z - a)f(z)$. Then

$$\frac{1}{2\pi i}\int_\Gamma f(z)\,dz = \frac{1}{2\pi i}\int_\Gamma \frac{F(z)}{z - a}\,dz = F(a)\cdot \mathrm{Ind}_\Gamma(a) = 0, \tag{12}$$

because $F(a) = 0$.

Finally, (5) follows from (4) if (3) is applied to the cycle $\Gamma = \Gamma_1 - \Gamma_0$. This completes the proof. ////

10.36 Remarks

(*a*) If γ is a closed path in a convex region Ω and if $\alpha \notin \Omega$, an application of Theorem 10.14 to $f(z) = (z - \alpha)^{-1}$ shows that $\mathrm{Ind}_\gamma(\alpha) = 0$. Hypothesis (1) of Theorem 10.35 is therefore satisfied by every cycle in Ω if Ω is convex. This shows that Theorem 10.35 generalizes Theorems 10.14 and 10.15.

(*b*) The last part of Theorem 10.35 shows under what circumstances integration over one cycle can be replaced by integration over another, without changing the value of the integral. For example, let Ω be the plane with three disjoint closed discs D_i removed. If Γ, γ_1, γ_2, γ_3 are positively oriented circles in Ω such that Γ surrounds $D_1 \cup D_2 \cup D_3$ and γ_i surrounds D_i but not D_j for $j \neq i$, then

$$\int_\Gamma f(z)\,dz = \sum_{i=1}^{3}\int_{\gamma_i} f(z)\,dz$$

for every $f \in H(\Omega)$.

(*c*) In order to apply Theorem 10.35, it is desirable to have a reasonably efficient method of finding the index of a point with respect to a closed path. The following theorem does this for all paths that occur in practice.

It says, essentially, that the index *increases* by 1 when the path is crossed "from right to left." If we recall that $\text{Ind}_\gamma(\alpha) = 0$ if α is in the unbounded component of the complement W of γ^*, we can then successively determine $\text{Ind}_\gamma(\alpha)$ in the other components of W, provided that W has only finitely many components and that γ traverses no arc more than once.

10.37 Theorem *Suppose γ is a closed path in the plane, with parameter interval $[\alpha, \beta]$. Suppose $\alpha < u < v < \beta$, a and b are complex numbers, $|b| = r > 0$, and*

(i) $\gamma(u) = a - b$, $\gamma(v) = a + b$,
(ii) $|\gamma(s) - a| < r$ *if and only if* $u < s < v$,
(iii) $|\gamma(s) - a| = r$ *if and only if* $s = u$ *or* $s = v$.

Assume furthermore that $D(a; r) - \gamma^*$ is the union of two regions, D_+ and D_-, labeled so that $a + bi \in \bar{D}_+$ and $a - bi \in \bar{D}_-$. Then

$$\text{Ind}_\gamma(z) = 1 + \text{Ind}_\gamma(w)$$

if $x \in D_+$ and $w \in D_-$.

As $\gamma(t)$ traverses $D(a; r)$ from $a - b$ to $a + b$, D_- is "on the right" and D_+ is "on the left" of the path.

PROOF To simplify the writing, reparametrize γ so that $u = 0$ and $v = \pi$. Define

$$C(s) = a - be^{is} \qquad (0 \le s \le 2\pi)$$

$$f(s) = \begin{cases} C(s) & (0 \le s \le \pi) \\ \gamma(2\pi - s) & (\pi \le s \le 2\pi) \end{cases}$$

$$g(s) = \begin{cases} \gamma(s) & (0 \le s \le \pi) \\ C(s) & (\pi \le s \le 2\pi) \end{cases}$$

$$h(s) = \begin{cases} \gamma(s) & (\alpha \le s \le 0 \text{ or } \pi \le s \le \beta) \\ C(s) & (0 \le s \le \pi). \end{cases}$$

Since $\gamma(0) = C(0)$ and $\gamma(\pi) = C(\pi)$, f, g, and h are closed paths.

If $E \subset \bar{D}(a; r)$, $|\zeta - a| = r$, and $\zeta \notin E$, then E lies in the disc $D(2a - \zeta; 2r)$ which does not contain ζ. Apply this to $E = g^*$, $\zeta = a - bi$, to see [from Remark 10.36(*a*)] that $\text{Ind}_g(a - bi) = 0$. Since $\bar{D}_-$ is connected and D_- does not intersect g^*, it follows that

$$\text{Ind}_g(w) = 0 \qquad \text{if } w \in D_-. \tag{1}$$

The same reasoning shows that

$$\text{Ind}_f(z) = 0 \qquad \text{if } z \in D_+. \tag{2}$$

We conclude that

$$\begin{aligned}\operatorname{Ind}_\gamma(z) &= \operatorname{Ind}_h(z) = \operatorname{Ind}_h(w)\\ &= \operatorname{Ind}_C(w) + \operatorname{Ind}_\gamma(w) = 1 + \operatorname{Ind}_\gamma(w).\end{aligned}$$

The first of these equalities follows from (2), since $h = \gamma \dotplus f$. The second holds because z and w lie in $D(a; r)$, a connected set which does not intersect h^*. The third follows from (1), since $h \dotplus g = C \dotplus \gamma$, and the fourth is a consequence of Theorem 10.11. This completes the proof. ////

We now turn to a brief discussion of another topological concept that is relevant to Cauchy's theorem.

10.38 Homotopy Suppose γ_0 and γ_1 are closed curves in a topological space X, both with parameter interval $I = [0, 1]$. We say that γ_0 and γ_1 are *X-homotopic* if there is a continuous mapping H of the unit square $I^2 = I \times I$ into X such that

$$H(s, 0) = \gamma_0(s), \qquad H(s, 1) = \gamma_1(s), \qquad H(0, t) = H(1, t) \tag{1}$$

for all $s \in I$ and $t \in I$. Put $\gamma_t(s) = H(s, t)$. Then (1) defines a *one-parameter family of closed curves γ_t in X*, which *connects* γ_0 and γ_1. Intuitively, this means that γ_0 can be continuously deformed to γ_1, within X.

If γ_0 is X-homotopic to a constant mapping γ_1 (i.e., if γ_1^* consists of just one point), we say that γ_0 is *null-homotopic in X*. If X is connected and if every closed curve in X is null-homotopic, X is said to be *simply connected.*

For example, every convex region Ω is simply connected. To see this, let γ_0 be a closed curve in Ω, fix $z_1 \in \Omega$, and define

$$H(s, t) = (1 - t)\gamma_0(s) + tz_1 \qquad (0 \le s \le 1, \quad 0 \le t \le 1). \tag{2}$$

Theorem 10.40 will show that condition (4) of Cauchy's theorem 10.35 holds whenever Γ_0 and Γ_1 are Ω-homotopic closed paths. As a special case of this, note that *condition* (1) *of Theorem* 10.35 *holds for every closed path Γ in Ω if Ω is simply connected.*

10.39 Lemma *If γ_0 and γ_1 are closed paths with parameter interval* $[0, 1]$, *if α is a complex number, and if*

$$|\gamma_1(s) - \gamma_0(s)| < |\alpha - \gamma_0(s)| \qquad (0 \le s \le 1) \tag{1}$$

then $\operatorname{Ind}_{\gamma_1}(\alpha) = \operatorname{Ind}_{\gamma_0}(\alpha)$.

PROOF Note first that (1) implies that $\alpha \notin \gamma_0^*$ and $\alpha \notin \gamma_1^*$. Hence one can define $\gamma = (\gamma_1 - \alpha)/(\gamma_0 - \alpha)$. Then

$$\frac{\gamma'}{\gamma} = \frac{\gamma_1'}{\gamma_1 - \alpha} - \frac{\gamma_0'}{\gamma_0 - \alpha} \tag{2}$$

and $|1 - \gamma| < 1$, by (1). Hence $\gamma^* \subset D(1; 1)$, which implies that $\text{Ind}_\gamma(0) = 0$. Integration of (2) over [0, 1] now gives the desired result. ////

10.40 Theorem *If Γ_0 and Γ_1 are Ω-homotopic closed paths in a region Ω, and if $\alpha \notin \Omega$, then*

$$\text{Ind}_{\Gamma_1}(\alpha) = \text{Ind}_{\Gamma_0}(\alpha). \tag{1}$$

PROOF By definition, there is a continuous $H: I^2 \to \Omega$ such that

$$H(s, 0) = \Gamma_0(s), \qquad H(s, 1) = \Gamma_1(s), \qquad H(0, t) = H(1, t). \tag{2}$$

Since I^2 is compact, so is $H(I^2)$. Hence there exists $\epsilon > 0$ such that

$$|\alpha - H(s, t)| > 2\epsilon \qquad \text{if} \qquad (s, t) \in I^2. \tag{3}$$

Since H is uniformly continuous, there is a positive integer n such that

$$|H(s, t) - H(s', t')| < \epsilon \qquad \text{if} \qquad |s - s'| + |t - t'| \leq 1/n. \tag{4}$$

Define polygonal closed paths $\gamma_0, \ldots, \gamma_n$ by

$$\gamma_k(s) = H\left(\frac{i}{n}, \frac{k}{n}\right)(ns + 1 - i) + H\left(\frac{i-1}{n}, \frac{k}{n}\right)(i - ns) \tag{5}$$

if $i - 1 \leq ns \leq i$ and $i = 1, \ldots, n$. By (4) and (5),

$$|\gamma_k(s) - H(s, k/n)| < \epsilon \qquad (k = 0, \ldots, n;\ 0 \leq s \leq 1). \tag{6}$$

In particular, taking $k = 0$ and $k = n$,

$$|\gamma_0(s) - \Gamma_0(s)| < \epsilon, \qquad |\gamma_n(s) - \Gamma_1(s)| < \epsilon. \tag{7}$$

By (6) and (3),

$$|\alpha - \gamma_k(s)| > \epsilon \qquad (k = 0, \ldots, n;\ 0 \leq s \leq 1). \tag{8}$$

On the other hand, (4) and (5) also give

$$|\gamma_{k-1}(s) - \gamma_k(s)| < \epsilon \qquad (k = 1, \ldots, n;\ 0 \leq s \leq 1). \tag{9}$$

Now it follows from (7), (8), (9), and $n + 2$ applications of Lemma 10.39 that α has the same index with respect to each of the paths $\Gamma_0, \gamma_0, \gamma_1, \ldots, \gamma_n, \Gamma_1$. This proves the theorem. ////

Note: If $\Gamma_t(s) = H(s, t)$ in the preceding proof, then each Γ_t is a closed *curve*, but not necessarily a *path*, since H is not assumed to be differentiable. The paths γ_k were introduced for this reason. Another (and perhaps more satisfactory) way to circumvent this difficulty is to extend the definition of index to closed curves. This is sketched in Exercise 28.

The Calculus of Residues

10.41 Definition A function f is said to be *meromorphic* in an open set Ω if there is a set $A \subset \Omega$ such that

(*a*) A has no limit point in Ω,
(*b*) $f \in H(\Omega - A)$,
(*c*) f has a pole at each point of A.

Note that the possibility $A = \varnothing$ is not excluded. Thus every $f \in H(\Omega)$ is meromorphic in Ω.

Note also that (*a*) implies that no compact subset of Ω contains infinitely many points of A, and that A is therefore at most countable.

If f and A are as above, if $a \in A$, and if

$$Q(z) = \sum_{k=1}^{m} c_k (z - a)^{-k} \tag{1}$$

is the principal part of f at a, as defined in Theorem 10.21 (i.e., if $f - Q$ has a removable singularity at a), then the number c_1 is called the *residue* of f at a:

$$c_1 = \operatorname{Res}(f; a). \tag{2}$$

If Γ is a cycle and $a \notin \Gamma^*$, (1) implies

$$\frac{1}{2\pi i} \int_\Gamma Q(z)\, dz = c_1 \operatorname{Ind}_\Gamma(a) = \operatorname{Res}(Q; a) \operatorname{Ind}_\Gamma(a). \tag{3}$$

This very special case of the following theorem will be used in its proof.

10.42 The Residue Theorem *Suppose f is a meromorphic function in Ω. Let A be the set of points in Ω at which f has poles. If Γ is a cycle in $\Omega - A$ such that*

$$\operatorname{Ind}_\Gamma(\alpha) = 0 \qquad \textit{for all} \qquad \alpha \notin \Omega, \tag{1}$$

then

$$\frac{1}{2\pi i} \int_\Gamma f(z)\, dz = \sum_{a \in A} \operatorname{Res}(f; a) \operatorname{Ind}_\Gamma(a). \tag{2}$$

PROOF Let $B = \{a \in A : \operatorname{Ind}_\Gamma(a) \neq 0\}$. Let W be the complement of Γ^*. Then $\operatorname{Ind}_\Gamma(z)$ is constant in each component V of W. If V is unbounded, or if V intersects Ω^c, (1) implies that $\operatorname{Ind}_\Gamma(z) = 0$ for every $z \in V$. Since A has no limit point in Ω, we conclude that B is a finite set.

The sum in (2), though formally infinite, is therefore actually finite.

Let $a_1, \ldots, a_n$ be the points of B, let $Q_1, \ldots, Q_n$ be the principal parts of f at $a_1, \ldots, a_n$, and put $g = f - (Q_1 + \cdots + Q_n)$. (If $B = \varnothing$, a possibility which is not excluded, then $g = f$.) Put $\Omega_0 = \Omega - (A - B)$. Since g has removable

singularities at $a_1, \ldots, a_n$, Theorem 10.35, applied to the function g and the open set Ω_0, shows that

$$\int_\Gamma g(z)\,dz = 0. \tag{3}$$

Hence

$$\frac{1}{2\pi i}\int_\Gamma f(z)\,dz = \sum_{i=1}^{n}\frac{1}{2\pi i}\int_\Gamma Q_k(z)\,dz = \sum_{k=1}^{n}\operatorname{Res}(Q_k; a_k)\operatorname{Ind}_\Gamma(a_k),$$

and since f and Q_k have the same residue at a_k, we obtain (2). ////

We conclude this chapter with two typical applications of the residue theorem. The first one concerns zeros of holomorphic functions, the second is the evaluation of a certain integral.

10.43 Theorem *Suppose γ is a closed path in a region Ω, such that $\operatorname{Ind}_\gamma(\alpha) = 0$ for every α not in Ω. Suppose also that $\operatorname{Ind}_\gamma(\alpha) = 0$ or 1 for every $\alpha \in \Omega - \gamma^*$, and let Ω_1 be the set of all α with $\operatorname{Ind}_\gamma(\alpha) = 1$.*

For any $f \in H(\Omega)$ let N_f be the number of zeros of f in Ω_1, counted according to their multiplicities.

(*a*) *If $f \in H(\Omega)$ and f has no zeros on γ^* then*

$$N_f = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \operatorname{Ind}_\Gamma(0) \tag{1}$$

where $\Gamma = f \circ \gamma$.

(*b*) *If also $g \in H(\Omega)$ and*

$$|f(z) - g(z)| < |f(z)| \qquad \text{for all } z \in \gamma^* \tag{2}$$

then $N_g = N_f$.

Part (*b*) is usually called Rouché's theorem. It says that two holomorphic functions have the same number of zeros in Ω_1 if they are close together on the boundary of Ω_1, as specified by (2).

PROOF Put $\varphi = f'/f$, a meromorphic function in Ω. If $a \in \Omega$ and f has a zero of order $m = m(a)$ at a, then $f(z) = (z-a)^m h(z)$, where h and $1/h$ are holomorphic in some neighborhood V of a. In $V - \{a\}$,

$$\varphi(z) = \frac{f'(z)}{f(z)} = \frac{m}{z-a} + \frac{h'(z)}{h(z)}. \tag{3}$$

Thus

$$\operatorname{Res}(\varphi; a) = m(a). \tag{4}$$

Let $A = \{a \in \Omega_1 : f(a) = 0\}$. If our assumptions about the index of γ are combined with the residue theorem one obtains

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz = \sum_{a\in A} \operatorname{Res}(\varphi; a) = \sum_{a\in A} m(a) = N_f.$$

This proves one half of (1). The other half is a matter of direct computation:

$$\begin{aligned}\operatorname{Ind}_\Gamma(0) &= \frac{1}{2\pi i}\int_\Gamma \frac{dz}{z} = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\Gamma'(s)}{\Gamma(s)}\,ds \\ &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f'(\gamma(s))}{f(\gamma(s))}\,\gamma'(s)\,ds = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}\,dz.\end{aligned}$$

The parameter interval of γ was here taken to be $[0, 2\pi]$.

Next, (2) shows that g has no zero on γ^*. Hence (1) holds with g in place of f. Put $\Gamma_0 = g \circ \gamma$. Then it follows from (1), (2), and Lemma 10.39 that

$$N_g = \operatorname{Ind}_{\Gamma_0}(0) = \operatorname{Ind}_\Gamma(0) = N_f. \qquad ////$$

10.44 Problem *For real t, find the limit, as $A \to \infty$, of*

$$\int_{-A}^{A} \frac{\sin x}{x}\,e^{ixt}\,dx. \tag{1}$$

SOLUTION Since $z^{-1} \cdot \sin z \cdot e^{itz}$ is entire, its integral over $[-A, A]$ equals that over the path Γ_A obtained by going from $-A$ to -1 along the real axis, from -1 to 1 along the lower half of the unit circle, and from 1 to A along the real axis. This follows from Cauchy's theorem. Γ_A avoids the origin, and we may therefore use the identity

$$2i \sin z = e^{iz} - e^{-iz}$$

to see that (1) equals $\varphi_A(t+1) - \varphi_A(t-1)$, where

$$\frac{1}{\pi}\varphi_A(s) = \frac{1}{2\pi i}\int_{\Gamma_A} \frac{e^{isz}}{z}\,dz. \tag{2}$$

Complete Γ_A to a closed path in two ways: First, by the semicircle from A to $-Ai$ to $-A$; secondly, by the semicircle from A to Ai to $-A$. The function e^{isz}/z has a single pole, at $z = 0$, where its residue is 1. It follows that

$$\frac{1}{\pi}\varphi_A(s) = \frac{1}{2\pi}\int_{-\pi}^{0} \exp(isAe^{i\theta})\,d\theta \tag{3}$$

and

$$\frac{1}{\pi}\varphi_A(s) = 1 - \frac{1}{2\pi}\int_0^{\pi} \exp(isAe^{i\theta})\,d\theta. \tag{4}$$

Note that

$$|\exp(isAe^{i\theta})| = \exp(-As\sin\theta), \tag{5}$$

and that this is <1 and tends to 0 as $A \to \infty$ if s and $\sin\theta$ have the same sign. The dominated convergence theorem shows therefore that the integral in (3) tends to 0 if $s < 0$, and the one in (4) tends to 0 if $s > 0$. Thus

$$\lim_{A\to\infty} \varphi_A(s) = \begin{cases} \pi & \text{if } s > 0, \\ 0 & \text{if } s < 0, \end{cases} \tag{6}$$

and if we apply (6) to $s = t + 1$ and to $s = t - 1$, we get

$$\lim_{A\to\infty} \int_{-A}^{A} \frac{\sin x}{x} e^{itx}\, dx = \begin{cases} \pi & \text{if } -1 < t < 1, \\ 0 & \text{if } |t| > 1. \end{cases} \tag{7}$$

Since $\varphi_A(0) = \pi/2$, the limit in (7) is $\pi/2$ when $t = \pm 1$. ////

Note that (7) gives the Fourier transform of $(\sin x)/x$. We leave it as an exercise to check the result against the inversion theorem.

Exercises

1 The following fact was tacitly used in this chapter: If A and B are disjoint subsets of the plane, if A is compact, and if B is closed, then there exists a $\delta > 0$ such that $|\alpha - \beta| \geq \delta$ for all $\alpha \in A$ and $\beta \in B$. Prove this, with an arbitrary metric space in place of the plane.

2 Suppose that f is an entire function, and that in every power series

$$f(z) = \sum_{n=0}^{\infty} c_n(z-a)^n$$

at least one coefficient is 0. Prove that f is a polynomial.

Hint: $n!\, c_n = f^{(n)}(a)$.

3 Suppose f and g are entire functions, and $|f(z)| \leq |g(z)|$ for every z. What conclusion can you draw?

4 Suppose f is an entire function, and

$$|f(z)| \leq A + B|z|^k$$

for all z, where A, B, and k are positive numbers. Prove that f must be a polynomial.

5 Suppose $\{f_n\}$ is a uniformly bounded sequence of holomorphic functions in Ω such that $\{f_n(z)\}$ converges for every $z \in \Omega$. Prove that the convergence is uniform on every compact subset of Ω.

Hint: Apply the dominated convergence theorem to the Cauchy formula for $f_n - f_m$.

6 There is a region Ω that $\exp(\Omega) = D(1; 1)$. Show that exp is one-to-one in Ω, but that there are many such Ω. Fix one, and define $\log z$, for $|z - 1| < 1$, to be that $w \in \Omega$ for which $e^w = z$. Prove that $\log'(z) = 1/z$. Find the coefficients a_n in

$$\frac{1}{z} = \sum_{n=0}^{\infty} a_n(z-1)^n$$

and hence find the coefficients c_n in the expansion

$$\log z = \sum_{n=0}^{\infty} c_n(z-1)^n.$$

In what other discs can this be done?

7 If $f \in H(\Omega)$, the Cauchy formula for the derivatives of f,

$$f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^{n+1}}\,d\zeta \qquad (n = 1, 2, 3, \ldots)$$

is valid under certain conditions on z and Γ. State these, and prove the formula.

8 Suppose P and Q are polynomials, the degree of Q exceeds that of P by at least 2, and the rational function $R = P/Q$ has no pole on the real axis. Prove that the integral of R over $(-\infty, \infty)$ is $2\pi i$ times the sum of the residues of R in the upper half plane. [Replace the integral over $(-A, A)$ by one over a suitable semicircle, and apply the residue theorem.] What is the analogous statement for the lower half plane? Use this method to compute

$$\int_{-\infty}^{\infty} \frac{x^2}{1+x^4}\,dx.$$

9 Compute $\int_{-\infty}^{\infty} e^{itx}/(1+x^2)\,dx$ for real t, by the method described in Exercise 8. Check your answer against the inversion theorem for Fourier transforms.

10 Let γ be the positively oriented unit circle, and compute

$$\frac{1}{2\pi i}\int_\gamma \frac{e^z - e^{-z}}{z^4}\,dz.$$

11 Suppose α is a complex number, $|\alpha| \neq 1$, and compute

$$\int_0^{2\pi} \frac{d\theta}{1 - 2\alpha\cos\theta + \alpha^2}$$

by integrating $(z-\alpha)^{-1}(z - 1/\alpha)^{-1}$ over the unit circle.

12 Compute

$$\int_{-\infty}^{\infty} \left(\frac{\sin x}{x}\right)^2 e^{itx}\,dx \qquad \text{(for real } t\text{)}.$$

13 Compute

$$\int_0^{\infty} \frac{dx}{1+x^n} \qquad (n = 2, 3, 4, \ldots).$$

[For even n, the method of Exercise 8 can be used. However, a different path can be chosen, which simplifies the computation and which also works for odd n: from 0 to R to $R\exp(2\pi i/n)$ to 0.]

Answer: $(\pi/n)/\sin(\pi/n)$.

14 Suppose Ω_1 and Ω_2 are plane regions, f and g are nonconstant complex functions defined in Ω_1 and Ω_2, respectively, and $f(\Omega_1) \subset \Omega_2$. Put $h = g \circ f$. If f and g are holomorphic, we know that h is holomorphic. Suppose we know that f and h are holomorphic. Can we conclude anything about g? What if we know that g and h are holomorphic?

15 Suppose Ω is a region, $\varphi \in H(\Omega)$, φ' has no zero in Ω, $f \in H(\varphi(\Omega))$, $g = f \circ \varphi$, $z_0 \in \Omega$, and $w_0 = \varphi(z_0)$. Prove that if f has a zero of order m at w_0, then g also has a zero of order m at z_0. How is this modified if φ' has a zero of order k at z_0?

16 Suppose μ is a complex measure on a measure space X, Ω is an open set in the plane, φ is a bounded function on $\Omega \times X$ such that $\varphi(z, t)$ is a measurable function of t, for each $z \in \Omega$, and $\varphi(z, t)$ is holomorphic in Ω, for each $t \in X$. Define

$$f(z) = \int_X \varphi(z, t)\, d\mu(t)$$

for $z \in \Omega$. Prove that $f \in H(\Omega)$. *Hint*: Show that to every compact $K \subset \Omega$ there corresponds a constant $M < \infty$ such that

$$\left| \frac{\varphi(z, t) - \varphi(z_0, t)}{z - z_0} \right| < M \qquad (z \text{ and } z_0 \in K,\ t \in X).$$

17 Determine the regions in which the following functions are defined and holomorphic:

$$f(z) = \int_0^1 \frac{dt}{1 + tz}, \qquad g(z) = \int_0^\infty \frac{e^{tz}}{1 + t^2}\, dt, \qquad h(z) = \int_{-1}^1 \frac{e^{tz}}{1 + t^2}\, dt.$$

Hint: Either use Exercise 16, or combine Morera's theorem with Fubini's.

18 Suppose $f \in H(\Omega)$, $\bar{D}(a; r) \subset \Omega$, γ is the positively oriented circle with center at a and radius r, and f has no zero on γ^*. For $p = 0$, the integral

$$\frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, z^p\, dz$$

is equal to the number of zeros of f in $D(a; r)$. What is the value of this integral (in terms of the zeros of f) for $p = 1, 2, 3, \ldots$? What is the answer if z^p is replaced by any $\varphi \in H(\Omega)$?

19 Suppose $f \in H(U)$, $g \in H(U)$, and neither f nor g has a zero in U. If

$$\frac{f'}{f}\left(\frac{1}{n}\right) = \frac{g'}{g}\left(\frac{1}{n}\right) \qquad (n = 1, 2, 3, \ldots)$$

find another simple relation between f and g.

20 Suppose Ω is a region, $f_n \in H(\Omega)$ for $n = 1, 2, 3, \ldots$, none of the functions f_n has a zero in Ω, and $\{f_n\}$ converges to f uniformly on compact subsets of Ω. Prove that either f has no zero in Ω or $f(z) = 0$ for all $z \in \Omega$.

If Ω' is a region that contains every $f_n(\Omega)$, and if f is not constant, prove that $f(\Omega) \subset \Omega'$.

21 Suppose $f \in H(\Omega)$, Ω contains the closed unit disc, and $|f(z)| < 1$ if $|z| = 1$. How many fixed points must f have in the disc? That is, how many solutions does the equation $f(z) = z$ have there?

22 Suppose $f \in H(\Omega)$, Ω contains the closed unit disc, $|f(z)| > 2$ if $|z| = 1$, and $f(0) = 1$. Must f have a zero in the unit disc?

23 Suppose $P_n(z) = 1 + z/1! + \cdots + z^n/n!$, $Q_n(z) = P_n(z) - 1$, where $n = 1, 2, 3, \ldots$. What can you say about the location of the zeros of P_n and Q_n for large n? Be as specific as you can.

24 Prove the following general form of Rouché's theorem: Let Ω be the interior of a compact set K in the plane. Suppose f and g are continuous on K and holomorphic in Ω, and $|f(z) - g(z)| < |f(z)|$ for all $z \in K - \Omega$. Then f and g have the same number of zeros in Ω.

25 Let A be the annulus $\{z: r_1 < |z| < r_2\}$, where r_1 and r_2 are given positive numbers.

(*a*) Show that the Cauchy formula

$$f(z) = \frac{1}{2\pi i}\left(\int_{\gamma_1} + \int_{\gamma_2}\right) \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

is valid under the following conditions: $f \in H(A)$,

$$r_1 + \epsilon < |z| < r_2 - \epsilon,$$

and

$$\gamma_1(t) = (r_1 + \epsilon)e^{-it}, \qquad \gamma_2(t) = (r_2 - \epsilon)e^{it} \qquad (0 \le t \le 2\pi).$$

(*b*) Show by means of (*a*) that every $f \in H(A)$ can be decomposed into a sum $f = f_1 + f_2$, where f_1 is holomorphic outside $\bar{D}(0; r_1)$ and $f_2 \in H(D(0; r_2))$; the decomposition is unique if we require that $f_1(z) \to 0$ as $|z| \to \infty$.

(*c*) Use this decomposition to associate with each $f \in H(A)$ its so-called "Laurent series"

$$\sum_{-\infty}^{\infty} c_n z^n$$

which converges to f in A. Show that there is only one such series for each f. Show that it converges to f uniformly on compact subsets of A.

(*d*) If $f \in H(A)$ and f is bounded in A, show that the components f_1 and f_2 are also bounded.

(*e*) How much of the foregoing can you extend to the case $r_1 = 0$ (or $r_2 = \infty$, or both)?

(*f*) How much of the foregoing can you extend to regions bounded by finitely many (more than two) circles?

26 It is required to expand the function

$$\frac{1}{1 - z^2} + \frac{1}{3 - z}$$

in a series of the form $\sum_{-\infty}^{\infty} c_n z^n$.

How many such expansions are there? In which region is each of them valid? Find the coefficients c_n explicitly for each of these expansions.

27 Suppose Ω is a horizontal strip, determined by the inequalities $a < y < b$, say. Suppose $f \in H(\Omega)$, and $f(z) = f(z + 1)$ for all $z \in \Omega$. Prove that f has a Fourier expansion in Ω,

$$f(z) = \sum_{-\infty}^{\infty} c_n e^{2\pi i n z},$$

which converges uniformly in $\{z: a + \epsilon \le y \le b - \epsilon\}$, for every $\epsilon > 0$. *Hint*: The map $z \to e^{2\pi i z}$ converts f to a function in an annulus.

Find the integral formulas by means of which the coefficients c_n can be computed from f.

28 Suppose Γ is a closed curve in the plane, with parameter interval $[0, 2\pi]$. Take $\alpha \notin \Gamma^*$. Approximate Γ uniformly by trigonometric polynomials Γ_n. Show that $\mathrm{Ind}_{\Gamma_m}(\alpha) = \mathrm{Ind}_{\Gamma_n}(\alpha)$ if m and n are sufficiently large. Define this common value to be $\mathrm{Ind}_{\Gamma}(\alpha)$. Prove that the result does not depend on the choice of $\{\Gamma_n\}$; prove that Lemma 10.39 is now true for closed curves, and use this to give a different proof of Theorem 10.40.

29 Define

$$f(z) = \frac{1}{\pi} \int_0^1 r\, dr \int_{-\pi}^{\pi} \frac{d\theta}{re^{i\theta} + z}.$$

Show that $f(z) = \bar{z}$ if $|z| < 1$ and that $f(z) = 1/z$ if $|z| \ge 1$.

Thus f is not holomorphic in the unit disc, although the integrand is a holomorphic function of z. Note the contrast between this, on the one hand, and Theorem 10.7 and Exercise 16 on the other.

Suggestion: Compute the inner integral separately for $r < |z|$ and for $r > |z|$.

30 Let Ω be the plane minus two points, and show that some closed paths Γ in Ω satisfy assumption (1) of Theorem 10.35 without being null-homotopic in Ω.

CHAPTER

ELEVEN

HARMONIC FUNCTIONS

The Cauchy-Riemann Equations

11.1 The Operators ∂ and $\bar{\partial}$ Suppose f is a complex function defined in a plane open set Ω. Regard f as a transformation which maps Ω into R^2, and assume that f has a differential at some point $z_0 \in \Omega$, in the sense of Definition 7.22. For simplicity, suppose $z_0 = f(z_0) = 0$. Our differentiability assumption is then equivalent to the existence of two complex numbers α and β (the partial derivatives of f with respect to x and y at $z_0 = 0$) such that

$$f(z) = \alpha x + \beta y + \eta(z)z \qquad (z = x + iy), \tag{1}$$

where $\eta(z) \to 0$ as $z \to 0$.

Since $2x = z + \bar{z}$ and $2iy = z - \bar{z}$, (1) can be rewritten in the form

$$f(z) = \frac{\alpha - i\beta}{2} z + \frac{\alpha + i\beta}{2} \bar{z} + \eta(z)z. \tag{2}$$

This suggests the introduction of the differential operators

$$\partial = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{3}$$

Now (2) becomes

$$\frac{f(z)}{z} = (\partial f)(0) + (\bar{\partial} f)(0) \cdot \frac{\bar{z}}{z} + \eta(z) \qquad (z \neq 0). \tag{4}$$

For real z, $\bar{z}/z = 1$; for pure imaginary z, $\bar{z}/z = -1$. Hence $f(z)/z$ has a limit at 0 if and only if $(\bar{\partial} f)(0) = 0$, and we obtain the following characterization of holomorphic functions:

11.2 Theorem *Suppose f is a complex function in Ω that has a differential at every point of Ω. Then $f \in H(\Omega)$ if and only if the Cauchy-Riemann equation*

$$(\bar{\partial} f)(z) = 0 \tag{1}$$

holds for every $z \in \Omega$. In that case we have

$$f'(z) = (\partial f)(z) \qquad (z \in \Omega). \tag{2}$$

If $f = u + iv$, u and v real, (1) splits into the pair of equations

$$u_x = v_y, \qquad u_y = -v_x$$

where the subscripts refer to partial differentiation with respect to the indicated variable. These are the *Cauchy-Riemann equations* which must be satisfied by the real and imaginary parts of a holomorphic function.

11.3 The Laplacian Let f be a complex function in a plane open set Ω, such that f_{xx} and f_{yy} exist at every point of Ω. The *Laplacian* of f is then defined to be

$$\Delta f = f_{xx} + f_{yy}. \tag{1}$$

If f is continuous in Ω and if

$$\Delta f = 0 \tag{2}$$

at every point of Ω, then f is said to be *harmonic* in Ω.

Since the Laplacian of a real function is real (if it exists), it is clear that *a complex function is harmonic in Ω if and only if both its real part and its imaginary part are harmonic in Ω.*

Note that

$$\Delta f = 4\partial\bar{\partial} f \tag{3}$$

provided that $f_{xy} = f_{yx}$, and that this happens for all f which have continuous second-order derivatives.

If f is holomorphic, then $\bar{\partial} f = 0$, f has continuous derivatives of all orders, and therefore (3) shows:

11.4 Theorem *Holomorphic functions are harmonic.*

We shall now turn our attention to an integral representation of harmonic functions which is closely related to the Cauchy formula for holomorphic functions. It will show, among other things, that every real harmonic function is locally the real part of a holomorphic function, and it will yield information about the boundary behavior of certain classes of holomorphic functions in open discs.

The Poisson Integral

11.5 The Poisson Kernel This is the function

$$P_r(t) = \sum_{-\infty}^{\infty} r^{|n|} e^{int} \qquad (0 \le r < 1,\ t \text{ real}). \tag{1}$$

We may regard $P_r(t)$ as a function of two variables r and t or as a family of functions of t, indexed by r.

If $z = re^{i\theta}$ $(0 \le r < 1,\ \theta$ real), a simple calculation, made in Sec. 5.24, shows that

$$P_r(\theta - t) = \operatorname{Re}\left[\frac{e^{it} + z}{e^{it} - z}\right] = \frac{1 - r^2}{1 - 2r\cos(\theta - t) + r^2}. \tag{2}$$

From (1) we see that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(t)\,dt = 1 \qquad (0 \le r < 1). \tag{3}$$

From (2) it follows that $P_r(t) > 0$, $P_r(t) = P_r(-t)$, that

$$P_r(t) < P_r(\delta) \qquad (0 < \delta < |t| \le \pi), \tag{4}$$

and that

$$\lim_{r \to 1} P_r(\delta) = 0 \qquad (0 < \delta \le \pi). \tag{5}$$

These properties are reminiscent of the trigonometric polynomials $Q_k(t)$ that were discussed in Sec. 4.24.

The open unit disc $D(0; 1)$ will from now on be denoted by U. The unit circle - the boundary of U in the complex plane - will be denoted by T. Whenever it is convenient to do so, we shall identify the spaces $L^p(T)$ and $C(T)$ with the corresponding spaces of 2π-periodic functions on R^1, as in Sec. 4.23.

One can also regard $P_r(\theta - t)$ as a function of $z = re^{i\theta}$ and e^{it}. Then (2) becomes

$$P(z, e^{it}) = \frac{1 - |z|^2}{|e^{it} - z|^2} \tag{6}$$

for $z \in U$, $e^{it} \in T$.

11.6 The Poisson Integral If $f \in L^1(T)$ and

$$F(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(t)\,dt, \tag{1}$$

then the function F so defined in U is called the *Poisson integral* of f. We shall sometimes abbreviate the relation (1) to

$$F = P[f]. \tag{2}$$

If f is real, formula 11.5(2) shows that $P[f]$ is the real part of

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}f(t)\,dt, \tag{3}$$

which is a holomorphic function of $z = re^{i\theta}$ in U, by Theorem 10.7. Hence $P[f]$ is harmonic in U. Since linear combinations (with constant coefficients) of harmonic functions are harmonic, we see that the following is true:

11.7 Theorem *If $f \in L^1(T)$ then the Poisson integral $P[f]$ is a harmonic function in U.*

The following theorem shows that Poisson integrals of continuous functions behave particularly well near the boundary of U.

11.8 Theorem *If $f \in C(T)$ and if Hf is defined on the closed unit disc $\bar{U}$ by*

$$(Hf)(re^{i\theta}) = \begin{cases} f(e^{i\theta}) & \text{if } r = 1, \\ P[f](re^{i\theta}) & \text{if } 0 \le r < 1, \end{cases} \tag{1}$$

then $Hf \in C(\bar{U})$.

PROOF Since $P_r(t) > 0$, formula 11.5(3) shows, for every $g \in C(T)$, that

$$|P[g](re^{i\theta})| \le \|g\|_T \qquad (0 \le r < 1), \tag{2}$$

so that

$$\|Hg\|_{\bar{U}} = \|g\|_T \qquad (g \in C(T)). \tag{3}$$

(As in Sec. 5.22, we use the notation $\|g\|_E$ to denote the supremum of $|g|$ on the set E.) If

$$g(e^{i\theta}) = \sum_{n=-N}^{N} c_n e^{in\theta} \tag{4}$$

is any trigonometric polynomial, it follows from 11.5(1) that

$$(Hg)(re^{i\theta}) = \sum_{n=-N}^{N} c_n r^{|n|} e^{in\theta}, \tag{5}$$

so that $Hg \in C(\bar{U})$.

Finally, there are trigonometric polynomials g_k such that $\|g_k - f\|_T \to 0$ as $k \to \infty$. (See Sec. 4.24.) By (3), it follows that

$$\|Hg_k - Hf\|_{\bar{U}} = \|H(g_k - f)\|_{\bar{U}} \to 0 \tag{6}$$

as $k \to \infty$. This says that the functions $Hg_k \in C(\bar{U})$ converge, uniformly on $\bar{U}$, to Hf. Hence $Hf \in C(\bar{U})$. ////

Note: This theorem provides the solution of a boundary value problem (the *Dirichlet problem*): A continuous function f is given on T and it is required to find a harmonic function F in U "whose boundary values are f." The theorem exhibits a solution, by means of the Poisson integral of f, and it states the relation between f and F more precisely. The uniqueness theorem which corresponds to this existence theorem is contained in the following result.

11.9 Theorem *Suppose u is a continuous real function on the closed unit disc $\bar{U}$, and suppose u is harmonic in U. Then (in U) u is the Poisson integral of its restriction to T, and u is the real part of the holomorphic function*

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} u(e^{it})\, dt \qquad (z \in U). \tag{1}$$

PROOF Theorem 10.7 shows that $f \in H(U)$. If $u_1 = \operatorname{Re} f$, then (1) shows that u_1 is the Poisson integral of the boundary values of u, and the theorem will be proved as soon as we show that $u = u_1$.

Put $h = u - u_1$. Then h is continuous on $\bar{U}$ (apply Theorem 11.8 to u_1), h is harmonic in U, and $h = 0$ at all points of T. Assume (this will lead to a contradiction) that $h(z_0) > 0$ for some $z_0 \in U$. Fix ϵ so that $0 < \epsilon < h(z_0)$, and define

$$g(z) = h(z) + \epsilon |z|^2 \qquad (z \in \bar{U}). \tag{2}$$

Then $g(z_0) \geq h(z_0) > \epsilon$. Since $g \in C(\bar{U})$ and since $g = \epsilon$ at all points of T, there exists a point $z_1 \in U$ at which g has a local maximum. This implies that $g_{xx} \leq 0$ and $g_{yy} \leq 0$ at z_1. But (2) shows that the Laplacian of g is $4\epsilon > 0$, and we have a contradiction.

Thus $u - u_1 \leq 0$. The same argument shows that $u_1 - u \leq 0$. Hence $u = u_1$, and the proof is complete. ////

11.10 So far we have considered only the unit disc $U = D(0; 1)$. It is clear that the preceding work can be carried over to arbitrary circular discs, by a simple change of variables. Hence we shall merely summarize some of the results:

If u is a continuous real function on the boundary of the disc $D(a; R)$ and if u is defined in $D(a; R)$ by the Poisson integral

$$u(a + re^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{R^2 - r^2}{R^2 - 2Rr\cos(\theta - t) + r^2} u(a + Re^{it})\, dt \tag{1}$$

then u is continuous on $\bar{D}(a; R)$ and harmonic in $D(a; R)$.

If u is harmonic (and real) in an open set Ω and if $\bar{D}(a; R) \subset \Omega$, then u satisfies (1) in $D(a; R)$ and there is a holomorphic function f defined in $D(a; R)$ whose real part is u. This f is uniquely defined, up to a pure imaginary additive constant. For if two functions, holomorphic in the same region, have the same real part, their difference must be constant (a corollary of the open mapping theorem, or the Cauchy-Riemann equations).

We may summarize this by saying that *every real harmonic function is locally the real part of a holomorphic function.*

Consequently, every harmonic function has continuous partial derivatives of all orders.

The Poisson integral also yields information about sequences of harmonic functions:

11.11 Harnack's Theorem *Let $\{u_n\}$ be a sequence of harmonic functions in a region Ω.*

(*a*) *If $u_n \to u$ uniformly on compact subsets of Ω, then u is harmonic in Ω.*
(*b*) *If $u_1 \le u_2 \le u_3 \le \cdots$, then either $\{u_n\}$ converges uniformly on compact subsets of Ω, or $u_n(z) \to \infty$ for every $z \in \Omega$.*

PROOF To prove (*a*), assume $\bar{D}(a; R) \subset \Omega$, and replace u by u_n in the Poisson integral 11.10(1). Since $u_n \to u$ uniformly on the boundary of $\bar{D}(a; R)$, we conclude that u itself satisfies 11.10(1) in $D(a; R)$.

In the proof of (*b*), we may assume that $u_1 \ge 0$. (If not, replace u_n by $u_n - u_1$.) Put $u = \sup u_n$, let $A = \{z \in \Omega: u(z) < \infty\}$, and $B = \Omega - A$. Choose $\bar{D}(a; R) \subset \Omega$. The Poisson kernel satisfies the inequalities

$$\frac{R-r}{R+r} \le \frac{R^2 - r^2}{R^2 - 2rR\cos(\theta - t) + r} \le \frac{R+r}{R-r}$$

for $0 \le r < R$. Hence

$$\frac{R-r}{R+r}\, u_n(a) \le u_n(a + re^{i\theta}) \le \frac{R+r}{R-r}\, u_n(a).$$

The same inequalities hold with u in place of u_n. It follows that either $u(z) = \infty$ for all $z \in D(a; R)$ or $u(z) < \infty$ for all $z \in D(a; R)$.

Thus both A and B are open; and since Ω is connected, we have either $A = \varnothing$ (in which case there is nothing to prove) or $A = \Omega$. In the latter case, the monotone convergence theorem shows that the Poisson formula holds for u in every disc in Ω. Hence u is harmonic in Ω. Whenever a sequence of continuous functions converges monotonically to a continuous limit, the convergence is uniform on compact sets ([26], Theorem 7.13). This completes the proof. ////

The Mean Value Property

11.12 Definition We say that a continuous function u in an open set Ω has the *mean value property* if to every $z \in \Omega$ there corresponds a sequence $\{r_n\}$ such that $r_n > 0$, $r_n \to 0$ as $n \to \infty$, and

$$u(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(z + r_n e^{it})\, dt \qquad (n = 1, 2, 3, \ldots). \tag{1}$$

In other words, $u(z)$ is to be equal to the mean value of u on the circles of radius r_n and with center at z.

Note that the Poisson formula shows that (1) holds for every harmonic function u, and for *every* r such that $\bar{D}(z; r) \subset \Omega$. Thus harmonic functions satisfy a much stronger mean value property than the one that we just defined. The following theorem may therefore come as a surprise:

11.13 Theorem *If a continuous function u has the mean value property in an open set Ω, then u is harmonic in Ω.*

PROOF It is enough to prove this for real u. Fix $\bar{D}(a; R) \subset \Omega$. The Poisson integral gives us a continuous function h on $\bar{D}(a; R)$ which is harmonic in $D(a; R)$ and which coincides with u on the boundary of $D(a; R)$. Put $v = u - h$, and let $m = \sup \{v(z): z \in \bar{D}(a; R)\}$. Assume $m > 0$, and let E be the set of all $z \in \bar{D}(a; R)$ at which $v(z) = m$. Since $v = 0$ on the boundary of $D(a; R)$, E is a compact subset of $D(a; R)$. Hence there exists a $z_0 \in E$ such that

$$|z_0 - a| \geq |z - a|$$

for all $z \in E$. For *all* small enough r, at least half the circle with center z_0 and radius r lies outside E, so that the corresponding mean values of v are all less than $m = v(z_0)$. But v has the mean value property, and we have a contradiction. Thus $m = 0$, so $v \leq 0$. The same reasoning applies to $-v$. Hence $v = 0$, or $u = h$ in $D(a; R)$, and since $\bar{D}(a; R)$ was an arbitrary closed disc in Ω, u is harmonic in Ω. ////

Theorem 11.13 leads to a reflection theorem for holomorphic functions. By the *upper half plane* Π^+ we mean the set of all $z = x + iy$ with $y > 0$; the *lower half plane* Π^- consists of all z whose imaginary part is negative.

11.14 Theorem (The Schwarz reflection principle) *Suppose L is a segment of the real axis, Ω^+ is a region in Π^+, and every $t \in L$ is the center of an open disc D_t such that $\Pi^+ \cap D_t$ lies in Ω^+. Let Ω^- be the reflection of Ω^+:*

$$\Omega^- = \{z: \bar{z} \in \Omega^+\}. \tag{1}$$

Suppose $f = u + iv$ is holomorphic in Ω^+, and

$$\lim_{n\to\infty} v(z_n) = 0 \tag{2}$$

for every sequence $\{z_n\}$ in Ω^+ which converges to a point of L.

Then there is a function F, holomorphic in $\Omega^+ \cup L \cup \Omega^-$, such that $F(z) = f(z)$ in Ω^+; this F satisfies the relation

$$F(\bar{z}) = \overline{F(z)} \qquad (z \in \Omega^+ \cup L \cup \Omega^-). \tag{3}$$

The theorem asserts that f can be extended to a function which is holomorphic in a region symmetric with respect to the real axis, and (3) states that F preserves this symmetry. Note that the continuity hypothesis (2) is merely imposed on the imaginary part of f.

PROOF Put $\Omega = \Omega^+ \cup L \cup \Omega^-$. We extend v to Ω by defining $v(z) = 0$ for $z \in L$ and $v(z) = -v(\bar{z})$ for $z \in \Omega^-$. It is then immediate that v is continuous and that v has the mean value property in Ω, so that v is harmonic in Ω, by Theorem 11.13.

Hence v is locally the imaginary part of a holomorphic function. This means that to each of the discs D_t there corresponds an $f_t \in H(D_t)$ such that $\operatorname{Im} f_t = v$. Each f_t is determined by v up to a real additive constant. If this constant is chosen so that $f_t(z) = f(z)$ for *some* $z \in D_t \cap \Pi^+$, the same will hold for all $z \in D_t \cap \Pi^+$, since $f - f_t$ is constant in the region $D_t \cap \Pi^+$. We assume that the functions f_t are so adjusted.

The power series expansion of f_t in powers of $z - t$ has only real coefficients, since $v = 0$ on L, so that all derivatives of f_t are real at t. It follows that

$$f_t(\bar{z}) = \overline{f_t(z)} \qquad (z \in D_t). \tag{4}$$

Next, assume that $D_s \cap D_t \neq \varnothing$. Then $f_t = f = f_s$ in $D_t \cap D_s \cap \Pi^+$; and since $D_t \cap D_s$ is connected, Theorem 10.18 shows that

$$f_t(z) = f_s(z) \qquad (z \in D_t \cap D_s). \tag{5}$$

Thus it is consistent to define

$$F(z) = \begin{cases} f(z) & \text{for } z \in \Omega^+ \\ f_t(z) & \text{for } z \in D_t \\ \overline{f(\bar{z})} & \text{for } z \in \Omega^- \end{cases} \tag{6}$$

and it remains to show that F is holomorphic in Ω^-. If $D(a; r) \subset \Omega^-$, then $D(\bar{a}; r) \subset \Omega^+$, so for every $z \in D(a; r)$ we have

$$f(\bar{z}) = \sum_{n=0}^{\infty} c_n(\bar{z} - \bar{a})^n. \tag{7}$$

Hence

$$F(z) = \sum_{n=0}^{\infty} \bar{c}_n (z-a)^n \qquad (z \in D(a; r)). \tag{8}$$

This completes the proof. ////

Boundary Behavior of Poisson Integrals

11.15 Our next objective is to find analogues of Theorem 11.8 for Poisson integrals of L^p-functions and measures on T.

Let us associate to any function u in U a family of functions u_r on T, defined by

$$u_r(e^{i\theta}) = u(re^{i\theta}) \qquad (0 \le r < 1). \tag{1}$$

Thus u_r is essentially the restriction of u to the circle with radius r, center 0, but we shift the domain of u_r to T.

Using this terminology, Theorem 11.8 can be stated in the following form:

If $f \in C(T)$ and $F = P[f]$, then $F_r \to f$ uniformly on T, as $r \to 1$. In other words,

$$\lim_{r \to 1} \|F_r - f\|_\infty = 0, \tag{2}$$

which implies of course that

$$\lim_{r \to 1} F_r(e^{i\theta}) = f(e^{i\theta}) \tag{3}$$

at every point of T.

As regards (2), we shall now see (Theorem 11.16) that the corresponding norm-convergence result is just as easy in L^p. Instead of confining ourselves to investigating *radial limits*, as in (3), we shall then study *nontangential limits* of Poisson integrals of measures and L^p-functions; the differentiation theory developed in Chap. 7 will play an essential role in that study.

11.16 Theorem *If $1 \le p \le \infty$, $f \in L^p(T)$, and $u = P[f]$, then*

$$\|u_r\|_p \le \|f\|_p \qquad (0 \le r < 1). \tag{1}$$

If $1 \le p < \infty$, then

$$\lim_{r \to 1} \|u_r - f\|_p = 0. \tag{2}$$

PROOF If we apply Jensen's inequality (or Hölder's) to

$$u_r(e^{i\theta}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) P_r(\theta - t)\, dt \tag{3}$$

we obtain

$$|u_r(e^{i\theta})|^p \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(t)|^p P_r(\theta - t)\, dt. \tag{4}$$

If we integrate (4) with respect to θ, over $[-\pi, \pi]$ and use Fubini's theorem, we obtain (1).

Note that formula 11.5(3) was used twice in this argument.

To prove (2), choose $\epsilon > 0$, and choose $g \in C(T)$ so that $\|g - f\|_p < \epsilon$ (Theorem 3.14). Let $v = P[g]$. Then

$$u_r - f = (u_r - v_r) + (v_r - g) + (g - f). \tag{5}$$

By (1), $\|u_r - v_r\|_p = \|(u - v)_r\|_p \leq \|f - g\|_p < \epsilon$. Thus

$$\|u_r - f\|_p \leq 2\epsilon + \|v_r - g\|_p \tag{6}$$

for all $r < 1$. Also, $\|v_r - g\|_p \leq \|v_r - g\|_\infty$, and the latter converges to 0 as $r \to 1$, by Theorem 11.8. This proves (2). ////

11.17 Poisson Integrals of Measures If μ is a complex measure on T, and if we want to replace integrals over T by integrals over intervals of length 2π in R^1, these intervals have to be taken half open, because of the possible presence of point masses in μ. To avoid this (admittedly very minor) problem, we shall keep integration on the circle in what follows, and will write the Poisson integral $u = P[d\mu]$ of μ in the form

$$u(z) = \int_T P(z, e^{it})\, d\mu(e^{it}) \qquad (z \in U) \tag{1}$$

where $P(z, e^{it}) = (1 - |z|^2)/|e^{it} - z|^2$, as in formula 11.5(6).

The reasoning that led to Theorem 11.7 applies without change to Poisson integrals of measures. *Thus u, defined by* (1), *is harmonic in U.*

Setting $\|\mu\| = |\mu|(T)$, the analogue of the first half of Theorem 11.16 is

$$\|u_r\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |u(re^{i\theta})|\, d\theta \leq \|\mu\|. \tag{2}$$

To see this, replace μ by $|\mu|$ in (1), apply Fubini's theorem, and refer to formula 11.5(3).

11.18 Approach Regions For $0 < \alpha < 1$, we define Ω_α to be the union of the disc $D(0; \alpha)$ and the line segments from $z = 1$ to points of $D(0; \alpha)$.

In other words, Ω_α is the smallest convex open set that contains $D(0; \alpha)$ and has the point 1 in its boundary. Near $z = 1$, Ω_α is an angle, bisected by the radius of U that terminates at 1, of opening 2θ, where $\alpha = \sin\theta$. Curves that approach 1 within Ω_α cannot be tangent to T. Therefore Ω_α is called a *nontangential approach* region, with vertex 1.

The regions Ω_α expand when α increases. Their union is U, their intersection is the radius $[0, 1)$.

Rotated copies of Ω_α, with vertex at e^{it}, will be denoted by $e^{it}\Omega_\alpha$.

11.19 Maximal Functions If $0 < \alpha < 1$ and u is any complex function with domain U, its *nontangential maximal function* $N_\alpha u$ is defined on T by

$$(N_\alpha u)(e^{it}) = \sup \{ |u(z)| : z \in e^{it}\Omega_\alpha \}. \tag{1}$$

Similarly, the *radial maximal function* of u is

$$(M_{\text{rad}}\, u)(e^{it}) = \sup \{ |u(re^{it})| : 0 \le r < 1 \}. \tag{2}$$

If u is *continuous* and λ is a positive number, then the set where either of these maximal functions is $\le \lambda$ is a closed subset of T. Consequently, $N_\alpha u$ and $M_{\text{rad}}\, u$ are *lower semicontinuous* on T; in particular, they are measurable.

Clearly, $M_{\text{rad}}\, u \le N_\alpha u$, and the latter increases with α. If $u = P[d\mu]$, Theorem 11.20 will show that the size of $N_\alpha u$ is, in turn, controlled by the maximal function $M\mu$ that was defined in Sec. 7.2 (taking $k = 1$). However, it will simplify the notation if we replace ordinary Lebesgue measure m on T by $\sigma = m/2\pi$. Then σ is a rotation-invariant positive Borel measure on T, so normalized that $\sigma(T) = 1$.

Accordingly, $M\mu$ is now defined by

$$(M\mu)(e^{i\theta}) = \sup \frac{|\mu|(I)}{\sigma(I)}. \tag{3}$$

The supremum is taken over all open arcs $I \subset T$ whose centers are at $e^{i\theta}$, including T itself (even though T is of course not an arc).

Similarly, the derivative $D\mu$ of a measure μ on T is now

$$(D\mu)(e^{i\theta}) = \lim \frac{\mu(I)}{\sigma(I)}, \tag{4}$$

as the open arcs $I \subset T$ shrink to their center $e^{i\theta}$, and $e^{i\theta}$ is a *Lebesgue point* of $f \in L^1(T)$ if

$$\lim \frac{1}{\sigma(I)} \int_I |f - f(e^{i\theta})|\, d\sigma = 0, \tag{5}$$

where $\{I\}$ is as in (4).

If $d\mu = f\, d\sigma + d\mu_s$ is the Lebesgue decomposition of a complex Borel measure μ on T, where $f \in L^1(T)$ and $\mu_s \perp \sigma$, Theorems 7.4, 7.7, and 7.14 assert that

$$\sigma\{M\mu > \lambda\} \le \frac{3}{\lambda} \|\mu\|, \tag{6}$$

that almost every point of T is a Lebesgue point of f, and that $D\mu = f$, $D\mu_s = 0$ a.e. $[\sigma]$.

We will now see, for any complex Borel measure μ on T, that the nontangential and radial maximal functions of the harmonic function $P[d\mu]$ are controlled by $M\mu$. In fact, if any one of them is finite at some point of T, so are the others; this can be seen by combining Theorem 11.20 with Exercise 19.

11.20 Theorem *Assume* $0 < \alpha < 1$. *Then there is a constant* $c_\alpha > 0$ *with the following property: If* μ *is a positive finite Borel measure on* T *and* $u = P[d\mu]$ *is its Poisson integral, then the inequalities*

$$c_\alpha(N_\alpha u)(e^{i\theta}) \le (M_{\text{rad}}\, u)(e^{i\theta}) \le (M\mu)(e^{i\theta}) \tag{1}$$

hold at every point $e^{i\theta} \in T$.

PROOF We shall prove (1) for $\theta = 0$. The general case follows then if the special case is applied to the rotated measure $\mu_\theta(E) = \mu(e^{i\theta}E)$.

Since $u(z) = \int_T P(z, e^{it})\, d\mu(e^{it})$, the first inequality in (1) will follow if we can show that

$$c_\alpha P(z, e^{it}) \le P(|z|, e^{it}) \tag{2}$$

holds for all $z \in \Omega_\alpha$ and all $e^{it} \in T$. By formula 11.5(6), (2) is the same as

$$c_\alpha |e^{it} - r|^2 \le |e^{it} - z|^2 \tag{3}$$

where $r = |z|$.

The definition of Ω_α shows that $|z - r|/(1 - r)$ is bounded in Ω_α, say by γ_α. Hence

$$\begin{aligned} |e^{it} - r| &\le |e^{it} - z| + |z - r| \\ &\le |e^{it} - z| + \gamma_\alpha(1 - r) \le (1 + \gamma_\alpha)|e^{it} - z| \end{aligned}$$

so that (3) holds with $c_\alpha = (1 + \gamma_\alpha)^{-2}$. This proves the first half of (1).

For the second half, we have to prove that

$$\int_T P_r(t)\, d\mu(e^{it}) \le (M\mu)(1) \qquad (0 \le r \le 1). \tag{4}$$

Fix r. Choose open arcs $I_j \subset T$, centered at 1, so that $I_1 \subset I_2 \subset \cdots \subset I_{n-1}$, put $I_n = T$. For $1 \le j \le n$, let χ_j be the characteristic function of I_j, and let h_j be the largest positive number for which $h_j \chi_j \le P_r$ on T. Define

$$K = \sum_{j=1}^{n} (h_j - h_{j+1})\chi_j \tag{5}$$

where $h_{n+1} = 0$. Since $P_r(t)$ is an even function of t that decreases as t increases from 0 to π, we see that $h_j - h_{j+1} \ge 0$, that $K = h_j$ on $I_j - I_{j-1}$ (putting $I_0 = \varnothing$), and that $K \le P_r$. The definition of $M\mu$ shows that

$$\mu(I_j) \le (M\mu)(1)\sigma(I_j). \tag{6}$$

Hence, setting $(M\mu)(1) = M$,

$$\int_T K\, d\mu = \sum_{j=1}^{n} (h_j - h_{j+1})\mu(I_j) \le M \sum_{j=1}^{n} (h_j - h_{j+1})\sigma(I_j)$$

$$= M \int_T K\, d\sigma \le M \int_T P_r\, d\sigma = M. \tag{7}$$

Finally, if we choose the arcs I_j so that their endpoints form a sufficiently fine partition of T, we obtain step functions K that converge to P_r, uniformly on T. Hence (4) follows from (7). ////

11.21 Nontangential Limits A function F, defined in U, is said to *have nontangential limit* λ *at* $e^{i\theta} \in T$ if, for each $\alpha < 1$,

$$\lim_{j\to\infty} F(z_j) = \lambda$$

for every sequence $\{z_j\}$ that converges to $e^{i\theta}$ and that lies in $e^{i\theta}\Omega_\alpha$.

11.22 Theorem *If μ is a positive Borel measure on T and $(D\mu)(e^{i\theta}) = 0$ for some θ, then its Poisson integral $u = P[d\mu]$ has nontangential limit 0 at $e^{i\theta}$.*

PROOF By definition, the assumption $(D\mu)(e^{i\theta}) = 0$ means that

$$\lim \mu(I)/\sigma(I) = 0 \tag{1}$$

as the open arcs $I \subset T$ shrink to their center $e^{i\theta}$. Pick $\epsilon > 0$. One of these arcs, say I_0, is then small enough to ensure that

$$\mu(I) < \epsilon\sigma(I) \tag{2}$$

for every $I \subset I_0$ that has $e^{i\theta}$ as center.

Let μ_0 be the restriction of μ to I_0, put $\mu_1 = \mu - \mu_0$, and let u_i be the Poisson integral of μ_i $(i = 0, 1)$. Suppose z_j converges to $e^{i\theta}$ within some region $e^{i\theta}\Omega_\alpha$. Then z_j stays at a positive distance from $T - I_0$. The integrands in

$$u_1(z_j) = \int_{T-I_0} P(z_j, e^{it})\, d\mu(e^{it}) \tag{3}$$

converge therefore to 0 as $j \to \infty$, uniformly on $T - I_0$. Hence

$$\lim_{j\to\infty} u_1(z_j) = 0. \tag{4}$$

Next, use (2) together with Theorem 11.20 to see that

$$c_\alpha(N_\alpha u_0)(e^{i\theta}) \le (M\mu_0)(e^{i\theta}) \le \epsilon. \tag{5}$$

In $e^{i\theta}\Omega_\alpha$, $u_0(z) \le (N_\alpha u_0)(e^{i\theta})$. Hence (5) implies that

$$\limsup_{j\to\infty} u_0(z_j) \le \epsilon/c_\alpha. \tag{6}$$

Since $u = u_0 + u_1$ and ϵ was arbitrary, (4) and (6) give

$$\lim_{j\to\infty} u(z_j) = 0. \tag{7}$$

////

11.23 Theorem *If $f \in L^1(T)$, then $P[f]$ has nontangential limit $f(e^{i\theta})$ at every Lebesgue point $e^{i\theta}$ of f.*

PROOF Suppose $e^{i\theta}$ is a Lebesgue point of f. By subtracting a constant from f we may assume, without loss of generality, that $f(e^{i\theta}) = 0$. Then

$$\lim \frac{1}{\sigma(I)} \int_I |f|\, d\sigma = 0 \tag{1}$$

as the open arcs $I \subset T$ shrink to their center $e^{i\theta}$. Define a Borel measure μ on T by

$$\mu(E) = \int_E |f|\, d\sigma \tag{2}$$

Then (1) says that $(D\mu)(e^{i\theta}) = 0$; hence $P[d\mu]$ has nontangential limit 0 at $e^{i\theta}$, by Theorem 11.22. The same is true of $P[f]$, because

$$|P[f]| \le P[|f|] = P[d\mu]. \tag{3}$$

////

The last two theorems can be combined as follows.

11.24 Theorem *If $d\mu = f\, d\sigma + d\mu_s$ is the Lebesgue decomposition of a complex Borel measure μ on T, where $f \in L^1(T)$, $\mu_s \perp \sigma$, then $P[d\mu]$ has nontangential limit $f(e^{i\theta})$ at almost all points of T.*

PROOF Apply Theorem 11.22 to the positive and negative variations of the real and imaginary parts of μ_s, and apply Theorem 11.23 to f. ////

Here is another consequence of Theorem 11.20.

11.25 Theorem *For $0 < \alpha < 1$ and $1 \le p \le \infty$, there are constants $A(\alpha, p) < \infty$ with the following properties:*

(a) *If μ is a complex Borel measure on T, and $u = P[d\mu]$, then*

$$\sigma\{N_\alpha u > \lambda\} \leq \frac{A(\alpha, 1)}{\lambda} \|\mu\| \qquad (0 < \lambda < \infty).$$

(b) *If $1 < p \leq \infty$, $f \in L^p(T)$, and $u = P[f]$, then*

$$\|N_\alpha u\|_p \leq A(\alpha, p)\|f\|_p.$$

PROOF Combine Theorem 11.20 with Theorem 7.4 and the inequality (7) in the proof of Theorem 8.18. ////

The nontangential maximal functions $N_\alpha u$ are thus in weak L^1 if $u = P[d\mu]$, and they are in $L^p(T)$ if $u = P[f]$ for some $f \in L^p(T)$, $p > 1$. This latter result may be regarded as a strengthened form of the first part of Theorem 11.16.

Representation Theorems

11.26 How can one tell whether a harmonic function u in U is a Poisson integral or not? The preceding theorems (11.16 to 11.25) contain a number of necessary conditions. It turns out that the simplest of these, the L^p-boundedness of the family $\{u_r: 0 \leq r < 1\}$ is also sufficient! Thus, in particular, the boundedness of $\|u_r\|_1$, as $r \to 1$, implies the existence of nontangential limits a.e. on T, since, as we will see in Theorem 11.30, u can then be represented as the Poisson integral of a measure.

This measure will be obtained as a so-called "weak limit" of the functions u_r. Weak convergence is an important topic in functional analysis. We will approach it through another important concept, called equicontinuity, which we will meet again later, in connection with the so-called "normal families" of holomorphic functions.

11.27 Definition Let $\mathscr{F}$ be a collection of complex functions on a metric space X with metric ρ.

We say that $\mathscr{F}$ is *equicontinuous* if to every $\epsilon > 0$ corresponds a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for every $f \in \mathscr{F}$ and for all pairs of points x, y with $\rho(x, y) < \delta$. (In particular, every $f \in \mathscr{F}$ is then uniformly continuous.)

We say that $\mathscr{F}$ is *pointwise bounded* if to every $x \in X$ corresponds an $M(x) < \infty$ such that $|f(x)| \leq M(x)$ for every $f \in \mathscr{F}$.

11.28 Theorem (Arzela-Ascoli) *Suppose that $\mathscr{F}$ is a pointwise bounded equicontinuous collection of complex functions on a metric space X, and that X contains a countable dense subset E.*

Every sequence $\{f_n\}$ in $\mathscr{F}$ has then a subsequence that converges uniformly on every compact subset of X.

PROOF Let $x_1, x_2, x_3, \ldots$ be an enumeration of the points of E. Let S_0 be the set of all positive integers. Suppose $k \geq 1$ and an infinite set $S_{k-1} \subset S_0$ has been chosen. Since $\{f_n(x_k): n \in S_{k-1}\}$ is a bounded sequence of complex numbers, it has a convergent subsequence. In other words, there is an infinite set $S_k \subset S_{k-1}$ so that $\lim f_n(x_k)$ exists as $n \to \infty$ within S_k.

Continuing in this way, we obtain infinite sets $S_0 \supset S_1 \supset S_2 \supset \cdots$ with the property that $\lim f_n(x_j)$ exists, for $1 \leq j \leq k$, if $n \to \infty$ within S_k.

Let r_k be the kth term of S_k (with respect to the natural order of the positive integers) and put

$$S = \{r_1, r_2, r_3, \ldots\}.$$

For each k there are then at most $k - 1$ terms of S that are not in S_k.

Hence $\lim f_n(x)$ *exists, for every* $x \in E$, *as* $n \to \infty$ *within* S.

(The construction of S from $\{S_k\}$ is the so-called *diagonal process.*)

Now let $K \subset X$ be compact, pick $\epsilon > 0$. By equicontinuity, there is a $\delta > 0$ so that $\rho(p, q) < \delta$ implies $|f_n(p) - f_n(q)| < \epsilon$, for all n. Cover K with open balls $B_1, \ldots, B_M$ of radius $\delta/2$. Since E is dense in X there are points $p_i \in B_i \cap E$ for $1 \leq i \leq M$. Since $p_i \in E$, $\lim f_n(p_i)$ exists, as $n \to \infty$ within S. Hence there is an integer N such that

$$|f_m(p_i) - f_n(p_i)| < \epsilon$$

for $i = 1, \ldots, M$, if $m > N$, $n > N$, and m and n are in S.

To finish, pick $x \in K$. Then $x \in B_i$ for some i, and $\rho(x, p_i) < \delta$. Our choice of δ and N shows that

$$\begin{aligned} |f_m(x) - f_n(x)| &\leq |f_m(x) - f_m(p_i)| + |f_m(p_i) - f_n(p_i)| + |f_n(p_i) - f_n(x)| \\ &< \epsilon + \epsilon + \epsilon = 3\epsilon \end{aligned}$$

if $m > N$, $n > N$, $m \in S$, $n \in S$. ////

11.29 Theorem *Suppose that*

(*a*) X *is a separable Banach space,*
(*b*) $\{\Lambda_n\}$ *is a sequence of linear functionals on* X,
(*c*) $\sup_n \|\Lambda_n\| = M < \infty$.

Then there is a subsequence $\{\Lambda_{n_i}\}$ *such that the limit*

$$\Lambda x = \lim_{i \to \infty} \Lambda_{n_i} x \tag{1}$$

exists for every $x \in X$. *Moreover,* Λ *is linear, and* $\|\Lambda\| \leq M$.

(In this situation, Λ is said to be the *weak limit* of $\{\Lambda_{n_i}\}$; see Exercise 18.)

PROOF To say that X is separable means, by definition, that X has a countable dense subset. The inequalities

$$|\Lambda_n x| \leq M\|x\|, \qquad |\Lambda_n x' - \Lambda_n x''| \leq M\|x' - x''\|$$

show that $\{\Lambda_n\}$ is pointwise bounded and equicontinuous. Since each point of X is a compact set, Theorem 11.28 implies that there is a subsequence $\{\Lambda_{n_i}\}$ such that $\{\Lambda_{n_i} x\}$ converges, for every $x \in X$, as $i \to \infty$. To finish, define Λ by (1). It is then clear that Λ is linear and that $\|\Lambda\| \leq M$. ////

Let us recall, for the application that follows, that $C(T)$ and $L^p(T)$ $(p < \infty)$ are *separable* Banach spaces, because the trigonometric polynomials are dense in them, and because it is enough to confine ourselves to trigonometric polynomials whose coefficients lie in some prescribed countable dense subset of the complex field.

11.30 Theorem *Suppose u is harmonic in U, $1 \leq p \leq \infty$, and*

$$\sup_{0<r<1} \|u_r\|_p = M < \infty. \tag{1}$$

(a) If $p = 1$, it follows that there is a unique complex Borel measure μ on T so that $u = P[d\mu]$.
(b) If $p > 1$, it follows that there is a unique $f \in L^p(T)$ so that $u = P[f]$.
(c) Every positive harmonic function in U is the Poisson integral of a unique positive Borel measure on T.

PROOF Assume first that $p = 1$. Define linear functionals Λ_r on $C(T)$ by

$$\Lambda_r g = \int_T g u_r \, d\sigma \qquad (0 \leq r < 1). \tag{2}$$

By (1), $\|\Lambda_r\| \leq M$. By Theorems 11.29 and 6.19 there is a measure μ on T, with $\|\mu\| \leq M$, and a sequence $r_j \to 1$, so that

$$\lim_{j\to\infty} \int_T g u_{r_j} \, d\sigma = \int_T g \, d\mu \tag{3}$$

for every $g \in C(T)$.

Put $h_j(z) = u(r_j z)$. Then h_j is harmonic in U, continuous on $\bar{U}$, and is therefore the Poisson integral of its restriction to T (Theorem 11.9). Fix $z \in U$, and apply (3) with

$$g(e^{it}) = P(z, e^{it}). \tag{4}$$

Since $h_j(e^{it}) = u_{r_j}(e^{it})$, we obtain

$$u(z) = \lim_j u(r_j z) = \lim_j h_j(z)$$

$$= \lim_j \int_T P(z, e^{it}) h_j(e^{it})\, d\sigma(e^{it})$$

$$= \int_T P(z, e^{it})\, d\mu(e^{it}) = P[d\mu](z).$$

If $1 < p \leq \infty$, let q be the exponent conjugate to p. Then $L^q(T)$ is separable. Define Λ_r as in (2), but for all $g \in L^q(T)$. Again, $\|\Lambda_r\| \leq M$. Refer to Theorems 6.16 and 11.29 to deduce, as above, that there is an $f \in L^p(T)$, with $\|f\|_p \leq M$, so that (3) holds, with $f\, d\sigma$ in place of $d\mu$, for every $g \in L^q(T)$. The rest of the proof is as it was in the case $p = 1$.

This establishes the existence assertions in (*a*) and (*b*). To prove uniqueness, it suffices to show that $P[d\mu] = 0$ implies $\mu = 0$.

Pick $f \in C(T)$, put $u = P[f]$, $v = P[d\mu]$. By Fubini's theorem, and the symmetry $P(re^{i\theta}, e^{it}) = P(re^{it}, e^{i\theta})$,

$$\int_T u_r\, d\mu = \int_T v_r f\, d\sigma \qquad (0 \leq r < 1). \tag{5}$$

When $v = 0$ then $v_r = 0$, and since $u_r \to f$ uniformly, as $r \to 1$, we conclude that

$$\int_T f\, d\mu = 0 \tag{6}$$

for every $f \in C(T)$ if $P[d\mu] = 0$. By Theorem 6.19, (6) implies that $\mu = 0$.

Finally, (*c*) is a corollary of (*a*), since $u > 0$ implies (1) with $p = 1$:

$$\int_T |u_r|\, d\sigma = \int_T u_r\, d\sigma = u(0) \qquad (0 \leq r < 1) \tag{7}$$

by the mean value property of harmonic functions. The functionals Λ_r used in the proof of (*a*) are now positive, hence $\mu \geq 0$. ////

11.31 Since holomorphic functions are harmonic, all of the preceding results (of which Theorems 11.16, 11.24, 11.25, 11.30 are the most significant) apply to holomorphic functions in U. This leads to the study of the H^p-spaces, a topic that will be taken up in Chap. 17.

At present we shall only give one application, to functions in the space H^∞. This, by definition, is the space of all bounded holomorphic functions in U; the norm

$$\|f\|_\infty = \sup \{|f(z)| : z \in U\}$$

turns H^∞ into a Banach space.

As before, $L^\infty(T)$ is the space of all (equivalence classes of) essentially bounded functions on T, normed by the essential supremum norm, relative to Lebesgue measure. For $g \in L^\infty(T)$, $\|g\|_\infty$ stands for the essential supremum of $|g|$.

11.32 Theorem *To every $f \in H^\infty$ corresponds a function $f^* \in L^\infty(T)$, defined almost everywhere by*

$$f^*(e^{i\theta}) = \lim_{r \to 1} f(re^{i\theta}). \tag{1}$$

The equality $\|f\|_\infty = \|f^*\|_\infty$ holds.

If $f^*(e^{i\theta}) = 0$ for almost all $e^{i\theta}$ on some arc $I \subset T$, then $f(z) = 0$ for every $z \in U$.

(A considerably stronger uniqueness theorem will be obtained later, in Theorem 15.19. See also Theorem 17.18 and Sec. 17.19.)

PROOF By Theorem 11.30, there is a unique $g \in L^\infty(T)$ such that $f = P[g]$. By Theorem 11.23, (1) holds with $f^* = g$. The inequality $\|f\|_\infty \leq \|f^*\|_\infty$ follows from Theorem 11.16(1); the opposite inequality is obvious.

In particular, if $f^* = 0$ a.e., then $\|f^*\|_\infty = 0$, hence $\|f\|_\infty = 0$, hence $f = 0$.

Now choose a positive integer n so that the length of I is larger than $2\pi/n$. Let $\alpha = \exp\{2\pi i/n\}$ and define

$$F(z) = \prod_{k=1}^{n} f(\alpha^k z) \qquad (z \in U). \tag{2}$$

Then $F \in H^\infty$ and $F^* = 0$ a.e. on T, hence $F(z) = 0$ for all $z \in U$. If $Z(f)$, the zero set of f in U, were at most countable, the same would be true of $Z(F)$, since $Z(F)$ is the union of n sets obtained from $Z(f)$ by rotations. But $Z(F) = U$. Hence $f = 0$, by Theorem 10.18. ////

Exercises

1 Suppose u and v are real harmonic functions in a plane region Ω. Under what conditions is uv harmonic? (Note that the answer depends strongly on the fact that the question is one about *real* functions.) Show that u^2 cannot be harmonic in Ω, unless u is constant. For which $f \in H(\Omega)$ is $|f|^2$ harmonic?

2 Suppose f is a complex function in a region Ω, and both f and f^2 are harmonic in Ω. Prove that either f or $\bar{f}$ is holomorphic in Ω.

3 If u is a harmonic function in a region Ω, what can you say about the set of points at which the gradient of u is 0? (This is the set on which $u_x = u_y = 0$.)

4 Prove that every partial derivative of every harmonic function is harmonic.

Verify, by direct computation, that $P_r(\theta - t)$ is, for each fixed t, a harmonic function of $re^{i\theta}$. Deduce (without referring to holomorphic functions) that the Poisson integral $P[d\mu]$ of every finite Borel measure μ on T is harmonic in U, by showing that every partial derivative of $P[d\mu]$ is equal to the integral of the corresponding partial derivative of the kernel.

5 Suppose $f \in H(\Omega)$ and f has no zero in Ω. Prove that $\log|f|$ is harmonic in Ω, by computing its Laplacian. Is there an easier way?

6 Suppose $f \in H(U)$, where U is the open unit disc, f is one-to-one in U, $\Omega = f(U)$, and $f(z) = \sum c_n z^n$. Prove that the area of Ω is

$$\pi \sum_{n=1}^{\infty} n|c_n|^2.$$

Hint: The Jacobian of f is $|f'|^2$.

7 (*a*) If $f \in H(\Omega)$, $f(z) \neq 0$ for $z \in \Omega$, and $-\infty < \alpha < \infty$, prove that

$$\Delta(|f|^\alpha) = \alpha^2 |f|^{\alpha-2} |f'|^2,$$

by proving the formula

$$\partial\bar{\partial}(\psi \circ (f\bar{f})) = (\varphi \circ |f|^2) \cdot |f'|^2,$$

in which ψ is twice differentiable on $(0, \infty)$ and

$$\varphi(t) = t\psi''(t) + \psi'(t).$$

(*b*) Assume $f \in H(\Omega)$ and Φ is a complex function with domain $f(\Omega)$, which has continuous second-order partial derivatives. Prove that

$$\Delta[\Phi \circ f] = [(\Delta\Phi) \circ f] \cdot |f'|^2.$$

Show that this specializes to the result of (*a*) if $\Phi(w) = \Phi(|w|)$.

8 Suppose Ω is a region, $f_n \in H(\Omega)$ for $n = 1, 2, 3, \ldots$, u_n is the real part of f_n, $\{u_n\}$ converges uniformly on compact subsets of Ω, and $\{f_n(z)\}$ converges for at least one $z \in \Omega$. Prove that then $\{f_n\}$ converges uniformly on compact subsets of Ω.

9 Suppose u is a Lebesgue measurable function in a region Ω, and u is locally in L^1. This means that the integral of $|u|$ over any compact subset of Ω is finite. Prove that u is harmonic if it satisfies the following form of the mean value property:

$$u(a) = \frac{1}{\pi r^2} \iint_{D(a;\, r)} u(x, y)\, dx\, dy$$

whenever $\bar{D}(a; r) \subset \Omega$.

10 Suppose $I = [a, b]$ is an interval on the real axis, φ is a continuous function on I, and

$$f(z) = \frac{1}{2\pi i} \int_a^b \frac{\varphi(t)}{t - z}\, dt \qquad (z \notin I).$$

Show that

$$\lim_{\epsilon \to 0} [f(x + i\epsilon) - f(x - i\epsilon)] \qquad (\epsilon > 0)$$

exists for every real x, and find it in terms of φ.

How is the result affected if we assume merely that $\varphi \in L^1$? What happens then at points x at which φ has right- and left-hand limits?

11 Suppose that $I = [a, b]$, Ω is a region, $I \subset \Omega$, f is continuous in Ω, and $f \in H(\Omega - I)$. Prove that actually $f \in H(\Omega)$.

Replace I by some other sets for which the same conclusion can be drawn.

12 (Harnack's Inequalities) Suppose Ω is a region, K is a compact subset of Ω, $z_0 \in \Omega$. Prove that there exist positive numbers α and β (depending on z_0, K, and Ω) such that

$$\alpha u(z_0) \leq u(z) \leq \beta u(z_0)$$

for every positive harmonic function u in Ω and for all $z \in K$.

If $\{u_n\}$ is a sequence of positive harmonic functions in Ω and if $u_n(z_0) \to 0$, describe the behavior of $\{u_n\}$ in the rest of Ω. Do the same if $u_n(z_0) \to \infty$. Show that the assumed positivity of $\{u_n\}$ is essential for these results.

13 Suppose u is a positive harmonic function in U and $u(0) = 1$. How large can $u(\frac{1}{2})$ be? How small? Get the best possible bounds.

14 For which pairs of lines L_1, L_2 do there exist real functions, harmonic in the whole plane, that are 0 at all points of $L_1 \cup L_2$ without vanishing identically?

15 Suppose u is a positive harmonic function in U, and $u(re^{i\theta}) \to 0$ as $r \to 1$, for every $e^{i\theta} \neq 1$. Prove that there is a constant c such that

$$u(re^{i\theta}) = cP_r(\theta).$$

16 Here is an example of a harmonic function in U which is not identically 0 but *all* of whose radial limits are 0:

$$u(z) = \operatorname{Im}\left[\left(\frac{1+z}{1-z}\right)^2\right].$$

Prove that this u is not the Poisson integral of any measure on T and that it is not the difference of two positive harmonic functions in U.

17 Let Φ be the set of all positive harmonic functions u in U such that $u(0) = 1$. Show that Φ is a convex set and find the extreme points of Φ. (A point x in a convex set Φ is called an *extreme point* of Φ if x lies on no segment both of whose end points lie in Φ and are different from x.) *Hint*: If C is the convex set whose members are the positive Borel measures on T, of total variation 1, show that the extreme points of C are precisely those $\mu \in C$ whose supports consist of only one point of T.

18 Let X^* be the dual space of the Banach space X. A sequence $\{\Lambda_n\}$ in X^* is said to *converge weakly* to $\Lambda \in X^*$ if $\Lambda_n x \to \Lambda x$ as $n \to \infty$, for every $x \in X$. Note that $\Lambda_n \to \Lambda$ weakly whenever $\Lambda_n \to \Lambda$ in the norm of X^*. (See Exercise 8, Chap. 5.) The converse need not be true. For example, the functionals $f \to \hat{f}(n)$ on $L^2(T)$ tend to 0 weakly (by the Bessel inequality), but each of these functionals has norm 1.

Prove that $\{\|\Lambda_n\|\}$ must be bounded if $\{\Lambda_n\}$ converges weakly.

19 (*a*) Show that $\delta P_r(\delta) > 1$ if $\delta = 1 - r$.

(*b*) If $\mu \geq 0$, $u = P[d\mu]$, and $I_\delta \subset T$ is the arc with center 1 and length 2δ, show that

$$\mu(I_\delta) \leq \delta u(1 - \delta)$$

and that therefore

$$(M\mu)(1) \leq \pi(M_{\text{rad}}\, u)(1).$$

(*c*) If, furthermore, $\mu \perp m$, show that

$$u(re^{i\theta}) \to \infty \qquad \text{a.e. } [\mu].$$

Hint: Use Theorem 7.15.

20 Suppose $E \subset T$, $m(E) = 0$. Prove that there is an $f \in H^\infty$, with $f(0) = 1$, that has

$$\lim_{r \to 1} f(re^{i\theta}) = 0$$

at every $e^{i\theta} \in E$.

Suggestion: Find a lower semicontinuous $\psi \in L^1(T)$, $\psi > 0$, $\psi = +\infty$ at every point of E. There is a holomorphic g whose real part is $P[\psi]$. Let $f = 1/g$.

21 Define $f \in H(U)$ and $g \in H(U)$ by $f(z) = \exp\{(1+z)/(1-z)\}$, $g(z) = (1-z)\exp\{-f(z)\}$. Prove that

$$g^*(e^{i\theta}) = \lim_{r \to 1} g(re^{i\theta})$$

exists at every $e^{i\theta} \in T$, that $g^* \in C(T)$, but that g is not in H^∞.

Suggestion: Fix s, put

$$z_t = \frac{t + is - 1}{t + is + 1} \qquad (0 < t < \infty).$$

For certain values of s, $|g(z_t)| \to \infty$ as $t \to \infty$.

22 Suppose u is harmonic in U, and $\{u_r : 0 \le r < 1\}$ is a uniformly integrable subset of $L^1(T)$. (See Exercise 10, Chap. 6.) Modify the proof of Theorem 11.30 to show that $u = P[f]$ for some $f \in L^1(T)$.

23 Put $\theta_n = 2^{-n}$ and define

$$u(z) = \sum_{n=1}^{\infty} n^{-2}\{P(z, e^{i\theta_n}) - P(z, e^{-i\theta_n})\},$$

for $z \in U$. Show that u is the Poisson integral of a measure on T, that $u(x) = 0$ if $-1 < x < 1$, but that

$$u(1 - \epsilon + i\epsilon)$$

is unbounded, as ϵ decreases to 0. (Thus u has a radial limit, but no nontangential limit, at 1.)

Hint: If $\epsilon = \sin\theta$ is small and $z = 1 - \epsilon + i\epsilon$, then

$$P(z, e^{i\theta}) - P(z, e^{-i\theta}) > 1/\epsilon.$$

24 Let $D_n(t)$ be the Dirichlet kernel, as in Sec. 5.11, define the Fejér kernel by

$$K_N = \frac{1}{N+1}(D_0 + D_1 + \cdots + D_N),$$

put $L_N(t) = \min(N, \pi^2/Nt^2)$. Prove that

$$K_{N-1}(t) = \frac{1}{N} \cdot \frac{1 - \cos Nt}{1 - \cos t} \le L_N(t)$$

and that $\int_T L_N \, d\sigma \le 2$.

Use this to prove that the arithmetic means

$$\sigma_N = \frac{S_0 + S_1 + \cdots + S_N}{N+1}$$

of the partial sums s_n of the Fourier series of a function $f \in L^1(T)$ converge to $f(e^{i\theta})$ at every Lebesgue point of f. (Show that $\sup |\sigma_N|$ is dominated by Mf, then proceed as in the proof of Theorem 11.23.)

25 If $1 \le p \le \infty$ and $f \in L^p(R^1)$, prove that $(f * h_\lambda)(x)$ is a harmonic function of $x + i\lambda$ in the upper half plane. (h_λ is defined in Sec. 9.7; it is the Poisson kernel for the half plane.)

CHAPTER

TWELVE

THE MAXIMUM MODULUS PRINCIPLE

Introduction

12.1 The maximum modulus theorem (10.24) asserts that the constants are the only homomorphic functions in a region Ω whose absolute values have a local maximum at any point of Ω.

Here is a restatement: *If K is the closure of a bounded region Ω, if f is continuous on K and holomorphic in Ω, then*

$$|f(z)| \leq \|f\|_{\partial\Omega} \tag{1}$$

for every $z \in \Omega$. If equality holds at one point $z \in \Omega$, then f is constant.

[The right side of (1) is the supremum of $|f|$ on the boundary $\partial\Omega$ of Ω.]

For if $|f(z)| \geq \|f\|_{\partial\Omega}$ at some $z \in \Omega$, then the maximum of $|f|$ on K (which is attained at some point of K, since K is compact) is actually attained at some point of Ω, so f is constant, by Theorem 10.24.

The equality $\|f\|_\infty = \|f^*\|_\infty$, which is part of Theorem 11.32, implies that

$$|f(z)| \leq \|f^*\|_\infty \qquad (z \in U, f \in H^\infty(U)). \tag{2}$$

This says (roughly speaking) that $|f(z)|$ is no larger than the supremum of the boundary values of f, a statement similar to (1). But this time boundedness on U is enough; we do not need continuity on $\bar{U}$.

This chapter contains further generalizations of the maximum modulus theorem, as well as some rather striking applications of it, and it concludes with a theorem which shows that the maximum property "almost" characterizes the class of holomorphic functions.

The Schwarz Lemma

This is the name usually given to the following theorem. We use the notation established in Sec. 11.31.

12.2 Theorem *Suppose* $f \in H^\infty$, $\|f\|_\infty \leq 1$, *and* $f(0) = 0$. *Then*

$$|f(z)| \leq |z| \qquad (z \in U), \tag{1}$$

$$|f'(0)| \leq 1; \tag{2}$$

if equality holds in (1) *for one* $z \in U - \{0\}$, *or if equality holds in* (2), *then* $f(z) = \lambda z$, *where* λ *is a constant,* $|\lambda| = 1$.

In geometric language, the hypothesis is that f is a holomorphic mapping of U into U which keeps the origin fixed; part of the conclusion is that either f is a rotation or f moves each $z \in U - \{0\}$ closer to the origin than it was.

PROOF Since $f(0) = 0$, $f(z)/z$ has a removable singularity at $z = 0$. Hence there exists $g \in H(U)$ such that $f(z) = zg(z)$. If $z \in U$ and $|z| < r < 1$, then

$$|g(z)| \leq \max_\theta \frac{|f(re^{i\theta})|}{r} \leq \frac{1}{r}.$$

Letting $r \to 1$, we see that $|g(z)| \leq 1$ at every $z \in U$. This gives (1). Since $f'(0) = g(0)$, (2) follows. If $|g(z)| = 1$ for some $z \in U$, then g is constant, by another application of the maximum modulus theorem. ////

Many variants of the Schwarz lemma can be obtained with the aid of the following mappings of U onto U:

12.3 Definition *For any* $\alpha \in U$, *define*

$$\varphi_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha}z}.$$

12.4 Theorem *Fix* $\alpha \in U$. *Then* φ_α *is a one-to-one mapping which carries* T *onto* T, U *onto* U, *and* α *to* 0. *The inverse of* φ_α *is* $\varphi_{-\alpha}$. *We have*

$$\varphi_\alpha'(0) = 1 - |\alpha|^2, \qquad \varphi_\alpha'(\alpha) = \frac{1}{1 - |\alpha|^2}. \tag{1}$$

PROOF φ_α is holomorphic in the whole plane, except for a pole at $1/\bar{\alpha}$ which lies outside $\bar{U}$. Straightforward substitution shows that

$$\varphi_{-\alpha}(\varphi_\alpha(z)) = z. \tag{2}$$

Thus φ_α is one-to-one, and $\varphi_{-\alpha}$ is its inverse. Since, for real t,

$$\left|\frac{e^{it}-\alpha}{1-\bar{\alpha}e^{it}}\right| = \frac{|e^{it}-\alpha|}{|e^{-it}-\bar{\alpha}|} = 1 \tag{3}$$

(z and $\bar{z}$ have the same absolute value), φ_α maps T into T; the same is true of $\varphi_{-\alpha}$; hence $\varphi_\alpha(T) = T$. It now follows from the maximum modulus theorem that $\varphi_\alpha(U) \subset U$, and consideration of $\varphi_{-\alpha}$ shows that actually $\varphi_\alpha(U) = U$. ////

12.5 An Extremal Problem Suppose α and β are complex numbers, $|\alpha| < 1$, and $|\beta| < 1$. *How large can* $|f'(\alpha)|$ *be if* f *is subject to the conditions* $f \in H^\infty$, $\|f\|_\infty \leq 1$, *and* $f(\alpha) = \beta$?

To solve this, put

$$g = \varphi_\beta \circ f \circ \varphi_{-\alpha}. \tag{1}$$

Since $\varphi_{-\alpha}$ and φ_β map U onto U, we see that $g \in H^\infty$ and $\|g\|_\infty \leq 1$; also, $g(0) = 0$. The passage from f to g has reduced our problem to the Schwarz lemma, which gives $|g'(0)| \leq 1$. By (1), the chain rule gives

$$g'(0) = \varphi'_\beta(\beta) f'(\alpha) \varphi'_{-\alpha}(0). \tag{2}$$

If we use Eqs. 12.4(1), we obtain the inequality

$$|f'(\alpha)| \leq \frac{1-|\beta|^2}{1-|\alpha|^2}. \tag{3}$$

This solves our problem, since equality can occur in (3). This happens if and only if $|g'(0)| = 1$, in which case g is a rotation (Theorem 12.2), so that

$$f(z) = \varphi_{-\beta}(\lambda \varphi_\alpha(z)) \qquad (z \in U) \tag{4}$$

for some constant λ with $|\lambda| = 1$.

A remarkable feature of the solution should be stressed. We imposed no smoothness conditions (such as continuity on $\bar{U}$, for instance) on the behavior of f near the boundary of U. Nevertheless, it turns out that the functions f which maximize $|f'(\alpha)|$ under the stated restrictions are actually rational functions. Note also that these extremal functions map U onto U (not just into) and that they are one-to-one. This observation may serve as the motivation for the proof of the Riemann mapping theorem in Chap. 14.

At present, we shall merely show how this extremal problem can be used to characterize the one-to-one holomorphic mappings of U onto U.

12.6 Theorem *Suppose* $f \in H(U)$, f *is one-to-one,* $f(U) = U$, $\alpha \in U$, *and* $f(\alpha) = 0$. *Then there is a constant* λ, $|\lambda| = 1$, *such that*

$$f(z) = \lambda \varphi_\alpha(z) \qquad (z \in U). \tag{1}$$

In other words, we obtain f by composing the mapping φ_α with a rotation.

PROOF Let g be the inverse of f, defined by $g(f(z)) = z$, $z \in U$. Since f is one-to-one, f' has no zero in U, so $g \in H(U)$, by Theorem 10.33. By the chain rule,

$$g'(0)f'(\alpha) = 1. \tag{2}$$

The solution of 12.5, applied to f and to g, yields the inequalities

$$|f'(\alpha)| \leq \frac{1}{1 - |\alpha|^2}, \qquad |g'(0)| \leq 1 - |\alpha|^2. \tag{3}$$

By (2), equality must hold in (3). As we observed in the preceding problem (with $\beta = 0$), this forces f to satisfy (1). ////

The Phragmen-Lindelöf Method

12.7 For a bounded region Ω, we saw in Sec. 12.1 that if f is continuous on the closure of Ω and if $f \in H(\Omega)$, the maximum modulus theorem implies

$$\|f\|_\Omega = \|f\|_{\partial\Omega}. \tag{1}$$

For unbounded regions, this is no longer true.

To see an example, let

$$\Omega = \left\{z = x + iy: -\frac{\pi}{2} < y < \frac{\pi}{2}\right\}; \tag{2}$$

Ω is the open strip bounded by the parallel lines $y = \pm\pi/2$; its boundary $\partial\Omega$ is the union of these two lines. Put

$$f(z) = \exp(\exp(z)). \tag{3}$$

For real x,

$$f\left(x \pm \frac{\pi i}{2}\right) = \exp(\pm ie^x) \tag{4}$$

since $\exp(\pi i/2) = i$, so $|f(z)| = 1$ for $z \in \partial\Omega$. But $f(z) \to \infty$ very rapidly as $x \to \infty$ along the positive real axis, which lies in Ω.

"Very" is the key word in the preceding sentence. A method developed by Phragmen and Lindelöf makes it possible to prove theorems of the following kind: If $f \in H(\Omega)$ and if $|f| < g$, where $g(z) \to \infty$ "slowly" as $z \to \infty$ in Ω (just what "slowly" means depends on Ω), then f is actually bounded in Ω, and this usually implies further conclusions about f, by the maximum modulus theorem.

Rather than describe the method by a theorem which would cover a large number of situations, we shall show how it works in two cases. In both, Ω will be a strip. In the first, f will be assumed to be bounded, and the theorem will improve the bound; in the second, a growth condition will be imposed on f which just excludes the function (3). In view of later applications, Ω will be a vertical strip in Theorem 12.8.

First, however, let us mention another example which also has this general flavor: *Suppose f is an entire function and*

$$|f(z)| < 1 + |z|^{1/2} \tag{5}$$

for all z. Then f is constant.

This follows immediately from the Cauchy estimates 10.26, since they show that $f^{(n)}(0) = 0$ for $n = 1, 2, 3, \ldots$.

12.8 Theorem *Suppose*

$$\Omega = \{x + iy: a < x < b\}, \qquad \bar{\Omega} = \{x + iy: a \leq x \leq b\}, \tag{1}$$

f is continuous on $\bar{\Omega}$, $f \in H(\Omega)$, and suppose that $|f(z)| < B$ for all $z \in \Omega$ and some fixed $B < \infty$. If

$$M(x) = \sup \{|f(x + iy)| : -\infty < y < \infty\} \qquad (a \leq x \leq b) \tag{2}$$

then we actually have

$$M(x)^{b-a} \leq M(a)^{b-x} M(b)^{x-a} \qquad (a < x < b). \tag{3}$$

Note: The conclusion (3) implies that the inequality $|f| < B$ can be replaced by $|f| \leq \max(M(a), M(b))$, so that $|f|$ is no larger in Ω than the supremum of $|f|$ on the *boundary* of Ω.

If we apply the theorem to strips bounded by lines $x = \alpha$ and $x = \beta$, where $a \leq \alpha < \beta \leq b$, the conclusion can be stated in the following way:

Corollary *Under the hypotheses of the theorem, log M is a convex function on (a, b).*

PROOF We assume first that $M(a) = M(b) = 1$. In this case we have to prove that $|f(z)| \leq 1$ for all $z \in \Omega$.

For each $\epsilon > 0$, we define an auxiliary function

$$h_\epsilon(z) = \frac{1}{1 + \epsilon(z - a)} \qquad (z \in \bar{\Omega}). \tag{4}$$

Since $\operatorname{Re}\{1 + \epsilon(z - a)\} = 1 + \epsilon(x - a) \geq 1$ in $\bar{\Omega}$, we have $|h_\epsilon| \leq 1$ in $\bar{\Omega}$, so that

$$|f(z)h_\epsilon(z)| \leq 1 \qquad (z \in \partial\Omega). \tag{5}$$

Also, $|1 + \epsilon(z - a)| \geq \epsilon |y|$, so that

$$|f(z)h_\epsilon(z)| \leq \frac{B}{\epsilon |y|} \qquad (z = x + iy \in \bar{\Omega}). \tag{6}$$

Let R be the rectangle cut off from $\bar{\Omega}$ by the lines $y = \pm B/\epsilon$. By (5) and (6), $|fh_\epsilon| \leq 1$ on ∂R, hence $|fh_\epsilon| \leq 1$ on R, by the maximum modulus theorem. But (6) shows that $|fh_\epsilon| \leq 1$ on the rest of $\bar{\Omega}$. Thus $|f(z)h_\epsilon(z)| \leq 1$

for all $z \in \Omega$ and all $\epsilon > 0$. If we fix $z \in \Omega$ and then let $\epsilon \to 0$, we obtain the desired result $|f(z)| \leq 1$.

We now turn to the general case. Put

$$g(z) = M(a)^{(b-z)/(b-a)} M(b)^{(z-a)/(b-a)}, \tag{7}$$

where, for $M > 0$ and w complex, M^w is defined by

$$M^w = \exp(w \log M), \tag{8}$$

and $\log M$ is real. Then g is entire, g has no zero, $1/g$ is bounded in $\bar{\Omega}$,

$$|g(a + iy)| = M(a), \qquad |g(b + iy)| = M(b), \tag{9}$$

and hence f/g satisfies our previous assumptions. Thus $|f/g| \leq 1$ in Ω, and this gives (3). (See Exercise 7.) ////

12.9 Theorem *Suppose*

$$\Omega = \left\{x + iy : |y| < \frac{\pi}{2}\right\}, \qquad \bar{\Omega} = \left\{x + iy : |y| \leq \frac{\pi}{2}\right\}. \tag{1}$$

Suppose f is continuous on $\bar{\Omega}$, $f \in H(\Omega)$, there are constants $\alpha < 1$, $A < \infty$, such that

$$|f(z)| < \exp\{A \exp(\alpha |x|)\} \qquad (z = x + iy \in \Omega), \tag{2}$$

and

$$\left|f\left(x \pm \frac{\pi i}{2}\right)\right| \leq 1 \qquad (-\infty < x < \infty). \tag{3}$$

Then $|f(z)| \leq 1$ for all $z \in \Omega$.

Note that the conclusion does not follow if $\alpha = 1$, as is shown by the function $\exp(\exp z)$.

PROOF Choose $\beta > 0$ so that $\alpha < \beta < 1$. For $\epsilon > 0$, define

$$h_\epsilon(z) = \exp\{-\epsilon(e^{\beta z} + e^{-\beta z})\}. \tag{4}$$

For $z \in \bar{\Omega}$,

$$\operatorname{Re}[e^{\beta z} + e^{-\beta z}] = (e^{\beta x} + e^{-\beta x}) \cos \beta y \geq \delta(e^{\beta x} + e^{-\beta x}) \tag{5}$$

where $\delta = \cos(\beta\pi/2) > 0$, since $|\beta| < 1$. Hence

$$|h_\epsilon(z)| \leq \exp\{-\epsilon\delta(e^{\beta x} + e^{-\beta x})\} < 1 \qquad (z \in \bar{\Omega}). \tag{6}$$

It follows that $|fh_\epsilon| \leq 1$ on $\partial\Omega$ and that

$$|f(z)h_\epsilon(z)| \leq \exp\{Ae^{\alpha|x|} - \epsilon\delta(e^{\beta x} + e^{-\beta x})\} \qquad (z \in \bar{\Omega}). \tag{7}$$

Fix $\epsilon > 0$. Since $\epsilon\delta > 0$ and $\beta > \alpha$, the exponent in (7) tends to $-\infty$ as $x \to \pm\infty$. Hence there exists an x_0 so that the right side of (7) is less than 1 for all $x > x_0$. Since $|fh_\epsilon| \leq 1$ on the boundary of the rectangle whose vertices are $\pm x_0 \pm (\pi i/2)$, the maximum modulus theorem shows that actually $|fh_\epsilon| \leq 1$ on this rectangle. Thus $|fh_\epsilon| \leq 1$ at every point of Ω, for every $\epsilon > 0$. As $\epsilon \to 0$, $h_\epsilon(z) \to 1$ for every z, so we conclude that $|f(z)| \leq 1$ for all $z \in \Omega$. ////

Here is a slightly different application of the same method. It will be used in the proof of Theorem 14.18.

12.10 Lindelöf's Theorem *Suppose Γ is a curve, with parameter interval $[0, 1]$, such that $|\Gamma(t)| < 1$ if $t < 1$ and $\Gamma(1) = 1$. If $g \in H^\infty$ and*

$$\lim_{t \to 1} g(\Gamma(t)) = L, \tag{1}$$

then g has radial limit L at 1.

(It follows from Exercise 14, Chap. 14, that g actually has nontangential limit L at 1.)

PROOF Assume $|g| < 1$, $L = 0$, without loss of generality. Let $\epsilon > 0$ be given. There exists $t_0 < 1$ so that, setting $r_0 = \operatorname{Re} \Gamma(t_0)$, we have

$$|g(\Gamma(t))| < \epsilon \quad \text{and} \quad \operatorname{Re} \Gamma(t) > r_0 > \frac{1}{2} \tag{2}$$

as soon as $t_0 < t < 1$.

Pick r, $r_0 < r < 1$.

Define h in $\Omega = D(0; 1) \cap D(2r; 1)$ by

$$h(z) = g(z)\overline{g(\bar{z})}g(2r - z)\overline{g(2r - \bar{z})}. \tag{3}$$

Then $h \in H(\Omega)$ and $|h| < 1$. We claim that

$$|h(r)| < \epsilon. \tag{4}$$

Since $h(r) = |g(r)|^4$, the theorem follows from (4).

To prove (4), let $E_1 = \Gamma([t_1, 1])$, where t_1 is the largest t for which $\operatorname{Re} \Gamma(t) = r$, let E_2 be the reflection of E_1 in the real axis, and let E be the union of $E_1 \cup E_2$ and its reflection in the line $x = r$. Then (2) and (3) imply that

$$|h(z)| < \epsilon \quad \text{if} \quad z \in \Omega \cap E. \tag{5}$$

Pick $c > 0$, define

$$h_c(z) = h(z)(1 - z)^c(2r - 1 - z)^c \tag{6}$$

for $z \in \Omega$, and put $h_c(1) = h_c(2r - 1) = 0$. If K is the union of E and the bounded components of the complement of E, then K is compact, h_c is continuous on K, holomorphic in the interior of K, and (5) implies that $|h_c| < \epsilon$ on the boundary of K. Since the construction of E shows that $r \in K$, the maximum modulus theorem implies that $|h_c(r)| < \epsilon$.

Letting $c \to 0$, we obtain (4). ////

An Interpolation Theorem

12.11 The convexity theorem 12.8 can sometimes be used to prove that certain linear transformations are bounded with respect to certain L^p-norms. Rather than discuss this in full generality, let us look at a particular situation of this kind.

Let X be a measure space, with a positive measure μ, and suppose $\{\psi_n\}$ $(n = 1, 2, 3, \ldots)$ is an orthonormal set of functions in $L^2(\mu)$; we recall what this means:

$$\int_X \psi_n \bar{\psi}_m \, d\mu = \begin{cases} 1 & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \tag{1}$$

Let us also assume that $\{\psi_n\}$ is a bounded sequence in $L^\infty(\mu)$: There exists an $M < \infty$ such that

$$|\psi_n(x)| \leq M \qquad (n = 1, 2, 3, \ldots; x \in X). \tag{2}$$

Then for any $f \in L^p(\mu)$, where $1 \leq p \leq 2$, the integrals

$$\hat{f}(n) = \int_X f\bar{\psi}_n \, d\mu \qquad (n = 1, 2, 3, \ldots) \tag{3}$$

exist and define a function $\hat{f}$ on the set of all positive integers.

There are now two very easy theorems: For $f \in L^1(\mu)$, (2) gives

$$\|\hat{f}\|_\infty \leq M\|f\|_1, \tag{4}$$

and for $f \in L^2(\mu)$, the Bessel inequality gives

$$\|\hat{f}\|_2 \leq \|f\|_2, \tag{5}$$

where the norms are defined as usual:

$$\|f\|_p = \left[\int |f|^p \, d\mu\right]^{1/p}, \qquad \|\hat{f}\|_q = [\sum |\hat{f}(n)|^q]^{1/q}, \tag{6}$$

and $\|\hat{f}\|_\infty = \sup_n |\hat{f}(n)|$.

Since $(1, \infty)$ and $(2, 2)$ are pairs of conjugate exponents, one may conjecture that $\|\hat{f}\|_q$ is finite whenever $f \in L^p(\mu)$ and $1 < p < 2$, $q = p/(p - 1)$. This is indeed true and can be proved by "interpolation" between the preceding trivial cases $p = 1$ and $p = 2$.

12.12 The Hausdorff-Young Theorem *Under the above assumptions, the inequality*

$$\|\hat{f}\|_q \leq M^{(2-p)/p}\|f\|_p \tag{1}$$

holds if $1 \leq p \leq 2$ *and if* $f \in L^p(\mu)$.

PROOF We first prove a reduced form of the theorem.

Fix p, $1 < p < 2$. Let f be a *simple* complex function such that $\|f\|_p = 1$, and let $b_1, \ldots, b_N$ be complex numbers such that $\sum |b_n|^p = 1$. Our objective is the inequality

$$\left|\sum_{n=1}^{N} b_n \hat{f}(n)\right| \leq M^{(2-p)/p}. \tag{2}$$

Put $F = |f|^p$, and put $B_n = |b_n|^p$. Then there is a function φ and there are complex numbers $\beta_1, \ldots, \beta_N$ such that

$$f = F^{1/p}\varphi, \qquad |\varphi| = 1, \qquad \int_X F\, d\mu = 1, \tag{3}$$

and

$$b_n = B_n^{1/p}\beta_n, \qquad |\beta_n| = 1, \qquad \sum_{n=1}^{N} B_n = 1. \tag{4}$$

If we use these relations and the definition of $\hat{f}(n)$ given in Sec. 12.11, we obtain

$$\sum_{n=1}^{N} b_n \hat{f}(n) = \sum_{n=1}^{N} B_n^{1/p}\beta_n \int_X F^{1/p}\varphi\bar{\psi}_n\, d\mu. \tag{5}$$

Now replace $1/p$ by z in (5), and define

$$\Phi(z) = \sum_{n=1}^{N} B_n^z \beta_n \int_X F^z \varphi\bar{\psi}_n\, d\mu \tag{6}$$

for any complex number z. Recall that $A^z = \exp(z \log A)$ if $A > 0$; if $A = 0$, we agree that $A^z = 0$. Since F is simple, since $F \geq 0$, and since $B_n \geq 0$, we see that Φ is a finite linear combination of such exponentials, so Φ is an entire function which is bounded on

$$\{z: a \leq \operatorname{Re}(z) \leq b\}$$

for any finite a and b. We shall take $a = \frac{1}{2}$ and $b = 1$, shall estimate Φ on the edges of this strip, and shall then apply Theorem 12.8 to estimate $\Phi(1/p)$.

For $-\infty < y < \infty$, define

$$c_n(y) = \int_X F^{1/2} F^{iy} \varphi\bar{\psi}_n\, d\mu. \tag{7}$$

The Bessel inequality gives

$$\sum_{n=1}^{N} |c_n(y)|^2 \leq \int_X |F^{1/2}F^{iy}\varphi|^2 \, d\mu = \int_X |F| \, d\mu = 1, \tag{8}$$

and then the Schwarz inequality shows that

$$|\Phi(\tfrac{1}{2} + iy)| = \left| \sum_{n=1}^{N} B_n^{1/2} B_n^{iy} \beta_n c_n \right| \leq \left\{ \sum_{n=1}^{N} B_n \cdot \sum_{n=1}^{N} |c_n|^2 \right\}^{1/2} \leq 1. \tag{9}$$

The estimate

$$|\Phi(1 + iy)| \leq M \qquad (-\infty < y < \infty) \tag{10}$$

follows trivially from (3), (4), and (6), since $\|\psi_n\|_\infty \leq M$.

We now conclude from (9), (10), and Theorem 12.8 that

$$|\Phi(x + iy)| \leq M^{2x-1} \qquad (\tfrac{1}{2} \leq x \leq 1, \ -\infty < y < \infty). \tag{11}$$

With $x = 1/p$ and $y = 0$, this gives the desired inequality (2).

The proof is now easily completed. Note first that

$$\left\{ \sum_{n=1}^{N} |\hat{f}(n)|^q \right\}^{1/q} = \sup \left| \sum_{n=1}^{N} b_n \hat{f}(n) \right|, \tag{12}$$

the supremum being taken over all $\{b_1, \ldots, b_N\}$ with $\sum |b_n|^p = 1$, since the L^q norm of any function on any measure space is equal to its norm as a linear functional on L^p. Hence (2) shows that

$$\left\{ \sum_{n=1}^{N} |\hat{f}(n)|^q \right\}^{1/q} \leq M^{(2-p)/p} \|f\|_p \tag{13}$$

for every simple complex $f \in L^p(\mu)$.

If now $f \in L^p(\mu)$, there are simple functions f_j such that $\|f_j - f\|_p \to 0$ as $j \to \infty$. Then $\hat{f}_j(n) \to \hat{f}(n)$ for every n, because $\psi_n \in L^q(\mu)$. Thus since (13) holds for each f_j, it also holds for f. Since N was arbitrary, we finally obtain (1). ////

A Converse of the Maximum Modulus Theorem

We now come to the theorem which was alluded to in the introduction of the present chapter.

The letter j will denote the identity function: $j(z) = z$.

The function which assigns the number 1 to each $z \in \bar{U}$ will be denoted by 1.

12.13 Theorem *Suppose M is a vector space of continuous complex functions on the closed unit disc $\bar{U}$, with the following properties:*

(a) $1 \in M$.

(b) *If $f \in M$, then also $jf \in M$.*

(c) *If $f \in M$, then $\|f\|_U = \|f\|_T$.*

Then every $f \in M$ is holomorphic in U.

Note that (*c*) is a rather weak form of the maximum modulus principle; (*c*) asserts only that the overall maximum of $|f|$ on $\bar{U}$ is attained at some point of the boundary T, but (*c*) does not a priori exclude the existence of *local* maxima of $|f|$ in U.

PROOF By (*a*) and (*b*), M contains all polynomials. In conjunction with (*c*), this shows that M satisfies the hypotheses of Theorem 5.25. Thus every $f \in M$ is harmonic in U. We shall use (*b*) to show that every $f \in M$ actually satisfies the Cauchy-Riemann equation.

Let ∂ and $\bar{\partial}$ be the differential operators introduced in Sec. 11.1. The product rule for differentiation gives

$$(\partial\bar{\partial})(fg) = f \cdot (\partial\bar{\partial}g) + (\partial f) \cdot (\bar{\partial}g) + (\bar{\partial}f) \cdot (\partial g) + (\partial\bar{\partial}f) \cdot g.$$

Fix $f \in M$, and take $g = j$. Then $fj \in M$. Hence f and fj are harmonic, so $\partial\bar{\partial}f = 0$ and $(\partial\bar{\partial})(fj) = 0$. Also, $\bar{\partial}j = 0$ and $\partial j = 1$. The above identity therefore reduces to $\bar{\partial}f = 0$. Thus $f \in H(U)$. ////

This result will be used in the following proof.

12.14 Radó's Theorem *Assume $f \in C(\bar{U})$, Ω is the set of all $z \in U$ at which $f(z) \neq 0$, and f is holomorphic in Ω. Then f is holomorphic in U.*

In particular, the theorem asserts that $U - \Omega$ is at most countable, unless $\Omega = \varnothing$.

PROOF Assume $\Omega \neq \varnothing$. We shall first prove that Ω is dense in U. If not, there exist $\alpha \in \Omega$ and $\beta \in U - \bar{\Omega}$ such that $2|\beta - \alpha| < 1 - |\beta|$. Choose n so that $2^n|f(\alpha)| > \|f\|_T$. Define $h(z) = (z - \beta)^{-n}f(z)$, for $z \in \bar{\Omega}$. If $z \in U \cap \partial\Omega$, then $f(z) = 0$, hence $h(z) = 0$. If $z \in T \cap \partial\Omega$, then

$$|h(z)| \leq (1 - |\beta|)^{-n}\|f\|_T < |\alpha - \beta|^{-n}|f(\alpha)| = |h(\alpha)|.$$

This contradicts the maximum modulus theorem.

Thus Ω is dense in U.

Next, let M be the vector space of all $g \in C(\bar{U})$ that are holomorphic in Ω. Fix $g \in M$. For $n = 1, 2, 3, \ldots$, $fg^n = 0$ on $U \cap \partial\Omega$. The maximum modulus theorem implies therefore, for every $\alpha \in \Omega$, that

$$|f(\alpha)|\,|g(\alpha)|^n \leq \|fg^n\|_{\partial\Omega} = \|fg^n\|_T \leq \|f\|_T\|g\|_T^n.$$

If we take nth roots and then let $n \to \infty$, we see that $|g(\alpha)| \leq \|g\|_T$, for every $\alpha \in \Omega$. Since Ω is dense in U, $\|g\|_U = \|g\|_T$.

It follows that M satisfies the hypotheses of Theorem 12.13. Since $f \in M$, f is holomorphic in U. ////

Exercises

1 Suppose Δ is a closed equilateral triangle in the plane, with vertices a, b, c. Find $\max (|z-a||z-b||z-c|)$ as z ranges over Δ.

2 Suppose $f \in H(\mathrm{II}^+)$, where II^+ is the upper half plane, and $|f| \leq 1$. How large can $|f'(i)|$ be? Find the extremal functions. (Compare the discussion in Sec. 12.5.)

3 Suppose $f \in H(\Omega)$. Under what conditions can $|f|$ have a local *minimum* in Ω?

4 (*a*) Suppose Ω is a region, D is a disc, $\bar{D} \subset \Omega$, $f \in H(\Omega)$, f is not constant, and $|f|$ is constant on the boundary of D. Prove that f has at least one zero in D.

(*b*) Find all entire functions f such that $|f(z)| = 1$ whenever $|z| = 1$.

5 Suppose Ω is a bounded region, $\{f_n\}$ is a sequence of continuous functions on $\bar{\Omega}$ which are holomorphic in Ω, and $\{f_n\}$ converges uniformly on the boundary of Ω. Prove that $\{f_n\}$ converges uniformly on $\bar{\Omega}$.

6 Suppose $f \in H(\Omega)$, Γ is a cycle in Ω such that $\mathrm{Ind}_\Gamma(\alpha) = 0$ for all $\alpha \notin \Omega$, $|f(\zeta)| \leq 1$ for every $\zeta \in \Gamma^*$, and $\mathrm{Ind}_\Gamma(z) \neq 0$. Prove that $|f(z)| \leq 1$.

7 In the proof of Theorem 12.8 it was tacitly assumed that $M(a) > 0$ and $M(b) > 0$. Show that the theorem is true if $M(a) = 0$, and that then $f(z) = 0$ for all $z \in \Omega$.

8 If $0 < R_1 < R_2 < \infty$, let $A(R_1, R_2)$ denote the annulus

$$\{z: R_1 < |z| < R_2\}.$$

There is a vertical strip which the exponential function maps onto $A(R_1, R_2)$. Use this to prove Hadamard's *three-circle theorem*: If $f \in H(A(R_1, R_2))$, if

$$M(r) = \max_\theta |f(re^{i\theta})| \qquad (R_1 < r < R_2),$$

and if $R_1 < a < r < b < R_2$, then

$$\log M(r) \leq \frac{\log (b/r)}{\log (b/a)} \log M(a) + \frac{\log (r/a)}{\log (b/a)} \log M(b).$$

[In other words, $\log M(r)$ is a convex function of $\log r$.] For which f does equality hold in this inequality?

9 Let Π be the open right half plane ($z \in \Pi$ if and only if $\mathrm{Re}\, z > 0$). Suppose f is continuous on the closure of Π ($\mathrm{Re}\, z \geq 0$), $f \in H(\Pi)$, and there are constants $A < \infty$ and $\alpha < 1$ such that

$$|f(z)| < A \exp(|z|^\alpha)$$

for all $z \in \Pi$. Furthermore, $|f(iy)| \leq 1$ for all real y. Prove that $|f(z)| \leq 1$ in Π.

Show that the conclusion is false for $\alpha = 1$.

How does the result have to be modified if Π is replaced by a region bounded by two rays through the origin, at an angle not equal to π?

10 Let Π be the open right half plane. Suppose that $f \in H(\Pi)$, that $|f(z)| < 1$ for all $z \in \Pi$, and that there exists α, $-\pi/2 < \alpha < \pi/2$, such that

$$\frac{\log |f(re^{i\alpha})|}{r} \to -\infty \quad \text{as} \quad r \to \infty.$$

Prove that $f = 0$.

Hint: Put $g_n(z) = f(z)e^{n z}$, $n = 1, 2, 3, \ldots$. Apply Exercise 9 to the two angular regions defined by $-\pi/2 < \theta < \alpha$, $\alpha < \theta < \pi/2$. Conclude that each g_n is bounded in Π, and hence that $|g_n| < 1$ in Π, for all n.

11 Suppose Γ is the boundary of an unbounded region Ω, $f \in H(\Omega)$, f is continuous on $\Omega \cup \Gamma$, and there are constants $B < \infty$ and $M < \infty$ such that $|f| \leq M$ on Γ and $|f| \leq B$ in Ω. Prove that we then actually have $|f| \leq M$ in Ω.

Suggestion: Show that it involves no loss of generality to assume that $U \cap \Omega = \varnothing$. Fix $z_0 \in \Omega$, let n be a large integer, let V be a large disc with center at 0, and apply the maximum modulus theorem to the function $f^n(z)/z$ in the component of $V \cap \Omega$ which contains z_0.

12 Let f be an entire function. If there is a continuous mapping γ of $[0, 1)$ into the complex plane such that $\gamma(t) \to \infty$ and $f(\gamma(t)) \to \alpha$ as $t \to 1$, we say that α is an *asymptotic value* of f. [In the complex plane, "$\gamma(t) \to \infty$ as $t \to 1$" means that to each $R < \infty$ there corresponds a $t_R < 1$ such that $|\gamma(t)| > R$ if $t_R < t < 1$.] Prove that every nonconstant entire function has ∞ as an asymptotic value.

Suggestion: Let $E_n = \{z: |f(z)| > n\}$. Each component of E_n is unbounded (proof?) and contains a component of E_{n+1}, by Exercise 11.

13 Show that exp has exactly two asymptotic values: 0 and ∞. How about sin and cos? *Note*: sin z and cos z are defined, for all complex z, by

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2}.$$

14 If f is entire and if α is not in the range of f, prove that α is an asymptotic value of f unless f is constant.

15 Suppose $f \in H(U)$. Prove that there is a sequence $\{z_n\}$ in U such that $|z_n| \to 1$ and $\{f(z_n)\}$ is bounded.

16 Suppose Ω is a bounded region, $f \in H(\Omega)$, and

$$\limsup_{n \to \infty} |f(z_n)| \leq M$$

for every sequence $\{z_n\}$ in Ω which converges to a boundary point of Ω. Prove that $|f(z)| \leq M$ for all $z \in \Omega$.

17 Let Φ be the set of all $f \in H(U)$ such that $0 < |f(z)| < 1$ for $z \in U$, and let Φ_c be the set of all $f \in \Phi$ that have $f(0) = c$. Define

$$M(c) = \sup \{|f'(0)| : f \in \Phi_c\}, \qquad M = \sup \{|f'(0)| : f \in \Phi\}.$$

Find M, and $M(c)$ for $0 < c < 1$. Find an $f \in \Phi$ with $f'(0) = M$, or prove that there is no such f.

Suggestion: $\log f$ maps U into the left half plane. Compose $\log f$ with a properly chosen map that takes this half plane to U. Apply the Schwarz lemma.

CHAPTER

THIRTEEN

APPROXIMATIONS BY RATIONAL FUNCTIONS

Preparation

13.1 The Riemann Sphere It is often convenient in the study of holomorphic functions to compactify the complex plane by the adjunction of a new point called ∞. The resulting set S^2 (the *Riemann sphere*, the union of R^2 and $\{\infty\}$) is topologized in the following manner. For any $r > 0$, let $D'(\infty; r)$ be the set of all complex numbers z such that $|z| > r$, put $D(\infty; r) = D'(\infty; r) \cup \{\infty\}$, and declare a subset of S^2 to be open if and only if it is the union of discs $D(a; r)$, where the a's are arbitrary points of S^2 and the r's are arbitrary positive numbers. On $S^2 - \{\infty\}$, this gives of course the ordinary topology of the plane. It is easy to see that S^2 is homeomorphic to a sphere (hence the notation). In fact, a homeomorphism φ of S^2 onto the unit sphere in R^3 can be explicitly exhibited: Put $\varphi(\infty) = (0, 0, 1)$, and put

$$\varphi(re^{i\theta}) = \left(\frac{2r\cos\theta}{r^2+1}, \frac{2r\sin\theta}{r^2+1}, \frac{r^2-1}{r^2+1}\right) \tag{1}$$

for all complex numbers $re^{i\theta}$. We leave it to the reader to construct the geometric picture that goes with (1).

If f is holomorphic in $D'(\infty; r)$, we say that f has an isolated singularity at ∞. The nature of this singularity is the same as that which the function $\tilde{f}$, defined in $D'(0; 1/r)$ by $\tilde{f}(z) = f(1/z)$, has at 0.

Thus if f is bounded in $D'(\infty; r)$, then $\lim_{z\to\infty} f(z)$ exists and is a complex number (as we see if we apply Theorem 10.20 to $\tilde{f}$), we define $f(\infty)$ to be this limit, and we thus obtain a function in $D(\infty; r)$ which we call holomorphic: note that this is defined in terms of the behavior of $\tilde{f}$ near 0, and not in terms of differentiability of f at ∞.

If $\tilde{f}$ has a pole of order m at 0, then f is said to have a pole of order m at ∞; the *principal part* of f at ∞ is then an ordinary polynomial of degree m (compare Theorem 10.21), and if we subtract this polynomial from f, we obtain a function with a removable singularity at ∞.

Finally, if $\tilde{f}$ has an essential singularity at 0, then f is said to have an essential singularity at ∞. For instance, every entire function which is not a polynomial has an essential singularity at ∞.

Later in this chapter we shall encounter the condition "$S^2 - \Omega$ is connected," where Ω is an open set in the plane. Note that this is not equivalent to the condition "the complement of Ω relative to the plane is connected." For example, if Ω consists of all complex $z = x + iy$ with $0 < y < 1$, the complement of Ω relative to the plane has two components, but $S^2 - \Omega$ is connected.

13.2 Rational Functions A rational function f is, by definition, a quotient of two polynomials P and Q: $f = P/Q$. It follows from Theorem 10.25 that every nonconstant polynomial is a product of factors of degree 1. We may assume that P and Q have no such factors in common. Then f has a pole at each zero of Q (the pole of f has the same order as the zero of Q). If we subtract the corresponding principal parts, we obtain a rational function whose only singularity is at ∞ and which is therefore a polynomial.

Every rational function $f = P/Q$ has thus a representation of the form

$$f(z) = A_0(z) + \sum_{j=1}^{k} A_j((z - a_j)^{-1}) \tag{1}$$

where $A_0, A_1, \ldots, A_k$ are polynomials, $A_1, \ldots, A_k$ have no constant term, and $a_1, \ldots, a_k$ are the distinct zeros of Q; (1) is called the *partial fractions decomposition* of f.

We turn to some topological considerations. We know that every open set in the plane is a countable union of compact sets (closed discs, for instance). However, it will be convenient to have some additional properties satisfied by these compact sets:

13.3 Theorem *Every open set Ω in the plane is the union of a sequence $\{K_n\}$, $n = 1, 2, 3, \ldots$, of compact sets such that*

(*a*) *K_n lies in the interior of K_{n+1}, for $n = 1, 2, 3, \ldots$.*
(*b*) *Every compact subset of Ω lies in some K_n.*
(*c*) *Every component of $S^2 - K_n$ contains a component of $S^2 - \Omega$, for $n = 1, 2, 3, \ldots$.*

Property (*c*) is, roughly speaking, that K_n has no holes except those which are forced upon it by the holes in Ω. Note that Ω is not assumed to be connected. The *interior* of a set E is, by definition, the largest open subset of E.

PROOF For $n = 1, 2, 3, \ldots$, put

$$V_n = D(\infty; n) \cup \bigcup_{a \notin \Omega} D\left(a; \frac{1}{n}\right) \tag{1}$$

and put $K_n = S^2 - V_n$. [Of course, $a \neq \infty$ in (1).] Then K_n is a closed and bounded (hence compact) subset of Ω, and $\Omega = \bigcup K_n$. If $z \in K_n$ and $r = n^{-1} - (n+1)^{-1}$, one verifies easily that $D(z; r) \subset K_{n+1}$. This gives (*a*). Hence Ω is the union of the interiors W_n of K_n. If K is a compact subset of Ω, then $K \subset W_1 \cup \cdots \cup W_N$ for some N, hence $K \subset K_N$.

Finally, each of the discs in (1) intersects $S^2 - \Omega$; each disc is connected; hence each component of V_n intersects $S^2 - \Omega$; since $V_n \supset S^2 - \Omega$, no component of $S^2 - \Omega$ can intersect two components of V_n. This gives (*c*). ////

13.4 Sets of Oriented Intervals Let Φ be a finite collection of oriented intervals in the plane. For each point p, let $m_I(p)[m_E(p)]$ be the number of members of Φ that have initial point [end point] p. If $m_I(p) = m_E(p)$ for every p, we shall say that Φ is *balanced.*

If Φ is balanced (and nonempty), the following construction can be carried out.

Pick $\gamma_1 = [a_0, a_1] \in \Phi$. Assume $k \geq 1$, and assume that distinct members $\gamma_1, \ldots, \gamma_k$ of Φ have been chosen in such a way that $\gamma_i = [a_{i-1}, a_i]$ for $1 \leq i \leq k$. If $a_k = a_0$, stop. If $a_k \neq a_0$, and if precisely r of the intervals $\gamma_1, \ldots, \gamma_k$ have a_k as end point, then only $r - 1$ of them have a_k as initial point; since Φ is balanced, Φ contains at least one other interval, say γ_{k+1}, whose initial point is a_k. Since Φ is finite, we must return to a_0 eventually, say at the nth step.

Then $\gamma_1, \ldots, \gamma_n$ join (in this order) to form a closed path.

The remaining members of Φ still form a balanced collection to which the above construction can be applied. It follows that the members of Φ can be so numbered that they form finitely many closed paths. The sum of these paths is a cycle. The following conclusion is thus reached.

If $\Phi = \{\gamma_1, \ldots, \gamma_N\}$ is a balanced collection of oriented intervals, and if

$$\Gamma = \gamma_1 \dot{+} \cdots \dot{+} \gamma_N$$

then Γ is a cycle.

13.5 Theorem *If K is a compact subset of a plane open set Ω ($\neq \varnothing$), then there is a cycle Γ in $\Omega - K$ such that the Cauchy formula*

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z}\, d\zeta \tag{1}$$

holds for every $f \in H(\Omega)$ and for every $z \in K$.

PROOF Since K is compact and Ω is open, there exists an $\eta > 0$ such that the distance from any point of K to any point outside Ω is at least 2η. Construct

a grid of horizontal and vertical lines in the plane, such that the distance between any two adjacent horizontal lines is η, and likewise for the vertical lines. Let $Q_1, \ldots, Q_m$ be those squares (closed 2-cells) of edge η which are formed by this grid and which intersect K. Then $Q_r \subset \Omega$ for $r = 1, \ldots, m$.

If a_r is the center of Q_r and $a_r + b$ is one of its vertices, let γ_{rk} be the oriented interval

$$\gamma_{rk} = [a_r + i^k b,\ a_r + i^{k+1} b] \tag{2}$$

and define

$$\partial Q_r = \gamma_{r1} \dot{+} \gamma_{r2} \dot{+} \gamma_{r3} \dot{+} \gamma_{r4} \qquad (r = 1, \ldots, m). \tag{3}$$

It is then easy to check (for example, as a special case of Theorem 10.37, or by means of Theorems 10.11 and 10.40) that

$$\operatorname{Ind}_{\partial Q_r}(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is in the interior of } Q_r, \\ 0 & \text{if } \alpha \text{ is not in } Q_r. \end{cases} \tag{4}$$

Let Σ be the collection of all γ_{rk} $(1 \leq r \leq m,\ 1 \leq k \leq 4)$. It is clear that Σ is balanced. Remove those members of Σ whose opposites (see Sec. 10.8) also belong to Σ. Let Φ be the collection of the remaining members of Σ. Then Φ is balanced. Let Γ be the cycle constructed from Φ, as in Sec. 13.4.

If an edge E of some Q_r intersects K, then the two squares in whose boundaries E lies intersect K. Hence Σ contains two oriented intervals which are each other's opposites and whose range is E. These intervals do not occur in Φ. Thus Γ is a cycle in $\Omega - K$.

The construction of Φ from Σ shows also that

$$\operatorname{Ind}_\Gamma(\alpha) = \sum_{r=1}^{m} \operatorname{Ind}_{\partial Q_r}(\alpha) \tag{5}$$

if α is not in the boundary of any Q_r. Hence (4) implies

$$\operatorname{Ind}_\Gamma(\alpha) = \begin{cases} 1 & \text{if } \alpha \text{ is in the interior of some } Q_r, \\ 0 & \text{if } \alpha \text{ lies in no } Q_r. \end{cases} \tag{6}$$

If $z \in K$, then $z \notin \Gamma^*$, and z is a limit point of the interior of some Q_r. Since the left side of (6) is constant in each component of the complement of Γ^*, (6) gives

$$\operatorname{Ind}_\Gamma(z) = \begin{cases} 1 & \text{if } z \in K, \\ 0 & \text{if } z \notin \Omega. \end{cases} \tag{7}$$

Now (1) follows from Cauchy's theorem 10.35. ////

Runge's Theorem

The main objective of this section is Theorem 13.9. We begin with a slightly different version in which the emphasis is on uniform approximation on one compact set.

13.6 Theorem *Suppose K is a compact set in the plane and $\{\alpha_j\}$ is a set which contains one point in each component of $S^2 - K$. If Ω is open, $\Omega \supset K$, $f \in H(\Omega)$, and $\epsilon > 0$, there exists a rational function R, all of whose poles lie in the prescribed set $\{\alpha_j\}$, such that*

$$|f(z) - R(z)| < \epsilon \tag{1}$$

for every $z \in K$.

Note that $S^2 - K$ has at most countably many components. Note also that the preassigned point in the unbounded component of $S^2 - K$ may very well be ∞; in fact, this happens to be the most interesting choice.

PROOF We consider the Banach space $C(K)$ whose members are the continuous complex functions on K, with the supremum norm. Let M be the subspace of $C(K)$ which consists of the restrictions to K of those rational functions which have all their poles in $\{\alpha_j\}$. The theorem asserts that f is in the closure of M. By Theorem 5.19 (a consequence of the Hahn-Banach theorem), this is equivalent to saying that every bounded linear functional on $C(K)$ which vanishes on M also vanishes at f, and hence the Riesz representation theorem (Theorem 6.19) shows that we must prove the following assertion:

If μ is a complex Borel measure on K such that

$$\int_K R \, d\mu = 0 \tag{2}$$

for every rational function R with poles only in the set $\{\alpha_j\}$, and if $f \in H(\Omega)$, then we also have

$$\int_K f \, d\mu = 0. \tag{3}$$

So let us assume that μ satisfies (2). Define

$$h(z) = \int_K \frac{d\mu(\zeta)}{\zeta - z} \qquad (z \in S^2 - K). \tag{4}$$

By Theorem 10.7 (with $X = K$, $\varphi(\zeta) = \zeta$), $h \in H(S^2 - K)$.

Let V_j be the component of $S^2 - K$ which contains α_j, and suppose $D(\alpha_j; r) \subset V_j$. If $\alpha_j \neq \infty$ and if z is fixed in $D(\alpha_j; r)$, then

$$\frac{1}{\zeta - z} = \lim_{N\to\infty} \sum_{n=0}^{N} \frac{(z - \alpha_j)^n}{(\zeta - \alpha_j)^{n+1}} \tag{5}$$

uniformly for $\zeta \in K$. Each of the functions on the right of (5) is one to which (2) applies. Hence $h(z) = 0$ for all $z \in D(\alpha_j; r)$. This implies that $h(z) = 0$ for all $z \in V_j$, by the uniqueness theorem 10.18.

If $\alpha_j = \infty$, (5) is replaced by

$$\frac{1}{\zeta - z} = -\lim_{N\to\infty} \sum_{n=0}^{N} z^{-n-1}\zeta^n \qquad (\zeta \in K, |z| > r), \tag{6}$$

which implies again that $h(z) = 0$ in $D(\infty; r)$, hence in V_j. We have thus proved from (2) that

$$h(z) = 0 \qquad (z \in S^2 - K). \tag{7}$$

Now choose a cycle Γ in $\Omega - K$, as in Theorem 13.5, and integrate this Cauchy integral representation of f with respect to μ. An application of Fubini's theorem (legitimate, since we are dealing with Borel measures and continuous functions on compact spaces), combined with (7), gives

$$\begin{aligned}\int_K f\,d\mu &= \int_K d\mu(\zeta)\left[\frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w - \zeta}\,dw\right] \\ &= \frac{1}{2\pi i}\int_\Gamma f(w)\,dw \int_K \frac{d\mu(\zeta)}{w - \zeta} \\ &= -\frac{1}{2\pi i}\int_\Gamma f(w)h(w)\,dw = 0.\end{aligned}$$

The last equality depends on the fact that $\Gamma^* \subset \Omega - K$, where $h(w) = 0$.

Thus (3) holds, and the proof is complete. ////

The following special case is of particular interest.

13.7 Theorem *Suppose K is a compact set in the plane, $S^2 - K$ is connected, and $f \in H(\Omega)$, where Ω is some open set containing K. Then there is a sequence $\{P_n\}$ of polynomials such that $P_n(z) \to f(z)$ uniformly on K.*

PROOF Since $S^2 - K$ has now only one component, we need only one point α_j to apply Theorem 13.6, and we may take $\alpha_j = \infty$. ////

13.8 Remark The preceding result is false for *every* compact K in the plane such that $S^2 - K$ is not connected. For in that case $S^2 - K$ has a bounded component V. Choose $\alpha \in V$, put $f(z) = (z - \alpha)^{-1}$, and put

$m = \max \{|z - \alpha| : z \in K\}$. Suppose P is a polynomial, such that $|P(z) - f(z)| < 1/m$ for all $z \in K$. Then

$$|(z - \alpha)P(z) - 1| < 1 \qquad (z \in K). \tag{1}$$

In particular, (1) holds if z is in the boundary of V; since the closure of V is compact, the maximum modulus theorem shows that (1) holds for every $z \in V$; taking $z = \alpha$, we obtain $1 < 1$. Hence the uniform approximation is not possible.

The same argument shows that none of the α_j can be dispensed with in Theorem 13.6.

We now apply the preceding approximation theorems to approximation in open sets. Let us emphasize that K was *not* assumed to be connected in Theorems 13.6 and 13.7 and that Ω will *not* be assumed to be connected in the theorem which follows.

13.9 Theorem *Let Ω be an open set in the plane, let A be a set which has one point in each component of $S^2 - \Omega$, and assume $f \in H(\Omega)$. Then there is a sequence $\{R_n\}$ of rational functions, with poles only in A, such that $R_n \to f$ uniformly on compact subsets of Ω.*

In the special case in which $S^2 - \Omega$ is connected, we may take $A = \{\infty\}$ and thus obtain polynomials P_n such that $P_n \to f$ uniformly on compact subsets of Ω.

Observe that $S^2 - \Omega$ may have uncountably many components; for instance, we may have $S^2 - \Omega = \{\infty\} \cup C$, where C is a Cantor set.

PROOF Choose a sequence of compact sets K_n in Ω, with the properties specified in Theorem 13.3. Fix n, for the moment. Since each component of $S^2 - K_n$ contains a component of $S^2 - \Omega$, each component of $S^2 - K_n$ contains a point of A, so Theorem 13.6 gives us a rational function R_n with poles in A such that

$$|R_n(z) - f(z)| < \frac{1}{n} \qquad (z \in K_n). \tag{1}$$

If now K is any compact set in Ω, there exists an N such that $K \subset K_n$ for all $n \geq N$. It follows from (1) that

$$|R_n(z) - f(z)| < \frac{1}{n} \qquad (z \in K, n \geq N), \tag{2}$$

which completes the proof. ////

The Mittag-Leffler Theorem

Runge's theorem will now be used to prove that meromorphic functions can be constructed with arbitrarily preassigned poles.

13.10 Theorem *Suppose Ω is an open set in the plane, $A \subset \Omega$, A has no limit point in Ω, and to each $\alpha \in A$ there are associated a positive integer $m(\alpha)$ and a rational function*

$$P_\alpha(z) = \sum_{j=1}^{m(\alpha)} c_{j,\alpha}(z - \alpha)^{-j}.$$

Then there exists a meromorphic function f in Ω, whose principal part at each $\alpha \in A$ is P_α and which has no other poles in Ω.

PROOF We choose a sequence $\{K_n\}$ of compact sets in Ω, as in Theorem 13.3: For $n = 1, 2, 3, \ldots$, K_n lies in the interior of K_{n+1}, every compact subset of Ω lies in some K_n, and every component of $S^2 - K_n$ contains a component of $S^2 - \Omega$. Put $A_1 = A \cap K_1$, and $A_n = A \cap (K_n - K_{n-1})$ for $n = 2, 3, 4, \ldots$. Since $A_n \subset K_n$ and A has no limit point in Ω (hence none in K_n), each A_n is a finite set. Put

$$Q_n(z) = \sum_{\alpha \in A_n} P_\alpha(z) \qquad (n = 1, 2, 3, \ldots). \tag{1}$$

Since each A_n is finite, each Q_n is a rational function. The poles of Q_n lie in $K_n - K_{n-1}$, for $n \geq 2$. In particular, Q_n is holomorphic in an open set containing K_{n-1}. It now follows from Theorem 13.6 that there exist rational functions R_n, all of whose poles are in $S^2 - \Omega$, such that

$$|R_n(z) - Q_n(z)| < 2^{-n} \qquad (z \in K_{n-1}). \tag{2}$$

We claim that

$$f(z) = Q_1(z) + \sum_{n=2}^{\infty} (Q_n(z) - R_n(z)) \qquad (z \in \Omega) \tag{3}$$

has the desired properties.

Fix N. On K_N, we have

$$f = Q_1 + \sum_{n=2}^{N} (Q_n - R_n) + \sum_{N+1}^{\infty} (Q_n - R_n). \tag{4}$$

By (2), each term in the last sum in (4) is less than 2^{-n} on K_N; hence this last series converges uniformly on K_N, to a function which is holomorphic in the interior of K_N. Since the poles of each R_n are outside Ω,

$$f - (Q_1 + \cdots + Q_N)$$

is holomorphic in the interior of K_N. Thus f has precisely the prescribed principal parts in the interior of K_N, and hence in Ω, since N was arbitrary. ////

Simply Connected Regions

We shall now summarize some properties of simply connected regions (see Sec. 10.38) which illustrate the important role that they play in the theory of holomorphic functions. Of these properties, (*a*) and (*b*) are what one might call internal topological properties of Ω; (*c*) and (*d*) refer to the way in which Ω is embedded in S^2; properties (*e*) to (*h*) are analytic in character; (*j*) is an algebraic statement about the ring $H(\Omega)$. The Riemann mapping theorem 14.8 is another very important property of simply connected regions. In fact, we shall use it to prove the implication (*j*) $\rightarrow$ (*a*).

13.11 Theorem *For a plane region Ω, each of the following nine conditions implies all the others.*

(*a*) *Ω is homeomorphic to the open unit disc U.*
(*b*) *Ω is simply connected.*
(*c*) *$\operatorname{Ind}_\gamma(\alpha) = 0$ for every closed path γ in Ω and for every $\alpha \in S^2 - \Omega$.*
(*d*) *$S^2 - \Omega$ is connected.*
(*e*) *Every $f \in H(\Omega)$ can be approximated by polynomials, uniformly on compact subsets of Ω.*
(*f*) *For every $f \in H(\Omega)$ and every closed path γ in Ω,*

$$\int_\gamma f(z)\,dz = 0.$$

(*g*) *To every $f \in H(\Omega)$ corresponds an $F \in H(\Omega)$ such that $F' = f$.*
(*h*) *If $f \in H(\Omega)$ and $1/f \in H(\Omega)$, there exists a $g \in H(\Omega)$ such that $f = \exp(g)$.*
(*j*) *If $f \in H(\Omega)$ and $1/f \in H(\Omega)$, there exists a $\varphi \in H(\Omega)$ such that $f = \varphi^2$.*

The assertion of (*h*) is that f has a "holomorphic logarithm" g in Ω; (*j*) asserts that f has a "holomorphic square root" φ in Ω; and (*f*) says that the Cauchy theorem holds for every closed path in a simply connected region.

In Chapter 16 we shall see that the monodromy theorem describes yet another characteristic property of simply connected regions.

PROOF (*a*) implies (*b*). To say that Ω is homeomorphic to U means that there is a continuous one-to-one mapping ψ of Ω onto U whose inverse ψ^{-1} is also continuous. If γ is a closed curve in Ω, with parameter interval $[0, 1]$, put

$$H(s, t) = \psi^{-1}(t\psi(\gamma(s))).$$

Then $H: I^2 \rightarrow \Omega$ is continuous; $H(s, 0) = \psi^{-1}(0)$, a constant; $H(s, 1) = \gamma(s)$; and $H(0, t) = H(1, t)$ because $\gamma(0) = \gamma(1)$. Thus Ω is simply connected.

(*b*) implies (*c*). If (*b*) holds and γ is a closed path in Ω, then γ is (by definition of "simply connected") Ω-homotopic to a constant path. Hence (*c*) holds, by Theorem 10.40.

(*c*) implies (*d*). Assume (*d*) is false. Then $S^2 - \Omega$ is a closed subset of S^2 which is not connected. As noted in Sec. 10.1, it follows that $S^2 - \Omega$ is the union of two nonempty disjoint closed sets H and K. Let H be the one that contains ∞. Let W be the complement of H, relative to the plane. Then $W = \Omega \cup K$. Since K is compact, Theorem 13.5 (with $f = 1$) shows that there is a cycle Γ in $W - K = \Omega$ such that $\mathrm{Ind}_\Gamma(z) = 1$ for $z \in K$. Since $K \neq \varnothing$, (*c*) fails.

(*d*) implies (*e*). This is part of Theorem 13.9.

(*e*) implies (*f*). Choose $f \in H(\Omega)$, let γ be a closed path in Ω, and choose polynomials P_n which converge to f, uniformly on γ^*. Since $\int_\gamma P_n(z)\,dz = 0$ for all n, we conclude that (*f*) holds.

(*f*) implies (*g*). Assume (*f*) holds, fix $z_0 \in \Omega$, and put

$$F(z) = \int_{\Gamma(z)} f(\zeta)\,d\zeta \qquad (z \in \Omega). \tag{1}$$

where $\Gamma(z)$ is any path in Ω from z_0 to z. This defines a function F in Ω. For if $\Gamma_1(z)$ is another path from z_0 to z (in Ω), then Γ followed by the opposite of Γ_1 is a closed path in Ω, the integral of f over this closed path is 0, so (1) is not affected if $\Gamma(z)$ is replaced by $\Gamma_1(z)$. We now verify that $F' = f$. Fix $a \in \Omega$. There exists an $r > 0$ such that $D(a; r) \subset \Omega$. For $z \in D(a; r)$ we can compute $F(z)$ by integrating f over a path $\Gamma(a)$, followed by the interval $[a, z]$. Hence, for $z \in D'(a; r)$,

$$\frac{F(z) - F(a)}{z - a} = \frac{1}{z - a}\int_{[a,\,z]} f(\zeta)\,d\zeta, \tag{2}$$

and the continuity of f at a implies now that $F'(a) = f(a)$, as in the proof of Theorem 10.14.

(*g*) implies (*h*). If $f \in H(\Omega)$ and f has no zero in Ω, then $f'/f \in H(\Omega)$, and (*g*) implies that there exists a $g \in H(\Omega)$ so that $g' = f'/f$. We can add a constant to g, so that $\exp\{g(z_0)\} = f(z_0)$ for some $z_0 \in \Omega$. Our choice of g shows that the derivative of fe^{-g} is 0 in Ω, hence fe^{-g} is constant (since Ω is connected), and it follows that $f = e^g$.

(*h*) implies (*j*). By (*h*), $f = e^g$. Put $\varphi = \exp(\frac{1}{2}g)$.

(*j*) implies (*a*). If Ω is the whole plane, then Ω is homeomorphic to U: map z to $z/(1 + |z|)$.

If Ω is a proper subregion of the plane which satisfies (*j*), then there actually exists a *holomorphic* homeomorphism of Ω onto U (a conformal mapping). This assertion is the Riemann mapping theorem, which is the main objective of the next chapter. Hence the proof of Theorem 13.11 will be complete as soon as the Riemann mapping theorem is proved. (See the note following the statement of Theorem 14.8.) ////

The fact that (*h*) holds in every simply connected region has the following consequence (which can also be proved by quite elementary means):

13.12 Theorem *If $f \in H(\Omega)$, where Ω is any open set in the plane, and if f has no zero in Ω, then* $\log |f|$ *is harmonic in* Ω.

PROOF To every disc $D \subset \Omega$ there corresponds a function $g \in H(D)$ such that $f = e^g$ in D. If $u = \operatorname{Re} g$, then u is harmonic in D, and $|f| = e^u$. Thus $\log |f|$ is harmonic in every disc in Ω, and this gives the desired conclusion. ////

Exercises

1 Prove that every meromorphic function on S^2 is rational.

2 Let $\Omega = \{z: |z| < 1 \text{ and } |2z - 1| > 1\}$, and suppose $f \in H(\Omega)$.

(*a*) Must there exist a sequence of polynomials P_n such that $P_n \to f$ uniformly on compact subsets of Ω?

(*b*) Must there exist such a sequence which converges to f uniformly in Ω?

(*c*) Is the answer to (*b*) changed if we require more of f, namely, that f be holomorphic in some open set which contains the closure of Ω?

3 Is there a sequence of polynomials P_n such that $P_n(0) = 1$ for $n = 1, 2, 3, \ldots$, but $P_n(z) \to 0$ for every $z \neq 0$, as $n \to \infty$?

4 Is there a sequence of polynomials P_n such that

$$\lim_{n\to\infty} P_n(z) = \begin{cases} 1 & \text{if } \operatorname{Im} z > 0, \\ 0 & \text{if } z \text{ is real}, \\ -1 & \text{if } \operatorname{Im} z < 0? \end{cases}$$

5 For $n = 1, 2, 3, \ldots$, let Δ_n be a closed disc in U, and let L_n be an arc (a homeomorphic image of $[0, 1]$) in $U - \Delta_n$ which intersects every radius of U. There are polynomials P_n which are very small on Δ_n and more or less arbitrary on L_n. Show that $\{\Delta_n\}$, $\{L_n\}$, and $\{P_n\}$ can be so chosen that the series $f = \Sigma P_n$ defines a function $f \in H(U)$ which has no radial limit at any point of T. In other words, for no real θ does $\lim_{r\to 1} f(re^{i\theta})$ exist.

6 Here is another construction of such a function. Let $\{n_k\}$ be a sequence of integers such that $n_1 > 1$ and $n_{k+1} > 2kn_k$. Define

$$h(z) = \sum_{k=1}^{\infty} 5^k z^{n_k}.$$

Prove that the series converges if $|z| < 1$ and prove that there is a constant $c > 0$ such that $|h(z)| > c \cdot 5^m$ for all z with $|z| = 1 - (1/n_m)$. [*Hint*: For such z the mth term in the series defining $h(z)$ is much larger than the sum of all the others.]

Hence h has no finite radial limits.

Prove also that h must have infinitely many zeros in U. (Compare with Exercise 15, Chap. 12.) In fact, prove that to every complex number α there correspond infinitely many $z \in U$ at which $h(z) = \alpha$.

7 Show that in Theorem 13.9 we need not assume that A intersects each component of $S^2 - \Omega$. It is enough to assume that the closure of A intersects each component of $S^2 - \Omega$.

8 Prove the Mittag-Leffler theorem for the case in which Ω is the whole plane, by a direct argument which makes no appeal to Runge's theorem.

9 Suppose Ω is a simply connected region, $f \in H(\Omega)$, f has no zero in Ω, and n is a positive integer. Prove that there exists a $g \in H(\Omega)$ such that $g^n = f$.

10 Suppose Ω is a region, $f \in H(\Omega)$, and $f \not\equiv 0$. Prove that f has a holomorphic logarithm in Ω if and only if f has holomorphic nth roots in Ω for every positive integer n.

11 Suppose that $f_n \in H(\Omega)$ $(n = 1, 2, 3, \ldots)$, f is a complex function in Ω, and $f(z) = \lim_{n\to\infty} f_n(z)$ for every $z \in \Omega$. Prove that Ω has a dense open subset V on which f is holomorphic. *Hint*: Put $\varphi = \sup |f_n|$. Use Baire's theorem to prove that every disc in Ω contains a disc on which φ is bounded. Apply Exercise 5, Chap. 10. (In general, $V \neq \Omega$. Compare Exercises 3 and 4.)

12 Suppose, however, that f is *any* complex-valued measurable function defined in the complex plane, and prove that there is a sequence of holomorphic polynomials P_n such that $\lim_{n\to\infty} P_n(z) = f(z)$ for *almost* every z (with respect to two-dimensional Lebesgue measure).

CHAPTER

FOURTEEN

CONFORMAL MAPPING

Preservation of Angles

14.1 Definition Each complex number $z \neq 0$ determines a *direction* from the origin, defined by the point

$$A[z] = \frac{z}{|z|} \tag{1}$$

on the unit circle.

Suppose f is a mapping of a region Ω into the plane, $z_0 \in \Omega$, and z_0 has a deleted neighborhood $D'(z_0; r) \subset \Omega$ in which $f(z) \neq f(z_0)$. We say that f *preserves angles at* z_0 if

$$\lim_{r \to 0} e^{-i\theta} A[f(z_0 + re^{i\theta}) - f(z_0)] \qquad (r > 0) \tag{2}$$

exists and *is independent of* θ.

In less precise language, the requirement is that for any two rays L' and L'', starting at z_0, the angle which their images $f(L')$ and $f(L'')$ make at $f(z_0)$ is the same as that made by L' and L'', in size as well as in orientation.

The property of preserving angles at each point of a region is characteristic of holomorphic functions whose derivative has no zero in that region. This is a corollary of Theorem 14.2 and is the reason for calling holomorphic functions with nonvanishing derivative *conformal mappings.*

14.2 Theorem *Let f map a region Ω into the plane. If $f'(z_0)$ exists at some $z_0 \in \Omega$ and $f'(z_0) \neq 0$, then f preserves angles at z_0. Conversely, if the differential of f exists and is different from 0 at z_0, and if f preserves angles at z_0, then $f'(z_0)$ exists and is different from 0.*

Here $f'(z_0) = \lim [f(z) - f(z_0)]/(z - z_0)$, as usual. The differential of f at z_0 is a linear transformation L of R^2 into R^2 such that, writing $z_0 = (x_0, y_0)$,

$$f(x_0 + x, y_0 + y) = f(x_0, y_0) + L(x, y) + (x^2 + y^2)^{1/2}\eta(x, y), \tag{1}$$

where $\eta(x, y) \to 0$ as $x \to 0$ and $y \to 0$, as in Definition 7.22.

PROOF Take $z_0 = f(z_0) = 0$, for simplicity. If $f'(0) = a \neq 0$, then it is immediate that

$$e^{-i\theta}A[f(re^{i\theta})] = \frac{e^{-i\theta}f(re^{i\theta})}{|f(re^{i\theta})|} \to \frac{a}{|a|} \qquad (r \to 0), \tag{2}$$

so f preserves angles at 0. Conversely, if the differential of f exists at 0 and is different from 0, then (1) can be rewritten in the form

$$f(z) = \alpha z + \beta\bar{z} + |z|\eta(z), \tag{3}$$

where $\eta(z) \to 0$ as $z \to 0$, and α and β are complex numbers, not both 0. If f also preserves angles at 0, then

$$\lim_{r \to 0} e^{-i\theta}A[f(re^{i\theta})] = \frac{\alpha + \beta e^{-2i\theta}}{|\alpha + \beta e^{-2i\theta}|} \tag{4}$$

exists and is independent of θ. We may exclude those θ for which the denominator in (4) is 0; there are at most two such θ in $[0, 2\pi)$. For all other θ, we conclude that $\alpha + \beta e^{-2i\theta}$ lies on a fixed ray through 0, and this is possible only when $\beta = 0$. Hence $\alpha \neq 0$, and (3) implies that $f'(0) = \alpha$. ////

Note: No holomorphic function preserves angles at any point where its derivative is 0. We omit the easy proof of this. However, the differential of a transformation may be 0 at a point where angles are preserved. Example: $f(z) = |z|z$, $z_0 = 0$.

Linear Fractional Transformations

14.3 If a, b, c, and d are complex numbers such that $ad - bc \neq 0$, the mapping

$$z \to \frac{az + b}{cz + d} \tag{1}$$

is called a *linear fractional transformation.* It is convenient to regard (1) as a mapping of the sphere S^2 into S^2, with the obvious conventions concerning the point ∞. For instance, $-d/c$ maps to ∞ and ∞ maps to a/c, if $c \neq 0$. It is then easy to see that each linear fractional transformation is a one-to-one mapping of S^2 onto S^2. Furthermore, each is obtained by a superposition of transformations of the following types:

(*a*) Translations: $z \to z + b$.
(*b*) Rotations: $z \to az$, $|a| = 1$.

(*c*) Homotheties: $z \to rz$, $r > 0$.
(*d*) Inversion: $z \to 1/z$.

If $c = 0$ in (1), this is obvious. If $c \neq 0$, it follows from the identity

$$\frac{az + b}{cz + d} = \frac{a}{c} + \frac{\lambda}{cz + d}, \qquad \lambda = \frac{bc - ad}{c}. \tag{2}$$

The first three types evidently carry lines to lines and circles to circles. This is not true of (*d*). But if we let $\mathscr{F}$ be the family consisting of all straight lines and all circles, then $\mathscr{F}$ *is* preserved by (*d*), and hence we have the important result that $\mathscr{F}$ *is preserved by every linear fractional transformation.* [It may be noted that when $\mathscr{F}$ is regarded as a family of subsets of S^2, then $\mathscr{F}$ consists of all circles on S^2, via the stereographic projection 13.1(1); we shall not use this property of $\mathscr{F}$ and omit its proof.]

The proof that $\mathscr{F}$ is preserved by inversion is quite easy. Elementary analytic geometry shows that every member of $\mathscr{F}$ is the locus of an equation

$$\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0, \tag{3}$$

where α and γ are real constants and β is a complex constant, provided that $\beta\bar{\beta} > \alpha\gamma$. If $\alpha \neq 0$, (3) defines a circle; $\alpha = 0$ gives the straight lines. Replacement of z by $1/z$ transforms (3) into

$$\alpha + \beta\bar{z} + \bar{\beta}z + \gamma z\bar{z} = 0, \tag{4}$$

which is an equation of the same type.

Suppose a, b, and c are distinct complex numbers. We construct a linear fractional transformation φ which maps the ordered triple $\{a, b, c\}$ into $\{0, 1, \infty\}$, namely,

$$\varphi(z) = \frac{(b - c)(z - a)}{(b - a)(z - c)} \tag{5}$$

There is only one such φ. For if $\varphi(a) = 0$, we must have $z - a$ in the numerator; if $\varphi(c) = \infty$, we must have $z - c$ in the denominator; and if $\varphi(b) = 1$, we are led to (5). If a or b or c is ∞, formulas analogous to (5) can easily be written down. If we follow (5) by the inverse of a transformation of the same type, we obtain the following result:

For any two ordered triples $\{a, b, c\}$ *and* $\{a', b', c'\}$ *in* S^2 *there is one and only one linear fractional transformation which maps* a *to* a', b *to* b', *and* c *to* c'.

(It is of course assumed that $a \neq b$, $a \neq c$, and $b \neq c$, and likewise for a', b', and c'.)

We conclude from this that every circle can be mapped onto every circle by a linear fractional transformation. Of more interest is the fact that every circle can be mapped onto every straight line (if ∞ is regarded as part of the line), and hence that *every open disc can be conformally mapped onto every open half plane.*

Let us discuss one such mapping more explicitly, namely,

$$\varphi(z) = \frac{1+z}{1-z}. \tag{6}$$

This φ maps $\{-1, 0, 1\}$ to $\{0, 1, \infty\}$; the segment $(-1, 1)$ maps onto the positive real axis. The unit circle T passes through -1 and 1; hence $\varphi(T)$ is a straight line through $\varphi(-1) = 0$. Since T makes a right angle with the real axis at -1, $\varphi(T)$ makes a right angle with the real axis at 0. Thus $\varphi(T)$ is the imaginary axis. Since $\varphi(0) = 1$, it follows that *φ is a conformal one-to-one mapping of the open unit disc onto the open right half plane.*

The role of linear fractional transformations in the theory of conformal mapping is also well illustrated by Theorem 12.6.

14.4 Linear fractional transformations make it possible to transfer theorems concerning the behavior of holomorphic functions near straight lines to situations where circular arcs occur instead. It will be enough to illustrate the method with an informal discussion of the reflection principle.

Suppose Ω is a region in U, bounded in part by an arc L on the unit circle, and f is continuous on $\bar{\Omega}$, holomorphic in Ω, and real on L. The function

$$\psi(z) = \frac{z-i}{z+i} \tag{1}$$

maps the upper half plane onto U. If $g = f \circ \psi$, Theorem 11.14 gives us a holomorphic extension G of g, and then $F = G \circ \psi^{-1}$ gives a holomorphic extension F of f which satisfies the relation

$$f(z^*) = \overline{F(z)}, \tag{2}$$

where $z^* = 1/\bar{z}$.

The last assertion follows from a property of ψ: If $w = \psi(z)$ and $w_1 = \psi(\bar{z})$, then $w_1 = w^*$, as is easily verified by computation.

Exercises 2 to 5 furnish other applications of this technique.

Normal Families

The Riemann mapping theorem will be proved by exhibiting the mapping function as the solution of a certain extremum problem. The existence of this solution depends on a very useful compactness property of certain families of holomorphic functions which we now formulate.

14.5 Definition Suppose $\mathscr{F} \subset H(\Omega)$, for some region Ω. We call $\mathscr{F}$ a *normal family* if every sequence of members of $\mathscr{F}$ contains a subsequence which converges uniformly on compact subsets of Ω. The limit function is not required to belong to $\mathscr{F}$.

(Sometimes a wider definition is adopted, by merely requiring that every sequence in $\mathscr{F}$ either converges or tends to ∞, uniformly on compact subsets of Ω. This is well adapted for dealing with meromorphic functions.)

14.6 Theorem *Suppose $\mathscr{F} \subset H(\Omega)$ and $\mathscr{F}$ is uniformly bounded on each compact subset of the region Ω. Then $\mathscr{F}$ is a normal family.*

PROOF The hypothesis means that to each compact $K \subset \Omega$ there corresponds a number $M(K) < \infty$ such that $|f(z)| \leq M(K)$ for all $f \in \mathscr{F}$ and all $z \in K$.

Let $\{K_n\}$ be a sequence of compact sets whose union is Ω, such that K_n lies in the interior of K_{n+1}; such a sequence was constructed in Theorem 13.3. Then there exist positive numbers δ_n such that

$$D(z; 2\delta_n) \subset K_{n+1} \qquad (z \in K_n). \tag{1}$$

Consider two points z' and z'' in K_n, such that $|z' - z''| < \delta_n$, let γ be the positively oriented circle with center at z' and radius $2\delta_n$, and estimate $|f(z') - f(z'')|$ by the Cauchy formula. Since

$$\frac{1}{\zeta - z'} - \frac{1}{\zeta - z''} = \frac{z' - z''}{(\zeta - z')(\zeta - z'')},$$

we have

$$f(z') - f(z'') = \frac{z' - z''}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z')(\zeta - z'')}\, d\zeta, \tag{2}$$

and since $|\zeta - z'| = 2\delta_n$ and $|\zeta - z''| > \delta_n$ for all $\zeta \in \gamma^*$, (2) gives the inequality

$$|f(z') - f(z'')| < \frac{M(K_{n+1})}{\delta_n} |z' - z''|, \tag{3}$$

valid for all $f \in \mathscr{F}$ and all z' and $z'' \in K_n$, provided that $|z' - z''| < \delta_n$.

This was the crucial step in the proof: We have proved, for each K_n, that the restrictions of the members of $\mathscr{F}$ to K_n form an *equicontinuous family.*

If $f_j \in \mathscr{F}$, for $j = 1, 2, 3, \ldots$, Theorem 11.28 implies therefore that there are infinite sets S_n of positive integers, $S_1 \supset S_2 \supset S_3 \supset \cdots$, so that $\{f_j\}$ converges uniformly on K_n as $j \to \infty$ within S_n. The diagonal process yields then an infinite set S such that $\{f_j\}$ converges uniformly on every K_n (and hence on every compact $K \subset \Omega$) as $j \to \infty$ within S. ////

The Riemann Mapping Theorem

14.7 Conformal Equivalence We call two regions Ω_1 and Ω_2 *conformally equivalent* if there exists a $\varphi \in H(\Omega_1)$ such that φ is one-to-one in Ω_1 and such that $\varphi(\Omega_1) = \Omega_2$, i.e., if there exists a conformal one-to-one mapping of Ω_1 onto Ω_2.

Under these conditions, the inverse of φ is holomorphic in Ω_2 (Theorem 10.33) and hence is a conformal mapping of Ω_2 onto Ω_1.

It follows that conformally equivalent regions are homeomorphic. But there is a much more important relation between conformally equivalent regions: If φ is as above, $f \to f \circ \varphi$ is a one-to-one mapping of $H(\Omega_2)$ onto $H(\Omega_1)$ which preserves sums and products, i.e., which is a ring isomorphism of $H(\Omega_2)$ onto $H(\Omega_1)$. If Ω_1 has a simple structure, problems about $H(\Omega_2)$ can be transferred to problems in $H(\Omega_1)$ and the solutions can be carried back to $H(\Omega_2)$ with the aid of the mapping function φ. The most important case of this is based on the Riemann mapping theorem (where Ω_2 is the unit disc U), which reduces the study of $H(\Omega)$ to the study of $H(U)$, for any simply connected proper subregion of the plane. Of course, for explicit solutions of problems, it may be necessary to have rather precise information about the mapping function.

14.8 Theorem *Every simply connected region Ω in the plane (other than the plane itself) is conformally equivalent to the open unit disc U.*

Note: The case of the plane clearly has to be excluded, by Liouville's theorem. Thus the plane is not conformally equivalent to U, although the two regions are homeomorphic.

The only property of simply connected regions which will be used in the proof is that every holomorphic function which has no zero in such a region has a holomorphic square root there. This will furnish the conclusion "(*j*) implies (*a*)" in Theorem 13.11 and will thus complete the proof of that theorem.

PROOF Suppose Ω is a simply connected region in the plane and let w_0 be a complex number, $w_0 \notin \Omega$. Let Σ be the class of all $\psi \in H(\Omega)$ which are one-to-one in Ω and which map Ω into U. We have to prove that some $\psi \in \Sigma$ maps Ω *onto* U.

We first prove that Σ *is not empty*. Since Ω is simply connected, there exists a $\varphi \in H(\Omega)$ so that $\varphi^2(z) = z - w_0$ in Ω. If $\varphi(z_1) = \varphi(z_2)$, then also $\varphi^2(z_1) = \varphi^2(z_2)$, hence $z_1 = z_2$; thus φ is one-to-one. The same argument shows that there are no two distinct points z_1 and z_2 in Ω such that $\varphi(z_1) = -\varphi(z_2)$. Since φ is an open mapping, $\varphi(\Omega)$ contains a disc $D(a; r)$, with $0 < r < |a|$. The disc $D(-a; r)$ therefore fails to intersect $\varphi(\Omega)$, and if we define $\psi = r/(\varphi + a)$, we see that $\psi \in \Sigma$.

The next step consists in showing that *if $\psi \in \Sigma$, if $\psi(\Omega)$ does not cover all of U, and if $z_0 \in \Omega$, then there exists a $\psi_1 \in \Sigma$ with*

$$|\psi_1'(z_0)| > |\psi'(z_0)|.$$

It will be convenient to use the functions φ_α defined by

$$\varphi_\alpha(z) = \frac{z - \alpha}{1 - \bar{\alpha} z}.$$

For $\alpha \in U$, φ_α is a one-to-one mapping of U onto U; its inverse is $\varphi_{-\alpha}$ (Theorem 12.4).

Suppose $\psi \in \Sigma$, $\alpha \in U$, and $\alpha \notin \psi(\Omega)$. Then $\varphi_\alpha \circ \psi \in \Sigma$, and $\varphi_\alpha \circ \psi$ has no zero in Ω; hence there exists a $g \in H(\Omega)$ such that $g^2 = \varphi_\alpha \circ \psi$. We see that g is one-to-one (as in the proof that $\Sigma \neq \varnothing$), hence $g \in \Sigma$; and if $\psi_1 = \varphi_\beta \circ g$, where $\beta = g(z_0)$, it follows that $\psi_1 \in \Sigma$. With the notation $w^2 = s(w)$, we now have

$$\psi = \varphi_{-\alpha} \circ s \circ g = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta} \circ \psi_1.$$

Since $\psi_1(z_0) = 0$, the chain rule gives

$$\psi'(z_0) = F'(0)\psi_1'(z_0),$$

where $F = \varphi_{-\alpha} \circ s \circ \varphi_{-\beta}$. We see that $F(U) \subset U$ and that F is not one-to-one in U. Therefore $|F'(0)| < 1$, by the Schwarz lemma (see Sec. 12.5), so that $|\psi'(z_0)| < |\psi_1'(z_0)|$. [Note that $\psi'(z_0) \neq 0$, since ψ is one-to-one in Ω.]

Fix $z_0 \in \Omega$, and put

$$\eta = \sup \{|\psi'(z_0)| : \psi \in \Sigma\}.$$

The foregoing makes it clear that *any* $h \in \Sigma$ *for which* $|h'(z_0)| = \eta$ *will map* Ω *onto* U. Hence the proof will be completed as soon as we prove the existence of such an h.

Since $|\psi(z)| < 1$ for all $\psi \in \Sigma$ and $z \in \Omega$, Theorem 14.6 shows that Σ is a normal family. The definition of η shows that there is a sequence $\{\psi_n\}$ in Σ such that $|\psi_n'(z_0)| \to \eta$, and by normality of Σ we can extract a subsequence (again denoted by $\{\psi_n\}$, for simplicity) which converges, uniformly on compact subsets of Ω, to a limit $h \in H(\Omega)$. By Theorem 10.28, $|h'(z_0)| = \eta$. Since $\Sigma \neq \varnothing$, $\eta > 0$, so h is not constant. Since $\psi_n(\Omega) \subset U$, for $n = 1, 2, 3, \ldots$, we have $h(\Omega) \subset \bar{U}$, but the open mapping theorem shows that actually $h(\Omega) \subset U$.

So all that remains to be shown is that h is one-to-one. Fix distinct points z_1 and $z_2 \in \Omega$; put $\alpha = h(z_1)$ and $\alpha_n = \psi_n(z_1)$ for $n = 1, 2, 3, \ldots$; and let $\bar{D}$ be a closed circular disc in Ω with center at z_2, such that $z_1 \notin \bar{D}$ and such that $h - \alpha$ has no zero on the boundary of $\bar{D}$. This is possible, since the zeros of $h - \alpha$ have no limit point in Ω. The functions $\psi_n - \alpha_n$ converge to $h - \alpha$, uniformly on $\bar{D}$; they have no zero in D since they are one-to-one and have a zero at z_1; it now follows from Rouché's theorem that $h - \alpha$ has no zero in D; in particular, $h(z_2) \neq h(z_1)$.

Thus $h \in \Sigma$, and the proof is complete. ////

A more constructive proof is outlined in Exercise 26.

14.9 Remarks The preceding proof also shows that $h(z_0) = 0$. For if $h(z_0) = \beta$ and $\beta \neq 0$, then $\varphi_\beta \circ h \in \Sigma$, and

$$|(\varphi_\beta \circ h)'(z_0)| = |\varphi_\beta'(\beta)h'(z_0)| = \frac{|h'(z_0)|}{1 - |\beta|^2} > |h'(z_0)|.$$

It is interesting to observe that although h was obtained by maximizing $|\psi'(z_0)|$ for $\psi \in \Sigma$, h also maximizes $|f'(z_0)|$ if f is allowed to range over the class consisting of all holomorphic mappings of Ω into U (not necessarily one-to-one). For if f is such a function, then $g = f \circ h^{-1}$ maps U into U, hence $|g'(0)| \leq 1$, with equality holding (by the Schwarz lemma) if and only if g is a rotation, so the chain rule gives the following result:

If $f \in H(\Omega)$, $f(\Omega) \subset U$, and $z_0 \in \Omega$, then $|f'(z_0)| \leq |h'(z_0)|$. Equality holds if and only if $f(z) = \lambda h(z)$, for some constant λ with $|\lambda| = 1$.

The Class $\mathscr{S}$

14.10 Definition $\mathscr{S}$ is the class of all $f \in H(U)$ which are *one-to-one* in U and which satisfy

$$f(0) = 0, \qquad f'(0) = 1. \tag{1}$$

Thus every $f \in \mathscr{S}$ has a power series expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \qquad (z \in U). \tag{2}$$

The class $\mathscr{S}$ is not closed under addition or multiplication, but has many other interesting properties. We shall develop only a few of these in this section. Theorem 14.15 will be used in the proof of Mergelyan's theorem, in Chap. 20.

14.11 Example If $|\alpha| \leq 1$ and

$$f_\alpha(z) = \frac{z}{(1 - \alpha z)^2} = \sum_{n=1}^{\infty} n\alpha^{n-1} z^n$$

then $f_\alpha \in \mathscr{S}$.

For if $f_\alpha(z) = f_\alpha(w)$, then $(z - w)(1 - \alpha^2 zw) = 0$, and the second factor is not 0 if $|z| < 1$ and $|w| < 1$.

When $|\alpha| = 1$, f_α is called a Koebe function. We leave it as an exercise to find the regions $f_\alpha(U)$.

14.12 Theorem *(a) If $f \in \mathscr{S}$, $|\alpha| = 1$, and $g(z) = \bar{\alpha} f(\alpha z)$, then $g \in \mathscr{S}$. (b) If $f \in \mathscr{S}$ there exists a $g \in \mathscr{S}$ such that*

$$g^2(z) = f(z^2) \qquad (z \in U). \tag{1}$$

PROOF (*a*) is clear. To prove (*b*), write $f(z) = z\varphi(z)$. Then $\varphi \in H(U)$, $\varphi(0) = 1$, and φ has no zero in U, since f has no zero in $U - \{0\}$. Hence there exists an $h \in H(U)$ with $h(0) = 1$, $h^2(z) = \varphi(z)$. Put

$$g(z) = zh(z^2) \qquad (z \in U). \tag{2}$$

Then $g^2(z) = z^2h^2(z^2) = z^2\varphi(z^2) = f(z^2)$, so that (1) holds. It is clear that $g(0) = 0$ and $g'(0) = 1$. We have to show that g is one-to-one.

Suppose z and $w \in U$ and $g(z) = g(w)$. Since f is one-to-one, (1) implies that $z^2 = w^2$. So either $z = w$ (which is what we want to prove) or $z = -w$. In the latter case, (2) shows that $g(z) = -g(w)$; it follows that $g(z) = g(w) = 0$, and since g has no zero in $U - \{0\}$, we have $z = w = 0$. ////

14.13 Theorem *If $F \in H(U - \{0\})$, F is one-to-one in U, and*

$$F(z) = \frac{1}{z} + \sum_{n=0}^{\infty} \alpha_n z^n \qquad (z \in U), \tag{1}$$

then

$$\sum_{n=1}^{\infty} n|\alpha_n|^2 \leq 1. \tag{2}$$

s is usually called the *area theorem*, for reasons which will become apparent in proof.

PROOF The choice of α_0 is clearly irrelevant. So assume $\alpha_0 = 0$. Neither the hypothesis nor the conclusion is affected if we replace $F(z)$ by $\lambda F(\lambda z)$ ($|\lambda| = 1$). So we may assume that α_1 is real.

Put $U_r = \{z: |z| < r\}$, $C_r = \{z: |z| = r\}$, and $V_r = \{z: r < |z| < 1\}$, for $0 < r < 1$. Then $F(U_r)$ is a neighborhood of ∞ (by the open mapping theorem, applied to $1/F$); the sets $F(U_r)$, $F(C_r)$, and $F(V_r)$ are disjoint, since F is one-to-one. Write

$$F(z) = \frac{1}{z} + \alpha_1 z + \varphi(z) \qquad (z \in U), \tag{3}$$

$F = u + iv$, and

$$A = \frac{1}{r} + \alpha_1 r, \qquad B = \frac{1}{r} - \alpha_1 r. \tag{4}$$

For $z = re^{i\theta}$, we then obtain

$$u = A\cos\theta + \operatorname{Re}\varphi \quad \text{and} \quad v = -B\sin\theta + \operatorname{Im}\varphi. \tag{5}$$

Divide Eqs. (5) by A and B, respectively, square, and add:

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1 + \frac{2\cos\theta}{A}\operatorname{Re}\varphi + \left(\frac{\operatorname{Re}\varphi}{A}\right)^2 - \frac{2\sin\theta}{B}\operatorname{Im}\varphi + \left(\frac{\operatorname{Im}\varphi}{B}\right)^2.$$

By (3), φ has a zero of order at least 2 at the origin. If we keep account of (4), it follows that there exists an $\eta > 0$ such that, for all sufficiently small r,

$$\frac{u^2}{A^2} + \frac{v^2}{B^2} < 1 + \eta r^3 \qquad (z = re^{i\theta}). \tag{6}$$

This says that $F(C_r)$ is in the interior of the ellipse E_r whose semiaxes are $A\sqrt{1+\eta r^3}$ and $B\sqrt{1+\eta r^3}$, and which therefore bounds an area

$$\pi AB(1+\eta r^3) = \pi\left(\frac{1}{r} + \alpha_1 r\right)\left(\frac{1}{r} - \alpha_1 r\right)(1+\eta r^3) \le \frac{\pi}{r^2}(1+\eta r^3). \tag{7}$$

Since $F(C_r)$ is in the interior of E_r, we have $E_r \subset F(U_r)$; hence $F(V_r)$ is in the interior of E_r, so the area of $F(V_r)$ is no larger than (7). The Cauchy-Riemann equations show that the Jacobian of the mapping $(x, y) \to (u, v)$ is $|F'|^2$. Theorem 7.26 therefore gives the following result:

$$\begin{aligned}
\frac{\pi}{r^2}(1+\eta r^3) &\ge \iint_{V_r} |F'|^2 \\
&= \int_r^1 t\,dt \int_0^{2\pi} \left| -t^{-2}e^{-2i\theta} + \sum_1^\infty n\alpha_n t^{n-1} e^{i(n-1)\theta} \right|^2 d\theta \\
&= 2\pi \int_r^1 \left(t^{-3} + \sum_1^\infty n^2 |\alpha_n|^2 t^{2n-1}\right) dt \\
&= \pi\{r^{-2} - 1 + \sum_1^\infty n|\alpha_n|^2(1 - r^{2n})\}.
\end{aligned} \tag{8}$$

If we divide (8) by π and then subtract r^{-2} from each side, we obtain

$$\sum_{n=1}^{N} n|\alpha_n|^2(1 - r^{2n}) \le 1 + \eta r \tag{9}$$

for all sufficiently small r and for all positive integers N. Let $r \to 0$ in (9), then let $N \to \infty$. This gives (2). ////

Corollary *Under the same hypothesis,* $|\alpha_1| \le 1$.

That this is in fact best possible is shown by $F(z) = (1/z) + \alpha z$, $|\alpha| = 1$, which is one-to-one in U.

14.14 Theorem *If* $f \in \mathscr{S}$, *and*

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

then (*a*) $|a_2| \leq 2$, *and* (*b*) $f(U) \supset D(0; \frac{1}{4})$.

The second assertion is that $f(U)$ contains all w with $|w| < \frac{1}{4}$.

PROOF By Theorem 14.12, there exists a $g \in \mathscr{S}$ so that $g^2(z) = f(z$ $G = 1/g$, then Theorem 14.13 applies to G, and this will give (*a*). Since

$$f(z^2) = z^2(1 + a_2 z^2 + \cdots),$$

we have

$$g(z) = z(1 + \tfrac{1}{2}a_2 z^2 + \cdots),$$

and hence

$$G(z) = \frac{1}{z}(1 - \tfrac{1}{2}a_2 z^2 + \cdots) = \frac{1}{z} - \frac{a_2}{2} z + \cdots.$$

The Corollary to Theorem 14.13 shows now that $|a_2| \leq 2$.

To prove (*b*), suppose $w \notin f(U)$. Define

$$h(z) = \frac{f(z)}{1 - f(z)/w}.$$

Then $h \in H(U)$, h is one-to-one in U, and

$$h(z) = (z + a_2 z^2 + \cdots)\left(1 + \frac{z}{w} + \cdots\right) = z + \left(a_2 + \frac{1}{w}\right)z^2 + \cdots,$$

so that $h \in \mathscr{S}$. Apply (*a*) to h: We have $|a_2 + (1/w)| \leq 2$, and since $|a_2|$ we finally obtain $|1/w| \leq 4$. So $|w| \geq \frac{1}{4}$ for every $w \notin f(U)$.
This completes the proof.

Example 14.11 shows that both (*a*) and (*b*) are best possible.

Moreover, given any $\alpha \neq 0$, one can find entire functions f, with $f(0)$ $f'(0) = 1$, that omit the value α. For example,

$$f(z) = \alpha(1 - e^{-z/\alpha}).$$

Of course, no such f can be one-to-one in U if $|\alpha| < \frac{1}{4}$.

14.15 Theorem *Suppose* $F \in H(U - \{0\})$, F *is one-to-one in* U, F *has a po order* 1 *at* $z = 0$, *with residue* 1, *and neither* w_1 *nor* w_2 *are in* $F(U)$.

Then $|w_1 - w_2| \leq 4$.

PROOF If $f = 1/(F - w_1)$, then $f \in \mathscr{S}$, hence $f(U) \supset D(0, \frac{1}{4})$, so the image of U under $F - w_1$ contains all w with $|w| > 4$. Since $w_2 - w_1$ is not in this image, we have $|w_2 - w_1| \leq 4$. ////

Note that this too is best possible: If $F(z) = z^{-1} + z$, then $F(U)$ does not contain the points 2, -2. In fact, the complement of $F(U)$ is precisely the interval $[-2, 2]$ on the real axis.

Continuity at the Boundary

Under certain conditions, every conformal mapping of a simply connected region Ω onto U can be extended to a homeomorphism of its closure $\bar{\Omega}$ onto $\bar{U}$. The nature of the boundary of Ω plays a decisive role here.

14.16 Definition A boundary point β of a simply connected plane region Ω will be called a *simple boundary point of* Ω if β has the following property: To every sequence $\{\alpha_n\}$ in Ω such that $\alpha_n \to \beta$ as $n \to \infty$ there corresponds a curve γ, with parameter interval $[0, 1]$, and a sequence $\{t_n\}$, $0 < t_1 < t_2 < \cdots$, $t_n \to 1$, such that $\gamma(t_n) = \alpha_n$ $(n = 1, 2, 3, \ldots)$ and $\gamma(t) \in \Omega$ for $0 \leq t < 1$.

In other words, there is a curve in Ω which passes through the points α_n and which ends at β.

14.17 Examples Since examples of simple boundary points are obvious, let us look at some that are not simple.

If Ω is $U - \{x: 0 \leq x < 1\}$, then Ω is simply connected; and if $0 < \beta \leq 1$, β is a boundary point of Ω which is not simple.

To get a more complicated example, let Ω_0 be the interior of the square with vertices at the points 0, 1, $1 + i$, and i. Remove the intervals

$$\left[\frac{1}{2n}, \frac{1}{2n} + \frac{n-1}{n}\, i\right] \quad \text{and} \quad \left[\frac{1}{2n+1} + \frac{i}{n}, \frac{1}{2n+1} + i\right]$$

from Ω_0. The resulting region Ω is simply connected. If $0 \leq y \leq 1$, then iy is a boundary point which is not simple.

14.18 Theorem *Let Ω be a bounded simply connected region in the plane, and let f be a conformal mapping of Ω onto U.*

(a) If β is a simple boundary point of Ω, then f has a continuous extension to $\Omega \cup \{\beta\}$. If f is so extended, then $|f(\beta)| = 1$.

(b) If β_1 and β_2 are distinct simple boundary points of Ω and if f is extended to $\Omega \cup \{\beta_1\} \cup \{\beta_2\}$ as in (a), then $f(\beta_1) \neq f(\beta_2)$.

PROOF Let g be the inverse of f. Then $g \in H(U)$, by Theorem 10.33, $g(U) = \Omega$, g is one-to-one, and $g \in H^\infty$, since Ω is bounded.

Suppose (*a*) is false. Then there is a sequence $\{\alpha_n\}$ in Ω such that $\alpha_n \to \beta$, $f(\alpha_{2n}) \to w_1$, $f(\alpha_{2n+1}) \to w_2$, and $w_1 \neq w_2$. Choose γ as in Definition 14.16, and put $\Gamma(t) = f(\gamma(t))$, for $0 \leq t < 1$. Put $K_r = g(\bar{D}(0; r))$, for $0 < r < 1$. Then K_r is a compact subset of Ω. Since $\gamma(t) \to \beta$ as $t \to 1$, there exists a $t^* < 1$ (depending on r) such that $\gamma(t) \notin K_r$ if $t^* < t < 1$. Thus $|\Gamma(t)| > r$ if $t^* < t < 1$. This says that $|\Gamma(t)| \to 1$ as $t \to 1$. Since $\Gamma(t_{2n}) \to w_1$ and $\Gamma(t_{2n+1}) \to w_2$, we also have $|w_1| = |w_2| = 1$.

It now follows that one of the two open arcs J whose union is $T - (\{w_1\} \cup \{w_2\})$ has the property that every radius of U which ends at a point of J intersects the range of Γ in a set which has a limit point on T. Note that $g(\Gamma(t)) = \gamma(t)$ for $0 \leq t < 1$ and that g has radial limits a.e. on T, since $g \in H^\infty$. Hence

$$\lim_{r \to 1} g(re^{it}) = \beta \qquad \text{(a.e. on } J\text{)}, \tag{1}$$

since $g(\Gamma(t)) \to \beta$ as $t \to 1$. By Theorem 11.32, applied to $g - \beta$, (1) shows that g is constant. But g is one-to-one in U, and we have a contradiction. Thus $w_1 = w_2$, and (*a*) is proved.

Suppose (*b*) is false. If we multiply f by a suitable constant of absolute value 1, we then have $\beta_1 \neq \beta_2$ but $f(\beta_1) = f(\beta_2) = 1$.

Since β_1 and β_2 are simple boundary points of Ω, there are curves γ_i with parameter interval $[0, 1]$ such that $\gamma_i([0, 1)) \subset \Omega$ for $i = 1$ and 2 and $\gamma_i(1) = \beta_i$. Put $\Gamma_i(t) = f(\gamma_i(t))$. Then $\Gamma_i([0, 1)) \subset U$, and $\Gamma_1(1) = \Gamma_2(1) = 1$. Since $g(\Gamma_i(t)) = \gamma_i(t)$ on $[0, 1)$, we have

$$\lim_{t \to 1} g(\Gamma_i(t)) = \beta_i \qquad (i = 1, 2). \tag{2}$$

Theorem 12.10 implies therefore that the radial limit of g at 1 is β_1 as well as β_2. This is impossible if $\beta_1 \neq \beta_2$. ////

14.19 Theorem *If Ω is a bounded simply connected region in the plane and if every boundary point of Ω is simple, then every conformal mapping of Ω onto U extends to a homeomorphism of $\bar{\Omega}$ onto $\bar{U}$.*

PROOF Suppose $f \in H(\Omega)$, $f(\Omega) = U$, and f is one-to-one. By Theorem 14.18 we can extend f to a mapping of $\bar{\Omega}$ into $\bar{U}$ such that $f(\alpha_n) \to f(z)$ whenever $\{\alpha_n\}$ is a sequence in Ω which converges to z. If $\{z_n\}$ is a sequence in $\bar{\Omega}$ which converges to z, there exist points $\alpha_n \in \Omega$ such that $|\alpha_n - z_n| < 1/n$ and $|f(\alpha_n) - f(z_n)| < 1/n$. Thus $\alpha_n \to z$, hence $f(\alpha_n) \to f(z)$, and this shows that $f(z_n) \to f(z)$.

We have now proved that our extension of f is continuous on $\bar{\Omega}$. Also $U \subset f(\bar{\Omega}) \subset \bar{U}$. The compactness of $\bar{U}$ implies that $f(\bar{\Omega})$ is compact. Hence $f(\bar{\Omega}) = \bar{U}$.

Theorem 14.18(*b*) shows that f is one-to-one on $\bar{\Omega}$. Since every continuous one-to-one mapping of a compact set has a continuous inverse ([26], Theorem 4.17), the proof is complete. ////

14.20 Remarks

(*a*) *The preceding theorem has a purely topological corollary: If every boundary point of a bounded simply connected plane region Ω is simple, then the boundary of Ω is a Jordan curve, and $\bar{\Omega}$ is homeomorphic to $\bar{U}$.*

(A Jordan curve is, by definition, a homeomorphic image of the unit circle.)

The converse is true, but we shall not prove it: If the boundary of Ω is a Jordan curve, then every boundary point of Ω is simple.

(*b*) Suppose f is as in Theorem 14.19, a, b, and c are distinct boundary points of Ω, and A, B, and C are distinct points of T. There is a linear fractional transformation φ which maps the triple $\{f(a), f(b), f(c)\}$ to $\{A, B, C\}$; suppose the orientation of $\{A, B, C\}$ agrees with that of $\{f(a), f(b), f(c)\}$; then $\varphi(U) = U$, and the function $g = \varphi \circ f$ is a homeomorphism of $\bar{\Omega}$ onto $\bar{U}$ which is holomorphic in Ω and which maps $\{a, b, c\}$ to prescribed values $\{A, B, C\}$. It follows from Sec. 14.3 that g is uniquely determined by these requirements.

(*c*) Theorem 14.19, as well as the above remark (*b*), extends without difficulty to simply connected regions Ω in the Riemann sphere S_2, all of whose boundary points are simple, provided that $S^2 - \Omega$ has a nonempty interior, for then a linear fractional transformation brings us back to the case in which Ω is a bounded region in the plane. Likewise, U can be replaced, for instance, by a half plane.

(*d*) More generally, if f_1 and f_2 map Ω_1 and Ω_2 onto U, as in Theorem 14.19, then $f = f_2^{-1} \circ f_1$ is a homeomorphism of $\bar{\Omega}_1$ onto $\bar{\Omega}_2$ which is holomorphic in Ω_1.

Conformal Mapping of an Annulus

14.21 It is a consequence of the Riemann mapping theorem that any two simply connected proper subregions of the plane are conformally equivalent, since each of them is conformally equivalent to the unit disc. This is a very special property of simply connected regions. One may ask whether it extends to the next simplest situation, i.e., whether any two annuli are conformally equivalent. The answer is negative.

For $0 < r < R$, let

$$A(r, R) = \{z: r < |z| < R\} \tag{1}$$

be the annulus with inner radius r and outer radius R. If $\lambda > 0$, the mapping $z \to \lambda z$ maps $A(r, R)$ onto $A(\lambda r, \lambda R)$. *Hence $A(r, R)$ and $A(r_1, R_1)$ are conformally equivalent if $R/r = R_1/r_1$.* The surprising fact is that this sufficient condition is

also necessary; thus among the annuli there is a different conformal type associated with each real number greater than 1.

14.22 Theorem *$A(r_1, R_1)$ and $A(r_2, R_2)$ are conformally equivalent if and only if $R_1/r_1 = R_2/r_2$.*

PROOF Assume $r_1 = r_2 = 1$, without loss of generality. Put

$$A_1 = A(1, R_1), \qquad A_2 = A(1, R_2), \tag{1}$$

and assume there exists $f \in H(A_1)$ such that f is one-to-one and $f(A_1) = A_2$. Let K be the circle with center at 0 and radius $r = \sqrt{R_2}$. Since $f^{-1}: A_2 \to A_1$ is also holomorphic, $f^{-1}(K)$ is compact. Hence

$$A(1, 1 + \epsilon) \cap f^{-1}(K) = \varnothing \tag{2}$$

for some $\epsilon > 0$. Then $V = f(A(1, 1 + \epsilon))$ is a connected subset of A_2 which does not intersect K, so that $V \subset A(1, r)$ or $V \subset A(r, R_2)$. In the latter case, replace f by R_2/f. So we can assume that $V \subset A(1, r)$. If $1 < |z_n| < 1 + \epsilon$ and $|z_n| \to 1$, then $f(z_n) \in V$ and $\{f(z_n)\}$ has no limit point in A_2 (since f^{-1} is continuous); thus $|f(z_n)| \to 1$. In the same manner we see that $|f(z_n)| \to R_2$ if $|z_n| \to R_1$.

Now define

$$\alpha = \frac{\log R_2}{\log R_1} \tag{3}$$

and

$$u(z) = 2 \log |f(z)| - 2\alpha \log |z| \qquad (z \in A_1). \tag{4}$$

Let ∂ be one of the Cauchy-Riemann operators. Since $\partial \bar{f} = 0$ and $\partial f = f'$, the chain rule gives

$$\partial(2 \log |f|) = \partial(\log (f\bar{f})) = f'/f, \tag{5}$$

so that

$$(\partial u)(z) = \frac{f'(z)}{f(z)} - \frac{\alpha}{z} \qquad (z \in A_1). \tag{6}$$

Thus u is a harmonic function in A_1 which, by the first paragraph of this proof, extends to a continuous function on $\bar{A}_1$ which is 0 on the boundary of A_1. Since nonconstant harmonic functions have no local maxima or minima, we conclude that $u = 0$. Thus

$$\frac{f'(z)}{f(z)} = \frac{\alpha}{z} \qquad (z \in A_1). \tag{7}$$

Put $\gamma(t) = \sqrt{R_1}\, e^{it}$ $(-\pi \le t \le \pi)$; put $\Gamma = f \circ \gamma$. As in the proof of Theorem 10.43, (7) gives

$$\alpha = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)}\, dz = \operatorname{Ind}_\Gamma (0). \tag{8}$$

Thus α is an integer. By (3), $\alpha > 0$. By (7), the derivative of $z^{-\alpha} f(z)$ is 0 in A_1. Thus $f(z) = cz^\alpha$. Since f is one-to-one in A_1, $\alpha = 1$. Hence $R_2 = R_1$. ////

Exercises

1 Find necessary and sufficient conditions which the complex numbers a, b, c, and d have to satisfy so that the linear fractional transformation $z \to (az + b)/(cz + d)$ maps the upper half plane onto itself.

2 In Theorem 11.14 the hypotheses were, in simplified form, that $\Omega \subset \Pi^+$, L is on the real axis, and $\operatorname{Im} f(z) \to 0$ as $z \to L$. Use this theorem to establish analogous reflection theorems under the following hypotheses:

(*a*) $\Omega \subset \Pi^+$, L on real axis, $|f(z)| \to 1$ as $z \to L$.

(*b*) $\Omega \subset U$, $L \subset T$, $|f(z)| \to 1$ as $z \to L$.

(*c*) $\Omega \subset U$, $L \subset T$, $\operatorname{Im} f(z) \to 0$ as $z \to L$.

In case (*b*), if f has a zero at $\alpha \in \Omega$, show that its extension has a pole at $1/\bar{\alpha}$. What are the analogues of this in cases (*a*) and (*c*)?

3 Suppose R is a rational function such that $|R(z)| = 1$ if $|z| = 1$. Prove that

$$R(z) = cz^m \prod_{n=1}^{k} \frac{z - \alpha_n}{1 - \bar{\alpha}_n z}$$

where c is a constant, m is an integer, and $\alpha_1, \ldots, \alpha_k$ are complex numbers such that $\alpha_n \neq 0$ and $|\alpha_n| \neq 1$. Note that each of the above factors has absolute value 1 if $|z| = 1$.

4 Obtain an analogous description of those rational functions which are positive on T.

Hint: Such a function must have the same number of zeros as poles in U. Consider products of factors of the form

$$\frac{(z - \alpha)(1 - \bar{\alpha} z)}{(z - \beta)(1 - \bar{\beta} z)}$$

where $|\alpha| < 1$ and $|\beta| < 1$.

5 Suppose f is a trigonometric polynomial,

$$f(\theta) = \sum_{k=-n}^{n} a_k e^{ik\theta},$$

and $f(\theta) > 0$ for all real θ. Prove that there is a polynomial $P(z) = c_0 + c_1 z + \cdots + c_n z^n$ such that

$$f(\theta) = |P(e^{i\theta})|^2 \qquad (\theta \text{ real}).$$

Hint: Apply Exercise 4 to the rational function $\Sigma a_k z^k$. Is the result still valid if we assume $f(\theta) \ge 0$ instead of $f(\theta) > 0$?

6 Find the fixed points of the mappings φ_α (Definition 12.3). Is there a straight line which φ_α maps to itself?

7 Find all complex numbers α for which f_α is one-to-one in U, where

$$f_\alpha(z) = \frac{z}{1 + \alpha z^2}.$$

Describe $f_\alpha(U)$ for all these cases.

8 Suppose $f(z) = z + (1/z)$. Describe the families of ellipses and hyperbolas onto which f maps circles with center at 0 and rays through 0.

9 (*a*) Suppose $\Omega = \{z: -1 < \operatorname{Re} z < 1\}$. Find an explicit formula for the one-to-one conformal mapping f of Ω onto U for which $f(0) = 0$ and $f'(0) > 0$. Compute $f'(0)$.

(*b*) Note that the inverse of the function constructed in (*a*) has its real part bounded in U, whereas its imaginary part is unbounded. Show that this implies the existence of a continuous real function u on $\bar{U}$ which is harmonic in U and whose harmonic conjugate v is unbounded in U. [v is the function which makes $u + iv$ holomorphic in U; we can determine v uniquely by the requirement $v(0) = 0$.]

(*c*) Suppose $g \in H(U)$, $|\operatorname{Re} g| < 1$ in U, and $g(0) = 0$. Prove that

$$|g(re^{i\theta})| \leq \frac{2}{\pi} \log \frac{1+r}{1-r}.$$

Hint: See Exercise 10.

(*d*) Let Ω be the strip that occurs in Theorem 12.9. Fix a point $\alpha + i\beta$ in Ω. Let h be a conformal one-to-one mapping of Ω onto Ω that carries $\alpha + i\beta$ to 0. Prove that

$$|h'(\alpha + i\beta)| = 1/\cos \beta.$$

10 Suppose f and g are holomorphic mappings of U into Ω, f is one-to-one, $f(U) = \Omega$, and $f(0) = g(0)$. Prove that

$$g(D(0; r)) \subset f(D(0; r)) \qquad (0 < r < 1).$$

11 Let Ω be the upper half of the unit disc U. Find the conformal mapping f of Ω onto U that carries $\{-1, 0, 1\}$ to $\{-1, -i, 1\}$. Find $z \in \Omega$ such that $f(z) = 0$. Find $f(i/2)$. *Hint*: $f = \varphi \circ s \circ \psi$, where φ and ψ are linear fractional transformations and $s(\lambda) = \lambda^2$.

12 Suppose Ω is a convex region, $f \in H(\Omega)$, and $\operatorname{Re} f'(z) > 0$ for all $z \in \Omega$. Prove that f is one-to-one in Ω. Is the result changed if the hypothesis is weakened to $\operatorname{Re} f'(z) \geq 0$? (Exclude the trivial case $f =$ constant.) Show by an example that "convex" cannot be replaced by "simply connected."

13 Suppose Ω is a region, $f_n \in H(\Omega)$ for $n = 1, 2, 3, \ldots$, each f_n is one-to-one in Ω, and $f_n \to f$ uniformly on compact subsets of Ω. Prove that f is either constant or one-to-one in Ω. Show that both cases can occur.

14 Suppose $\Omega = \{x + iy: -1 < y < 1\}$, $f \in H(\Omega)$, $|f| < 1$, and $f(x) \to 0$ as $x \to \infty$. Prove that

$$\lim_{x \to \infty} f(x + iy) = 0 \qquad (-1 < y < 1)$$

and that the passage to the limit is uniform if y is confined to an interval $[-\alpha, \alpha]$, where $\alpha < 1$. *Hint*: Consider the sequence $\{f_n\}$, where $f_n(z) = z + n$, in the square $|x| < 1$, $|y| < 1$.

What does this theorem tell about the behavior of a function $g \in H^\infty$ near a boundary point of U at which the radial limit of g exists?

15 Let $\mathscr{F}$ be the class of all $f \in H(U)$ such that $\operatorname{Re} f > 0$ and $f(0) = 1$. Show that $\mathscr{F}$ is a normal family. Can the condition "$f(0) = 1$" be omitted? Can it be replaced by "$|f(0)| \leq 1$"?

16 Let $\mathscr{F}$ be the class of all $f \in H(U)$ for which

$$\iint_U |f(z)|^2 \, dx \, dy \leq 1.$$

Is this a normal family?

17 Suppose Ω is a region, $f_n \in H(\Omega)$ for $n = 1, 2, 3, \ldots$, $f_n \to f$ uniformly on compact subsets of Ω, and f is one-to-one in Ω. Does it follow that to each compact $K \subset \Omega$ there corresponds an integer $N(K)$ such that f_n is one-to-one on K for all $n > N(K)$? Give proof or counterexample.

18 Suppose Ω is a simply connected region, $z_0 \in \Omega$, and f and g are one-to-one conformal mappings of Ω onto U which carry z_0 to 0. What relation exists between f and g? Answer the same question if $f(z_0) = g(z_0) = a$, for some $a \in U$.

19 Find a homeomorphism of U onto U which cannot be extended to a continuous function on $\bar{U}$.

20 If $f \in \mathscr{S}$ (Definition 14.10) and n is a positive integer, prove that there exists a $g \in \mathscr{S}$ such that $g^n(z) = f(z^n)$ for all $z \in U$.

21 Find all $f \in \mathscr{S}$ such that (*a*) $f(U) \supset U$, (*b*) $f(U) \supset \bar{U}$, (*c*) $|a_2| = 2$.

22 Suppose f is a one-to-one conformal mapping of U onto a square with center at 0, and $f(0) = 0$. Prove that $f(iz) = if(z)$. If $f(z) = \Sigma c_n z^n$, prove that $c_n = 0$ unless $n - 1$ is a multiple of 4. Generalize this: Replace the square by other simply connected regions with rotational symmetry.

23 Let Ω be a bounded region whose boundary consists of two nonintersecting circles. Prove that there is a one-to-one conformal mapping of Ω onto an annulus. (This is true for every region Ω such that $S^2 - \Omega$ has exactly two components, each of which contains more than one point, but this general situation is harder to handle.)

24 Complete the details in the following proof of Theorem 14.22. Suppose $1 < R_2 < R_1$ and f is a one-to-one conformal mapping of $A(1, R_1)$ onto $A(1, R_2)$. Define $f_1 = f$ and $f_n = f \circ f_{n-1}$. Then a subsequence of $\{f_n\}$ converges uniformly on compact subsets of $A(1, R_1)$ to a function g. Show that the range of g cannot contain any nonempty open set (by the three-circle theorem, for instance). On the other hand, show that g cannot be constant on the circle $\{z: |z|^2 = R_1\}$. Hence f cannot exist.

25 Here is yet another proof of Theorem 14.22. If f is as in 14.22, repeated use of the reflection principle extends f to an entire function such that $|f(z)| = 1$ whenever $|z| = 1$. This implies $f(z) = \alpha z^n$, where $|\alpha| = 1$ and n is an integer. Complete the details.

26 Iteration of Step 2 in the proof of Theorem 14.8 leads to a proof (due to Koebe) of the Riemann mapping theorem which is constructive in the sense that it makes no appeal to the theory of normal families and so does not depend on the existence of some unspecified subsequence. For the final step of the proof it is convenient to assume that Ω has property (*h*) of Theorem 13.11. Then any region conformally equivalent to Ω will satisfy (*h*). Recall also that (*h*) implies (*j*), trivially.

By Step 1 in Theorem 14.8 we may assume, without loss of generality, that $0 \in \Omega$, $\Omega \subset U$, and $\Omega \neq U$. Put $\Omega = \Omega_0$. The proof consists in the construction of regions $\Omega_1, \Omega_2, \Omega_3, \ldots$ and of functions $f_1, f_2, f_3, \ldots$, so that $f_n(\Omega_{n-1}) = \Omega_n$ and so that the functions $f_n \circ f_{n-1} \circ \cdots \circ f_2 \circ f_1$ converge to a conformal mapping of Ω onto U.

Complete the details in the following outline.

(*a*) Suppose Ω_{n-1} is constructed, let r_n be the largest number such that $D(0; r_n) \subset \Omega_{n-1}$, let α_n be a boundary point of Ω_{n-1} with $|\alpha_n| = r_n$, choose β_n so that $\beta_n^2 = -\alpha_n$, and put

$$F_n = \varphi_{-\alpha_n} \circ s \circ \varphi_{-\beta_n}.$$

(The notation is as in the proof of Theorem 14.8.) Show that F_n has a holomorphic inverse G_n in Ω_{n-1}, and put $f_n = \lambda_n G_n$, where $\lambda_n = |c|/c$ and $c = G_n'(0)$. (This f_n is the *Koebe mapping* associated with Ω_{n-1}. Note that f_n is an elementary function. It involves only two linear fractional transformations and a square root.)

(*b*) Compute that $f_n'(0) = (1 + r_n)/2\sqrt{r_n} > 1$.

(*c*) Put $\psi_0(z) = z$ and $\psi_n(z) = f_n(\psi_{n-1}(z))$. Show that ψ_n is a one-to-one mapping of Ω onto a region $\Omega_n \subset U$, that $\{\psi_n'(0)\}$ is bounded, that

$$\psi_n'(0) = \prod_{k=1}^{n} \frac{1 + r_k}{2\sqrt{r_k}},$$

and that therefore $r_n \to 1$ as $n \to \infty$.

(*d*) Write $\psi_n(z) = z h_n(z)$, for $z \in \Omega$, show that $|h_n| \leq |h_{n+1}|$, apply Harnack's theorem and Exercise 8 of Chap. 11 to $\{\log |h_n|\}$ to prove that $\{\psi_n\}$ converges uniformly on compact subsets of Ω, and show that $\lim \psi_n$ is a one-to-one mapping of Ω onto U.

27 Prove that $\sum_{n=1}^{\infty}(1-r_n)^2 < \infty$, where $\{r_n\}$ is the sequence which occurs in Exercise 26. *Hint*:

$$\frac{1+r}{2\sqrt{r}} = 1 + \frac{(1-\sqrt{r})^2}{2\sqrt{r}}.$$

28 Suppose that in Exercise 26 we choose $\alpha_n \in U - \Omega_{n-1}$ without insisting that $|\alpha_n| = r_n$. For example, insist only that

$$|\alpha_n| \leq \frac{1+r_n}{2}.$$

Will the resulting sequence $\{\psi_n\}$ still converge to the desired mapping function?

29 Suppose Ω is a bounded region, $a \in \Omega, f \in H(\Omega), f(\Omega) \subset \Omega$, and $f(a) = a$.

(*a*) Put $f_1 = f$ and $f_n = f \circ f_{n-1}$, compute $f_n'(a)$, and conclude that $|f'(a)| \leq 1$.

(*b*) If $f'(a) = 1$, prove that $f(z) = z$ for all $z \in \Omega$. *Hint*: If

$$f(z) = z + c_m(z-a)^m + \cdots,$$

compute the coefficient of $(z-a)^m$ in the expansion of $f_n(z)$.

(*c*) If $|f'(a)| = 1$, prove that f is one-to-one and that $f(\Omega) = \Omega$.

Hint: If $\gamma = f'(a)$, there are integers $n_k \to \infty$ such that $\gamma^{n_k} \to 1$ and $f_{n_k} \to g$. Then $g'(a) = 1$, $g(\Omega) \subset \Omega$ (by Exercise 20, Chap. 10), hence $g(z) = z$, by part (*b*). Use g to draw the desired conclusions about f.

30 Let Λ be the set of all linear fractional transformations.

If $\{\alpha, \beta, \gamma, \delta\}$ is an ordered quadruple of distinct complex numbers, its *cross ratio* is defined to be

$$[\alpha, \beta, \gamma, \delta] = \frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\delta)(\gamma-\beta)}.$$

If one of these numbers is ∞, the definition is modified in the obvious way, by continuity. The same applies if α coincides with β or γ or δ.

(*a*) If $\varphi(z) = [z, \alpha, \beta, \gamma]$, show that $\varphi \in \Lambda$ and φ maps $\{\alpha, \beta, \gamma\}$ to $\{0, 1, \infty\}$.

(*b*) Show that the equation $[w, a, b, c] = [z, \alpha, \beta, \gamma]$ can be solved in the form $w = \varphi(z)$; then $\varphi \in \Lambda$ maps $\{\alpha, \beta, \gamma\}$ to $\{a, b, c\}$.

(*c*) If $\varphi \in \Lambda$, show that

$$[\varphi(\alpha), \varphi(\beta), \varphi(\gamma), \varphi(\delta)] = [\alpha, \beta, \gamma, \delta].$$

(*d*) Show that $[\alpha, \beta, \gamma, \delta]$ is real if and only if the four points lie on the same circle or straight line.

(*e*) Two points z and z^* are said to be *symmetric* with respect to the circle (or straight line) C through α, β, and γ if $[z^*, \alpha, \beta, \gamma]$ is the complex conjugate of $[z, \alpha, \beta, \gamma]$. If C is the unit circle, find a simple geometric relation between z and z^*. Do the same if C is a straight line.

(*f*) Suppose z and z^* are symmetric with respect to C. Show that $\varphi(z)$ and $\varphi(z^*)$ are symmetric with respect to $\varphi(C)$, for every $\varphi \in \Lambda$.

31 (*a*) Show that Λ (see Exercise 30) is a group, with composition as group operation. That is, if $\varphi \in \Lambda$ and $\psi \in \Lambda$, show that $\varphi \circ \psi \in \Lambda$ and that the inverse φ^{-1} of φ is in Λ. Show that Λ is not commutative.

(*b*) Show that each member of Λ (other than the identity mapping) has either one or two fixed points on S^2. [A fixed point of φ is a point α such that $\varphi(\alpha) = \alpha$.]

(*c*) Call two mappings φ and $\varphi_1 \in \Lambda$ *conjugate* if there exists a $\psi \in \Lambda$ such that $\varphi_1 = \psi^{-1} \circ \varphi \circ \psi$. Prove that every $\varphi \in \Lambda$ with a unique fixed point is conjugate to the mapping $z \to z + 1$. Prove that every $\varphi \in \Lambda$ with two distinct fixed points is conjugate to the mapping $z \to \alpha z$, where α is a complex number; to what extent is α determined by φ?

(*d*) Let α be a complex number. Show that to every $\varphi \in \Lambda$ which has α for its unique fixed point there corresponds a β such that

$$\frac{1}{\varphi(z) - \alpha} = \frac{1}{z - \alpha} + \beta.$$

Let G_α be the set of all these φ, plus the identity transformation. Prove that G_α is a subgroup of Λ and that G_α is isomorphic to the additive group of all complex numbers.

(*e*) Let α and β be distinct complex numbers, and let $G_{\alpha,\beta}$ be the set of all $\varphi \in \Lambda$ which have α and β as fixed points. Show that every $\varphi \in G_{\alpha,\beta}$ is given by

$$\frac{\varphi(z) - \alpha}{\varphi(z) - \beta} = \gamma \cdot \frac{z - \alpha}{z - \beta},$$

where γ is a complex number. Show that $G_{\alpha,\beta}$ is a subgroup of Λ which is isomorphic to the multiplicative group of all nonzero complex numbers.

(*f*) If φ is as in (*d*) or (*e*), for which circles C is it true that $\varphi(C) = C$? The answer should be in terms of the parameters α, β, and γ.

32 For $z \in \bar{U}$, $z^2 \neq 1$, define

$$f(z) = \exp\left\{i \log \frac{1+z}{1-z}\right\},$$

choosing the branch of log that has $\log 1 = 0$. Describe $f(E)$ if E is

(*a*) U,
(*b*) the upper half of T,
(*c*) the lower half of T,
(*d*) any circular arc (in U) from -1 to 1,
(*e*) the radius $[0, 1)$,
(*f*) any disc $\{z: |z - r| < 1 - r\}$, $0 < r < 1$.
(*g*) any curve in U tending to 1.

33 If φ_α is as in Definition 12.3, show that

(*a*) $$\frac{1}{\pi}\int_U |\varphi_\alpha'|^2 \, dm = 1,$$

(*b*) $$\frac{1}{\pi}\int_U |\varphi_\alpha'| \, dm = \frac{1 - |\alpha|^2}{|\alpha|^2} \log \frac{1}{1 - |\alpha|^2}.$$

Here m denotes Lebesgue measure in R^2.

CHAPTER

FIFTEEN

ZEROS OF HOLOMORPHIC FUNCTIONS

Infinite Products

15.1 So far we have met only one result concerning the zero set $Z(f)$ of a non-constant holomorphic function f in a region Ω, namely, $Z(f)$ has no limit point in Ω. We shall see presently that this is all that can be said about $Z(f)$, if no other conditions are imposed on f, because of the theorem of Weierstrass (Theorem 15.11) which asserts that every $A \subset \Omega$ without limit point in Ω is $Z(f)$ for some $f \in H(\Omega)$. If $A = \{\alpha_n\}$, a natural way to construct such an f is to choose functions $f_n \in H(\Omega)$ so that f_n has only one zero, at α_n, and to consider the limit of the products

$$p_n = f_1 f_2 \cdots f_n,$$

as $n \to \infty$. One has to arrange it so that the sequence $\{p_n\}$ converges to some $f \in H(\Omega)$ *and* so that the limit function f is not 0 except at the prescribed points α_n. It is therefore advisable to begin by studying some general properties of infinite products.

15.2 Definition Suppose $\{u_n\}$ is a sequence of complex numbers,

$$p_n = (1 + u_1)(1 + u_2) \cdots (1 + u_n), \tag{1}$$

and $p = \lim_{n \to \infty} p_n$ exists. Then we write

$$p = \prod_{n=1}^{\infty} (1 + u_n). \tag{2}$$

The p_n are the *partial products* of the *infinite product* (2). We shall say that the infinite product (2) converges if the sequence $\{p_n\}$ converges.

In the study of infinite series Σa_n it is of significance whether the a_n approach 0 rapidly. Analogously, in the study of infinite products it is of interest whether the factors are or are not close to 1. This accounts for the above notation: $1 + u_n$ is close to 1 if u_n is close to 0.

15.3 Lemma *If $u_1, \ldots, u_N$ are complex numbers, and if*

$$p_N = \prod_{n=1}^{N} (1 + u_n), \qquad p_N^* = \prod_{n=1}^{N} (1 + |u_n|), \tag{1}$$

then

$$p_N^* \leq \exp(|u_1| + \cdots + |u_N|) \tag{2}$$

and

$$|p_N - 1| \leq p_N^* - 1. \tag{3}$$

PROOF For $x \geq 0$, the inequality $1 + x \leq e^x$ is an immediate consequence of the expansion of e^x in powers of x. Replace x by $|u_1|, \ldots, |u_N|$ and multiply the resulting inequalities. This gives (2). For $N = 1$, (3) is trivial. The general case follows by induction: For $k = 1, \ldots, N - 1$,

$$p_{k+1} - 1 = p_k(1 + u_{k+1}) - 1 = (p_k - 1)(1 + u_{k+1}) + u_{k+1},$$

so that if (3) holds with k in place of N, then also

$$|p_{k+1} - 1| \leq (p_k^* - 1)(1 + |u_{k+1}|) + |u_{k+1}| = p_{k+1}^* - 1. \qquad ////$$

15.4 Theorem *Suppose $\{u_n\}$ is a sequence of bounded complex functions on a set S, such that $\Sigma |u_n(s)|$ converges uniformly on S. Then the product*

$$f(s) = \prod_{n=1}^{\infty} (1 + u_n(s)) \tag{1}$$

converges uniformly on S, and $f(s_0) = 0$ at some $s_0 \in S$ if and only if $u_n(s_0) = -1$ for some n.

Furthermore, if $\{n_1, n_2, n_3, \ldots\}$ is any permutation of $\{1, 2, 3, \ldots\}$, then we also have

$$f(s) = \prod_{k=1}^{\infty} (1 + u_{n_k}(s)) \qquad (s \in S). \tag{2}$$

PROOF The hypothesis implies that $\Sigma |u_n(s)|$ is bounded on S, and if p_N denotes the Nth partial product of (1), we conclude from Lemma 15.3 that there is a constant $C < \infty$ such that $|p_N(s)| \leq C$ for all N and all s.

Choose ϵ, $0 < \epsilon < \frac{1}{2}$. There exists an N_0 such that

$$\sum_{n=N_0}^{\infty} |u_n(s)| < \epsilon \qquad (s \in S).$$

Let $\{n_1, n_2, n_3, \ldots\}$ be a permutation of $\{1, 2, 3, \ldots\}$. If $N \geq N_0$, if so large that

$$\{1, 2, \ldots, N\} \subset \{n_1, n_2, \ldots, n_M\},$$

and if $q_M(s)$ denotes the Mth partial product of (2), then

$$q_M - p_N = p_N\{\prod (1 + u_{n_k}) - 1\}.$$

The n_k which occur in (5) are all distinct and are larger than N_0. Ther (3) and Lemma 15.3 show that

$$|q_M - p_N| \leq |p_N|(e^{\epsilon} - 1) \leq 2|p_N|\epsilon \leq 2C\epsilon.$$

If $n_k = k$ $(k = 1, 2, 3, \ldots)$, then $q_M = p_M$, and (6) shows that $\{p_N\}$ verges uniformly to a limit function f. Also, (6) shows that

$$|p_M - p_{N_0}| \leq 2|p_{N_0}|\epsilon \qquad (M > N_0),$$

so that $|p_M| \geq (1 - 2\epsilon)|p_{N_0}|$. Hence

$$|f(s)| \geq (1 - 2\epsilon)|p_{N_0}(s)| \qquad (s \in S),$$

which shows that $f(s) = 0$ if and only if $p_{N_0}(s) = 0$.

Finally, (6) also shows that $\{q_M\}$ converges to the same limit as $\{p_N\}$.

15.5 Theorem *Suppose* $0 \leq u_n < 1$. *Then*

$$\prod_{n=1}^{\infty} (1 - u_n) > 0 \qquad \textit{if and only if} \qquad \sum_{n=1}^{\infty} u_n < \infty.$$

PROOF If $p_N = (1 - u_1) \cdots (1 - u_N)$, then $p_1 \geq p_2 \geq \cdots$, $p_N > 0$, hence $\lim p_N$ exists. If $\Sigma u_n < \infty$, Theorem 15.4 implies $p > 0$. On the other hand

$$p \leq p_N = \prod_{1}^{N} (1 - u_n) \leq \exp\{-u_1 - u_2 - \cdots - u_N\},$$

and the last expression tends to 0 as $N \to \infty$, if $\Sigma u_n = \infty$.

We shall frequently use the following consequence of Theorem 15.4:

15.6 Theorem *Suppose* $f_n \in H(\Omega)$ *for* $n = 1, 2, 3, \ldots$, *no* f_n *is identically zero in any component of* Ω, *and*

$$\sum_{n=1}^{\infty} |1 - f_n(z)|$$

converges uniformly on compact subsets of Ω. *Then the product*

$$f(z) = \prod_{n=1}^{\infty} f_n(z) \tag{2}$$

converges uniformly on compact subsets of Ω. *Hence* $f \in H(\Omega)$.

Furthermore, we have

$$m(f; z) = \sum_{n=1}^{\infty} m(f_n; z) \qquad (z \in \Omega), \tag{3}$$

where $m(f; z)$ *is defined to be the multiplicity of the zero of* f *at* z. [If $f(z) \neq 0$, then $m(f; z) = 0$.]

PROOF The first part follows immediately from Theorem 15.4. For the second part, observe that each $z \in \Omega$ has a neighborhood V in which at most finitely many of the f_n have a zero, by (1). Take these factors first. The product of the remaining ones has no zero in V, by Theorem 15.4, and this gives (3). Incidentally, we see also that at most finitely many terms in the series (3) can be positive for any given $z \in \Omega$. ////

The Weierstrass Factorization Theorem

15.7 Definition Put $E_0(z) = 1 - z$, and for $p = 1, 2, 3, \ldots$,

$$E_p(z) = (1 - z) \exp\left\{z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right\}.$$

These functions, introduced by Weierstrass, are sometimes called *elementary factors*. Their only zero is at $z = 1$. Their utility depends on the fact that they are close to 1 if $|z| < 1$ and p is large, although $E_p(1) = 0$.

15.8 Lemma *For* $|z| \leq 1$ *and* $p = 0, 1, 2, \ldots$,

$$|1 - E_p(z)| \leq |z|^{p+1}.$$

PROOF For $p = 0$, this is obvious. For $p \geq 1$, direct computation shows that

$$-E_p'(z) = z^p \exp\left\{z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right\}.$$

So $-E_p'$ has a zero of order p at $z = 0$, and the expansion of $-E_p'$ in powers of z has nonnegative real coefficients. Since

$$1 - E_p(z) = -\int_{[0, z]} E_p'(w)\, dw,$$

$1 - E_p$ has a zero of order $p + 1$ at $z = 0$, and if

$$\varphi(z) = \frac{1 - E_p(z)}{z^{p+1}},$$

then $\varphi(z) = \Sigma a_n z^n$, with all $a_n \geq 0$. Hence $|\varphi(z)| \leq \varphi(1) = 1$ if $|z| \leq 1$, and this gives the assertion of the lemma. ////

15.9 Theorem *Let $\{z_n\}$ be a sequence of complex numbers such that $z_n \neq 0$ and $|z_n| \to \infty$ as $n \to \infty$. If $\{p_n\}$ is a sequence of nonnegative integers such that*

$$\sum_{n=1}^{\infty} \left(\frac{r}{r_n}\right)^{1+p_n} < \infty \tag{1}$$

for every positive r (where $r_n = |z_n|$), then the infinite product

$$P(z) = \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right) \tag{2}$$

defines an entire function P which has a zero at each point z_n and which has no other zeros in the plane.

More precisely, if α occurs m times in the sequence $\{z_n\}$, then P has a zero of order m at α.

Condition (1) *is always satisfied if $p_n = n - 1$, for instance.*

PROOF For every r, $r_n > 2r$ for all but finitely many n, hence $r/r_n < \frac{1}{2}$ for these n, so (1) holds with $1 + p_n = n$.

Now fix r. If $|z| \leq r$, Lemma 15.8 shows that

$$\left|1 - E_{p_n}\left(\frac{z}{z_n}\right)\right| \leq \left|\frac{z}{z_n}\right|^{1+p_n} \leq \left(\frac{r}{r_n}\right)^{1+p_n}$$

if $r_n \geq r$, which holds for all but finitely many n. It now follows from (1) that the series

$$\sum_{n=1}^{\infty} \left|1 - E_{p_n}\left(\frac{z}{z_n}\right)\right|$$

converges uniformly on compact sets in the plane, and Theorem 15.6 gives the desired conclusion. ////

Note: For certain sequences $\{r_n\}$, (1) holds for a constant sequence $\{p_n\}$. It is of interest to take this constant as small as possible; the resulting function (2) is then called the *canonical product* corresponding to $\{z_n\}$. For instance, if $\Sigma 1/r_n < \infty$, we can take $p_n = 0$, and the canonical product is simply

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right).$$

If $\Sigma 1/r_n = \infty$ but $\Sigma 1/r_n^2 < \infty$, the canonical product is

$$\prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n}.$$

Canonical products are of great interest in the study of entire functions of finite order. (See Exercise 2 for the definition.)

We now state the Weierstrass factorization theorem.

15.10 Theorem *Let f be an entire function, suppose $f(0) \neq 0$, and let $z_1, z_2, z_3, \ldots$ be the zeros of f, listed according to their multiplicities. Then there exist an entire function g and a sequence $\{p_n\}$ of nonnegative integers, such that*

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right). \tag{1}$$

Note: (*a*) If f has a zero of order k at $z = 0$, the preceding applies to $f(z)/z^k$. (*b*) The factorization (1) is not unique; a unique factorization can be associated with those f whose zeros satisfy the condition required for the convergence of a canonical product.

PROOF Let P be the product in Theorem 15.9, formed with the zeros of f. Then f/P has only removable singularities in the plane, hence is (or can be extended to) an entire function. Also, f/P has no zero, and since the plane is simply connected, $f/P = e^g$ for some entire function g. ////

The proof of Theorem 15.9 is easily adapted to any open set:

15.11 Theorem *Let Ω be an open set in S^2, $\Omega \neq S^2$. Suppose $A \subset \Omega$ and A has no limit point in Ω. With each $\alpha \in A$ associate a positive integer $m(\alpha)$. Then there exists an $f \in H(\Omega)$ all of whose zeros are in A, and such that f has a zero of order $m(\alpha)$ at each $\alpha \in A$.*

PROOF It simplifies the argument, and causes no loss of generality, to assume that $\infty \in \Omega$ but $\infty \notin A$. (If this is not so, a linear fractional transformation will make it so.) Then $S^2 - \Omega$ is a nonempty compact subset of the plane, and ∞ is not a limit point of A.

If A is finite, we can take a rational function for f.

If A is infinite, then A is countable (otherwise there would be a limit point in Ω). Let $\{\alpha_n\}$ be a sequence whose terms are in A and in which each $\alpha \in A$ is listed precisely $m(\alpha)$ times. Associate with each α_n a point $\beta_n \in S^2 - \Omega$ such that $|\beta_n - \alpha_n| \leq |\beta - a_n|$ for all $\beta \in S^2 - \Omega$; this is possible since $S^2 - \Omega$ is compact. Then

$$|\beta_n - \alpha_n| \to 0$$

as $n \to \infty$; otherwise A would have a limit point in Ω. We claim that

$$f(z) = \prod_{n=1}^{\infty} E_n\left(\frac{\alpha_n - \beta_n}{z - \beta_n}\right)$$

has the desired properties.

Put $r_n = 2\,|\alpha_n - \beta_n|$. Let K be a compact subset of Ω. Since $r_n \to 0$, there exists an N such that $|z - \beta_n| > r_n$ for all $z \in K$ and all $n \geq N$. Hence

$$\left|\frac{\alpha_n - \beta_n}{z - \beta_n}\right| \leq \frac{1}{2},$$

which implies, by Lemma 15.8, that

$$\left|1 - E_n\left(\frac{\alpha_n - \beta_n}{z - \beta_n}\right)\right| \leq \left(\frac{1}{2}\right)^{n+1} \qquad (z \in K,\ n \geq N),$$

and this again completes the proof, by Theorem 15.6. ////

As a consequence, we can now obtain a characterization of meromorphic functions (see Definition 10.41):

15.12 Theorem *Every meromorphic function in an open set Ω is a quotient of two functions which are holomorphic in Ω.*

The converse is obvious: If $g \in H(\Omega)$, $h \in H(\Omega)$, and h is not identically 0 in any component of Ω, then g/h is meromorphic in Ω.

PROOF Suppose f is meromorphic in Ω; let A be the set of all poles of f in Ω; and for each $\alpha \in A$, let $m(\alpha)$ be the order of the pole of f at α. By Theorem 15.11 there exists an $h \in H(\Omega)$ such that h has a zero of multiplicity $m(\alpha)$ at each $\alpha \in A$, and h has no other zeros. Put $g = fh$. The singularities of g at the points of A are removable, hence we can extend g so that $g \in H(\Omega)$. Clearly, $f = g/h$ in $\Omega - A$. ////

An Interpolation Problem

The Mittag-Leffler theorem may be combined with the Weierstrass theorem 15.11 to give a solution of the following problem: Can we take an arbitrary set $A \subset \Omega$, without limit point in Ω, and find a function $f \in H(\Omega)$ which has prescribed values at every point of A? The answer is affirmative. In fact, we can do even better, and also prescribe finitely many derivatives at each point of A:

15.13 Theorem *Suppose Ω is an open set in the plane, $A \subset \Omega$, A has no limit point in Ω, and to each $\alpha \in A$ there are associated a nonnegative integer $m(\alpha)$*

and complex numbers $w_{n,\alpha}$, $0 \le n \le m(\alpha)$. *Then there exists an* $f \in H(\Omega)$ *such that*

$$f^{(n)}(\alpha) = n!\, w_{n,\alpha} \qquad (\alpha \in A,\ 0 \le n \le m(\alpha)). \tag{1}$$

PROOF By Theorem 15.11, there exists a $g \in H(\Omega)$ whose only zeros are in A and such that g has a zero of order $m(\alpha) + 1$ at each $\alpha \in A$. We claim we can associate to each $\alpha \in A$ a function P_α of the form

$$P_\alpha(z) = \sum_{j=1}^{1+m(\alpha)} c_{j,\alpha}(z - \alpha)^{-j} \tag{2}$$

such that gP_α has the power series expansion

$$g(z)P_\alpha(z) = w_{0,\alpha} + w_{1,\alpha}(z - \alpha) + \cdots + w_{m(\alpha),\alpha}(z - \alpha)^{m(\alpha)} + \cdots \tag{3}$$

in some disc with center at α.

To simplify the writing, take $\alpha = 0$ and $m(\alpha) = m$, and omit the subscripts α. For z near 0, we have

$$g(z) = b_1 z^{m+1} + b_2 z^{m+2} + \cdots, \tag{4}$$

where $b_1 \neq 0$. If

$$P(z) = c_1 z^{-1} + \cdots + c_{m+1} z^{-m-1}, \tag{5}$$

then

$$g(z)P(z) = (c_{m+1} + c_m z + \cdots + c_1 z^m)\quad (b_1 + b_2 z + b_3 z^2 + \cdots). \tag{6}$$

The b's are given, and we want to choose the c's so that

$$g(z)P(z) = w_0 + w_1 z + \cdots + w_m z^m + \cdots. \tag{7}$$

If we compare the coefficients of $1, z, \ldots, z^m$ in (6) and (7), we can solve the resulting equations successively for $c_{m+1}, c_m, \ldots, c_1$, since $b_1 \neq 0$.

In this way we obtain the desired P_α's. The Mittag-Leffler theorem now gives us a meromorphic h in Ω whose principal parts are these P_α's, and if we put $f = gh$ we obtain a function with the desired properties. ////

The solution of this interpolation problem can be used to determine the structure of all finitely generated ideals in the rings $H(\Omega)$.

15.14 Definition The *ideal* $[g_1, \ldots, g_n]$ *generated by the functions* $g_1, \ldots, g_n \in H(\Omega)$ is the set of all functions of the form $\Sigma f_i g_i$, where $f_i \in H(\Omega)$ for $i = 1, \ldots, n$ A *principal ideal* is one that is generated by a single function. Note that $[1] = H(\Omega)$.

If $f \in H(\Omega)$, $\alpha \in \Omega$, and f is not identically 0 in a neighborhood of α, the multiplicity of the zero of f at α will be denoted by $m(f; \alpha)$. If $f(\alpha) \neq 0$, then $m(f; \alpha) = 0$, as in Theorem 15.6.

15.15 Theorem *Every finitely generated ideal in $H(\Omega)$ is principal.*

More explicitly: If $g_1, \ldots, g_n \in H(\Omega)$, then there exist functions $g, f_i, h_i \in H(\Omega)$ such that

$$g = \sum_{i=1}^{n} f_i g_i \quad \text{and} \quad g_i = h_i g \qquad (1 \leq i \leq n).$$

PROOF We shall assume that Ω is a region. This is done to avoid problems posed by functions that are identically 0 in some components of Ω but not in all. Once the theorem is proved for regions, that case can be applied to each component of an arbitrary open set Ω, and the full theorem can be deduced. We leave the details of this as an exercise.

Let $P(n)$ be the following proposition:

If $g_1, \ldots, g_n \in H(\Omega)$, if no g_i is identically 0, and if no point of Ω is a zero of every g_i, then $[g_1, \ldots, g_n] = [1]$.

$P(1)$ is trivial. Assume that $n > 1$ and that $P(n-1)$ is true. Take $g_1, \ldots, g_n \in H(\Omega)$, without common zero. By the Weierstrass theorem 15.11 there exists $\varphi \in H(\Omega)$ such that

$$m(\varphi; \alpha) = \min \{m(g_i; \alpha): 1 \leq i \leq n-1\} \qquad (\alpha \in \Omega). \tag{1}$$

The functions $f_i = g_i/\varphi$ $(1 \leq i \leq n-1)$ are in $H(\Omega)$ and have no common zero in Ω. Since $P(n-1)$ is true, $[f_1, \ldots, f_{n-1}] = [1]$. Hence

$$[g_1, \ldots, g_{n-1}, g_n] = [\varphi, g_n]. \tag{2}$$

Moreover, our choice of φ shows that $g_n(\alpha) \neq 0$ at every point of the set $A = \{\alpha \in \Omega: \varphi(\alpha) = 0\}$. Hence it follows from Theorem 15.13 that there exists $h \in H(\Omega)$ such that

$$m(1 - hg_n; \alpha) \geq m(\varphi; \alpha) \qquad (\alpha \in \Omega). \tag{3}$$

Such an h is obtained by a suitable choice of the prescribed values of $h^{(k)}(\alpha)$ for $\alpha \in A$ and for $0 \leq k \leq m(\varphi; \alpha)$.

By (3), $(1 - hg_n)/\varphi$ has removable singularities. Thus

$$1 = hg_n + f\varphi \tag{4}$$

for some $f \in H(\Omega)$. By (2) and (4), $1 \in [g_1, \ldots, g_n]$.

We have shown that $P(n-1)$ implies $P(n)$. Hence $P(n)$ is true for all n.

Finally, suppose $G_1, \ldots, G_n \in H(\Omega)$, and no G_i is identically 0. (This involves no loss of generality.) Another application of Theorem 15.11 yields $\varphi \in H(\Omega)$ with $m(\varphi; \alpha) = \min m(G_i; \alpha)$ for all $\alpha \in \Omega$. Put $g_i = G_i/\varphi$. Then $g_i \in H(\Omega)$, and the functions $g_1, \ldots, g_n$ have no common zeros in Ω. By $P(n)$, $[g_1, \ldots, g_n] = [1]$. Hence $[G_1, \ldots, G_n] = [\varphi]$. This completes the proof. ////

Jensen's Formula

15.16 As we see from Theorem 15.11, the location of the zeros of a holomorphic function in a region Ω is subject to no restriction except the obvious one concerning the absence of limit points in Ω. The situation is quite different if we replace $H(\Omega)$ by certain subclasses which are defined by certain growth conditions. In those situations the distribution of the zeros has to satisfy certain quantitative conditions. The basis of most of these theorems is Jensen's formula (Theorem 15.18). We shall apply it to certain classes of entire functions and to certain subclasses of $H(U)$.

The following lemma affords an opportunity to apply Cauchy's theorem to the evaluation of a definite integral.

15.17 Lemma $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi} \log|1 - e^{i\theta}|\, d\theta = 0.$

PROOF Let $\Omega = \{z: \operatorname{Re} z < 1\}$. Since $1 - z \neq 0$ in Ω and Ω is simply connected, there exists an $h \in H(\Omega)$ such that

$$\exp\{h(z)\} = 1 - z$$

in Ω, and this h is uniquely determined if we require that $h(0) = 0$. Since $\operatorname{Re}(1 - z) > 0$ in Ω, we then have

$$\operatorname{Re} h(z) = \log|1 - z|, \qquad |\operatorname{Im} h(z)| < \frac{\pi}{2} \qquad (z \in \Omega). \tag{1}$$

For small $\delta > 0$, let Γ be the path

$$\Gamma(t) = e^{it} \qquad (\delta \leq t \leq 2\pi - \delta), \tag{2}$$

and let γ be the circular arc whose center is at 1 and which passes from $e^{i\delta}$ to $e^{-i\delta}$ within U. Then

$$\frac{1}{2\pi}\int_\delta^{2\pi-\delta} \log|1 - e^{i\theta}|\, d\theta = \operatorname{Re}\left[\frac{1}{2\pi i}\int_\Gamma h(z)\frac{dz}{z}\right] = \operatorname{Re}\left[\frac{1}{2\pi i}\int_\gamma h(z)\frac{dz}{z}\right]. \tag{3}$$

The last equality depended on Cauchy's theorem; note that $h(0) = 0$.

The length of γ is less than $\pi\delta$, so (1) shows that the absolute value of the last integral in (3) is less than $C\delta \log(1/\delta)$, where C is a constant. This gives the result if $\delta \to 0$ in (3). ////

15.18 Theorem *Suppose* $\Omega = D(0; R)$, $f \in H(\Omega)$, $f(0) \neq 0$, $0 < r < R$, *and* $\alpha_1, \ldots, \alpha_N$ *are the zeros of* f *in* $\bar{D}(0; r)$, *listed according to their multiplicities. Then*

$$|f(0)| \prod_{n=1}^{N} \frac{r}{|\alpha_n|} = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f(re^{i\theta})|\, d\theta\right\}. \tag{1}$$

This is known as *Jensen's formula*. The hypothesis $f(0) \neq 0$ causes no harm in applications, for if f has a zero of order k at 0, the formula can be applied to $f(z)/z^k$.

PROOF Order the points α_j so that $\alpha_1, \ldots, \alpha_m$ are in $D(0; r)$ and $|\alpha_{m+1}| = \cdots = |\alpha_N| = r$. (Of course, we may have $m = N$ or $m = 0$.) Put

$$g(z) = f(z) \prod_{n=1}^{m} \frac{r^2 - \bar{\alpha}_n z}{r(\alpha_n - z)} \prod_{n=m+1}^{N} \frac{\alpha_n}{\alpha_n - z}. \tag{2}$$

Then $g \in H(D)$, where $D = D(0; r + \epsilon)$ for some $\epsilon > 0$, g has no zero in D, hence $\log |g|$ is harmonic in D (Theorem 13.12), and so

$$\log |g(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |g(re^{i\theta})| \, d\theta. \tag{3}$$

By (2),

$$|g(0)| = |f(0)| \prod_{n=1}^{m} \frac{r}{|\alpha_n|}. \tag{4}$$

For $1 \leq n \leq m$, the factors in (2) have absolute value 1 if $|z| = r$. If $\alpha_n = re^{i\theta_n}$ for $m < n \leq N$, it follows that

$$\log |g(re^{i\theta})| = \log |f(re^{i\theta})| - \sum_{n=m+1}^{N} \log |1 - e^{i(\theta - \theta_n)}|. \tag{5}$$

Lemma 15.17 shows therefore that the integral in (3) is unchanged if g is replaced by f. Comparison with (4) now gives (1). ////

Jensen's formula gives rise to an inequality which involves the boundary values of bounded holomorphic functions in U (we recall that the class of these functions has been denoted by H^∞):

15.19 Theorem *If $f \in H^\infty$, f not identically 0, define*

$$\mu_r(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})| \, d\theta \qquad (0 < r < 1) \tag{1}$$

and

$$\mu^*(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f^*(e^{i\theta})| \, d\theta \tag{2}$$

where f^* is the radial limit function of f, as in Theorem 11.32. Then

$$\mu_r(f) \leq \mu_s(f) \quad \text{if} \quad 0 < r < s < 1, \tag{3}$$

$$\mu_r(f) \to \log |f(0)| \quad \text{as} \quad r \to 0, \tag{4}$$

and

$$\mu_r(f) \le \mu^*(f) \quad \textit{if} \quad 0 < r < 1. \tag{5}$$

Note the following consequence: One can choose r so that $f(z) \neq 0$ if $|z| = r$; then $\mu_r(f)$ is finite, and so is $\mu^*(f)$, by (5). *Thus* $\log |f^*| \in L^1(T)$, *and* $f^*(e^{i\theta}) \neq 0$ *at almost every point of* T.

PROOF There is an integer $m \ge 0$ such that $f(z) = z^m g(z)$, $g \in H^\infty$, and $g(0) \neq 0$. Apply Jensen's formula 15.18(1) to g in place of f. Its left side obviously cannot decrease if r increases. Thus $\mu_r(g) \le \mu_s(g)$ if $r < s$. Since

$$\mu_r(f) = \mu_r(g) + m \log r,$$

we have proved (3).

Let us now assume, without loss of generality, that $|f| \le 1$. Write $f_r(e^{i\theta})$ in place of $f(re^{i\theta})$. Then $f_r \to f(0)$ as $r \to 0$, and $f_r \to f^*$ a.e. as $r \to 1$. Since $\log (1/|f_r|) \ge 0$, two applications of Fatou's lemma, combined with (3), give (4) and (5). ////

15.20 Zeros of Entire Functions Suppose f is an entire function,

$$M(r) = \sup_\theta |f(re^{i\theta})| \qquad (0 < r < \infty), \tag{1}$$

and $n(r)$ is the number of zeros of f in $\bar{D}(0; r)$. Assume $f(0) = 1$, for simplicity. Jensen's formula gives

$$M(2r) \ge \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log |f(2re^{i\theta})|\, d\theta\right\} = \prod_{n=1}^{n(2r)} \frac{2r}{|\alpha_n|} \ge \prod_{n=1}^{n(r)} \frac{2r}{|\alpha_n|} \ge 2^{n(r)},$$

if $\{\alpha_n\}$ is the sequence of zeros of f, arranged so that $|\alpha_1| \le |\alpha_2| \le \cdots$. Hence

$$n(r) \log 2 \le \log M(2r). \tag{2}$$

Thus the rapidity with which $n(r)$ can increase (i.e., the density of the zeros of f) is controlled by the rate of growth of $M(r)$. Suppose, to look at a more specific situation, that for large r

$$M(r) < \exp\{Ar^k\} \tag{3}$$

where A and k are given positive numbers. Then (2) leads to

$$\limsup_{r\to\infty} \frac{\log n(r)}{\log r} \le k. \tag{4}$$

For example, if k is a positive integer and

$$f(z) = 1 - e^{z^k}, \tag{5}$$

then $n(r)$ is about $\pi^{-1}kr^k$, so that

$$\lim_{r\to\infty} \frac{\log n(r)}{\log r} = k. \tag{6}$$

This shows that the estimate (4) cannot be improved.

Blaschke Products

Jensen's formula makes it possible to determine the precise conditions which the zeros of a nonconstant $f \in H^\infty$ must satisfy.

15.21 Theorem *If $\{\alpha_n\}$ is a sequence in U such that $\alpha_n \neq 0$ and*

$$\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty, \tag{1}$$

if k is a nonnegative integer, and if

$$B(z) = z^k \prod_{n=1}^{\infty} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z} \frac{|\alpha_n|}{\alpha_n} \qquad (z \in U), \tag{2}$$

then $B \in H^\infty$, and B has no zeros except at the points α_n (and at the origin, if $k > 0$).

We call this function B a *Blaschke product.* Note that some of the α_n may be repeated, in which case B has multiple zeros at those points. Note also that each factor in (2) has absolute value 1 on T.

The term "Blaschke product" will also be used if there are only finitely many factors, and even if there are none, in which case $B(z) = 1$.

PROOF The nth term in the series

$$\sum_{n=1}^{\infty} \left| 1 - \frac{\alpha_n - z}{1 - \bar{\alpha}_n z} \cdot \frac{|\alpha_n|}{\alpha_n} \right|$$

is

$$\left| \frac{\alpha_n + |\alpha_n| z}{(1 - \bar{\alpha}_n z)\alpha_n} \right| (1 - |\alpha_n|) \leq \frac{1 + r}{1 - r} (1 - |\alpha_n|)$$

if $|z| \leq r$. Hence Theorem 15.6 shows that $B \in H(U)$ and that B has only the prescribed zeros. Since each factor in (2) has absolute value less than 1 in U, it follows that $|B(z)| < 1$, and the proof is complete. ////

15.22 The preceding theorem shows that

$$\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty \tag{1}$$

is a *sufficient* condition for the existence of an $f \in H^\infty$ which has only the prescribed zeros $\{\alpha_n\}$. This condition also turns out to be necessary: *If $f \in H^\infty$ and f is not identically zero, the zeros of f must satisfy* (1). This is a special case of Theorem 15.23. It is interesting that (1) is a necessary condition in a much larger class of functions, which we now describe.

For any real number t, define $\log^+ t = \log t$ if $t \geq 1$ and $\log^+ t = 0$ if $t < 1$. We let N (for Nevanlinna) be the class of all $f \in H(U)$ for which

$$\sup_{0<r<1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})|\, d\theta < \infty. \tag{2}$$

It is clear that $H^\infty \subset N$. Note that (2) imposes a restriction on the rate of growth of $|f(z)|$ as $|z| \to 1$, whereas the boundedness of the integrals

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})|\, d\theta \tag{3}$$

imposes no such restriction. For instance, (3) is independent of r if $f = e^g$ for any $g \in H(U)$. The point is that (3) can stay small because $\log |f|$ assumes large negative values as well as large positive ones, whereas $\log^+ |f| \geq 0$. The class N will be discussed further in Chap. 17.

15.23 Theorem *Suppose $f \in N$, f is not identically 0 in U, and $\alpha_1, \alpha_2, \alpha_3, \ldots$ are the zeros of f, listed according to their multiplicities. Then*

$$\sum_{n=1}^{\infty} (1 - |\alpha_n|) < \infty. \tag{1}$$

(We tacitly assume that f has infinitely many zeros in U. If there are only finitely many, the above sum has only finitely many terms, and there is nothing to prove. Also, $|\alpha_n| \leq |\alpha_{n+1}|$.)

PROOF If f has a zero of order m at the origin, and $g(z) = z^{-m} f(z)$, then $g \in N$, and g has the same zeros as f, except at the origin. Hence we may assume, without loss of generality, that $f(0) \neq 0$. Let $n(r)$ be the number of zeros of f in $\bar{D}(0; r)$, fix k, and take $r < 1$ so that $n(r) > k$. Then Jensen's formula

$$|f(0)| \prod_{n=1}^{n(r)} \frac{r}{|\alpha_n|} = \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f(re^{i\theta})|\, d\theta\right\} \tag{2}$$

implies that

$$|f(0)| \prod_{n=1}^{k} \frac{r}{|\alpha_n|} \leq \exp\left\{\frac{1}{2\pi} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})|\, d\theta\right\}. \tag{3}$$

Our assumption that $f \in N$ is equivalent to the existence of a constant $C < \infty$ which exceeds the right side of (3) for all r, $0 < r < 1$. It follows that

$$\prod_{n=1}^{k} |\alpha_n| \geq C^{-1} |f(0)| r^k. \tag{4}$$

The inequality persists, for every k, as $r \to 1$. Hence

$$\prod_{n=1}^{\infty} |\alpha_n| \geq C^{-1} |f(0)| > 0. \tag{5}$$

By Theorem 15.5, (5) implies (1). ////

Corollary *If $f \in H^\infty$ (or even if $f \in N$), if $\alpha_1, \alpha_2, \alpha_3, \ldots$ are the zeros of f in U, and if $\Sigma(1 - |\alpha_n|) = \infty$, then $f(z) = 0$ for all $z \in U$.*

For instance, no nonconstant bounded holomorphic function in U can have a zero at each of the points $(n - 1)/n$ $(n = 1, 2, 3, \ldots)$.

We conclude this section with a theorem which describes the behavior of a Blaschke product near the boundary of U. Recall that as a member of H^∞, B has radial limits $B^*(e^{i\theta})$ at almost all points of T.

15.24 Theorem *If B is a Blaschke product, then $|B^*(e^{i\theta})| = 1$ a.e. and*

$$\lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |B(re^{i\theta})| \, d\theta = 0. \tag{1}$$

PROOF The existence of the limit is a consequence of the fact that the integral is a monotonic function of r. Suppose $B(z)$ is as in Theorem 15.21, and put

$$B_N(z) = \prod_{n=N}^{\infty} \frac{\alpha_n - z}{1 - \bar{\alpha}_n z} \cdot \frac{|\alpha_n|}{\alpha_n} \tag{2}$$

Since $\log (|B/B_N|)$ is continuous in an open set containing T, the limit (1) is unchanged if B is replaced by B_N. If we apply Theorem 15.19 to B_N we therefore obtain

$$\log |B_N(0)| \leq \lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |B(re^{i\theta})| \, d\theta \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |B^*(e^{i\theta})| \, d\theta \leq 0. \tag{3}$$

As $N \to \infty$, the first term in (3) tends to 0. This gives (1), and shows that $\int \log |B^*| = 0$. Since $\log |B^*| \leq 0$ a.e., Theorem 1.39(*a*) now implies that $\log |B^*| = 0$ a.e. ////

The Müntz-Szasz Theorem

15.25 A classical theorem of Weierstrass ([26], Theorem 7.26) states that the polynomials are dense in $C(I)$, the space of all continuous complex functions on

the closed interval $I = [0, 1]$, with the supremum norm. In other words, the set of all finite linear combinations of the functions

$$1, t, t^2, t^3, \ldots \tag{1}$$

is dense in $C(I)$. This is sometimes expressed by saying that the functions (1) *span* $C(I)$.

This suggests a question: If $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$, under what conditions is it true that the functions

$$1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, \ldots \tag{2}$$

span $C(I)$?

It turns out that this problem has a very natural connection with the problem of the distribution of the zeros of a bounded holomorphic function in a half plane (or in a disc; the two are conformally equivalent). The surprisingly neat answer is that *the functions* (2) *span* $C(I)$ *if and only if* $\Sigma 1/\lambda_n = \infty$.

Actually, the proof gives an even more precise conclusion:

15.26 Theorem *Suppose* $0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$ *and let* X *be the closure in* $C(I)$ *of the set of all finite linear combinations of the functions*

$$1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, \ldots.$$

(*a*) *If* $\Sigma 1/\lambda_n = \infty$, *then* $X = C(I)$.

(*b*) *If* $\Sigma 1/\lambda_n < \infty$, *and if* $\lambda \notin \{\lambda_n\}$, $\lambda \neq 0$, *then* X *does not contain the function* t^λ.

PROOF It is a consequence of the Hahn-Banach theorem (Theorem 5.19) that $\varphi \in C(I)$ but $\varphi \notin X$ if and only if there is a bounded linear functional on $C(I)$ which does not vanish at φ but which vanishes on all of X. Since every bounded linear functional on $C(I)$ is given by integration with respect to a complex Borel measure on I, (*a*) will be a consequence of the following proposition:

If $\Sigma 1/\lambda_n = \infty$ *and if* μ *is a complex Borel measure on* I *such that*

$$\int_I t^{\lambda_n}\, d\mu(t) = 0 \qquad (n = 1, 2, 3, \ldots), \tag{1}$$

then also

$$\int_I t^k\, d\mu(t) = 0 \qquad (k = 1, 2, 3, \ldots). \tag{2}$$

For if this is proved, the preceding remark shows that X contains all functions t^k; since $1 \in X$, all polynomials are then in X, and the Weierstrass theorem therefore implies that $X = C(I)$.

So assume that (1) holds. Since the integrands in (1) and (2) vanish at 0, we may as well assume that μ is concentrated on (0, 1]. We associate with μ the function

$$f(z) = \int_I t^z \, d\mu(t). \tag{3}$$

For $t > 0$, $t^z = \exp(z \log t)$, by definition. We claim that f is holomorphic in the right half plane. The continuity of f is easily checked, and we can then apply Morera's theorem. Furthermore, if $z = x + iy$, if $x > 0$, and if $0 < t \leq 1$, then $|t^z| = t^x \leq 1$. Thus f is bounded in the right half plane, and (1) says that $f(\lambda_n) = 0$, for $n = 1, 2, 3, \ldots$. Define

$$g(z) = f\left(\frac{1+z}{1-z}\right) \qquad (z \in U). \tag{4}$$

Then $g \in H^\infty$ and $g(\alpha_n) = 0$, where $\alpha_n = (\lambda_n - 1)/(\lambda_n + 1)$. A simple computation shows that $\Sigma(1 - |\alpha_n|) = \infty$ if $\Sigma 1/\lambda_n = \infty$. The Corollary to Theorem 15.23 therefore tells us that $g(z) = 0$ for all $z \in U$. Hence $f = 0$. In particular, $f(k) = 0$ for $k = 1, 2, 3, \ldots$, and this is (2). We have thus proved part (*a*) of the theorem.

To prove (*b*) it will be enough to construct a measure μ on I such that (3) defines a function f which is holomorphic in the half plane $\operatorname{Re} z > -1$ (anything negative would do here), which is 0 at $0, \lambda_1, \lambda_2, \lambda_3, \ldots$ and which has no other zeros in this half plane. For the functional induced by this measure μ will then vanish on X but will not vanish at any function t^λ if $\lambda \neq 0$ and $\lambda \notin \{\lambda_n\}$.

We begin by constructing a function f which has these prescribed zeros, and we shall then show that this f can be represented in the form (3). Define

$$f(z) = \frac{z}{(2+z)^3} \prod_{n=1}^{\infty} \frac{\lambda_n - z}{2 + \lambda_n + z}. \tag{5}$$

Since

$$1 - \frac{\lambda_n - z}{2 + \lambda_n + z} = \frac{2z + 2}{2 + \lambda_n + z},$$

the infinite product in (5) converges uniformly on every compact set which contains none of the points $-\lambda_n - 2$. It follows that f is a meromorphic function in the whole plane, with poles at -2 and $-\lambda_n - 2$, and with zeros at $0, \lambda_1, \lambda_2, \lambda_3, \ldots$. Also, each factor in the infinite product (5) is less than 1 in absolute value if $\operatorname{Re} z > -1$. Thus $|f(z)| \leq 1$ if $\operatorname{Re} z \geq -1$. The factor $(2 + z)^3$ ensures that the restriction of f to the line $\operatorname{Re} z = -1$ is in L^1.

Fix z so that $\operatorname{Re} z > -1$, and consider the Cauchy formula for $f(z)$, where the path of integration consists of the semicircle with center at -1, radius $R > 1 + |z|$, from $-1 - iR$ to $-1 + R$ to $-1 + iR$, followed by the

interval from $-1 + iR$ to $-1 - iR$. The integral over the semicircle tends to 0 as $R \to \infty$, so we are left with

$$f(z) = -\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f(-1+is)}{-1+is-z}\,ds \qquad (\text{Re } z > -1). \tag{6}$$

But

$$\frac{1}{1+z-is} = \int_0^1 t^{z-is}\,dt \qquad (\text{Re } z > -1). \tag{7}$$

Hence (6) can be rewritten in the form

$$f(z) = \int_0^1 t^z \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} f(-1+is)e^{-is\log t}\,ds\right\} dt. \tag{8}$$

The interchange in the order of integration was legitimate: If the integrand in (8) is replaced by its absolute value, a finite integral results.

Put $g(s) = f(-1 + is)$. Then the inner integral in (8) is $\hat{g}(\log t)$, where $\hat{g}$ is the Fourier transform of g. This is a bounded continuous function on $(0, 1]$, and if we set $d\mu(t) = \hat{g}(\log t)\,dt$ we obtain a measure which represents f in the desired form (3).

This completes the proof. ////

15.27 Remark The theorem implies that whenever $\{1, t^{\lambda_1}, t^{\lambda_2}, \ldots\}$ spans $C(I)$, then some infinite subcollection of the t^{λ_i} can be removed without altering the span. In particular, $C(I)$ contains no minimal spanning sets of this type. This is in marked contrast to the behavior of orthonormal sets in a Hilbert space: if any element is removed from an orthonormal set, its span is diminished. Likewise, if $\{1, t^{\lambda_1}, t^{\lambda_2} \ldots\}$ does not span $C(I)$, removal of any of its elements will diminish the span; this follows from Theorem 15.26(*b*).

Exercises

1 Suppose $\{a_n\}$ and $\{b_n\}$ are sequences of complex numbers such that $\Sigma|a_n - b_n| < \infty$. On what sets will the product

$$\prod_{n=1}^{\infty} \frac{z-a_n}{z-b_n}$$

converge uniformly? Where will it define a holomorphic function?

2 Suppose f is entire, λ is a positive number, and the inequality

$$|f(z)| < \exp(|z|^{\lambda})$$

holds for all large enough $|z|$. (Such functions f are said to be of *finite order*. The greatest lower bound of all λ for which the above condition holds is the *order* of f.) If $f(z) = \Sigma a_n z^n$, prove that the inequality

$$|a_n| \le \left(\frac{e\lambda}{n}\right)^{n/\lambda}$$

holds for all large enough n. Consider the functions $\exp(z^k)$, $k = 1, 2, 3, \ldots$, to determine whether the above bound on $|a_n|$ is close to best possible.

3 Find all complex z for which $\exp(\exp(z)) = 1$. Sketch them as points in the plane. Show that there is no entire function of finite order which has a zero at each of these points (except, of course, $f \equiv 0$).

4 Show that the function

$$\pi \cot \pi z = \pi i \frac{e^{\pi i z} + e^{-\pi i z}}{e^{\pi i z} - e^{-\pi i z}}$$

has a simple pole with residue 1 at each integer. The same is true of the function

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} = \lim_{N \to \infty} \sum_{n=-N}^{N} \frac{1}{z - n}.$$

Show that both functions are periodic $[f(z+1) = f(z)]$, that their difference is a bounded entire function, hence a constant, and that this constant is actually 0, since

$$\lim_{y \to \infty} f(iy) = -2i \int_0^{\infty} \frac{dt}{1 + t^2} = -\pi i.$$

This gives the partial fractions decomposition

$$\pi \cot \pi z = \frac{1}{z} + \sum_{1}^{\infty} \frac{2z}{z^2 - n^2}.$$

(Compare with Exercise 12, Chap. 9.) Note that $\pi \cot \pi z$ is $(g'/g)(z)$ if $g(z) = \sin \pi z$. Deduce the product representation

$$\frac{\sin \pi z}{\pi z} = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right).$$

5 Suppose k is a positive integer, $\{z_n\}$ is a sequence of complex numbers such that $\Sigma |z_n|^{-k-1} < \infty$, and

$$f(z) = \prod_{n=1}^{\infty} E_k\left(\frac{z}{z_n}\right).$$

(See Definition 15.7.) What can you say about the rate of growth of

$$M(r) = \max_{\theta} |f(re^{i\theta})| ?$$

6 Suppose f is entire, $f(0) \neq 0$, $|f(z)| < \exp(|z|^p)$ for large $|z|$, and $\{z_n\}$ is the sequence of zeros of f, counted according to their multiplicities. Prove that $\Sigma |z_n|^{-p-\epsilon} < \infty$ for every $\epsilon > 0$. (Compare with Sec. 15.20.)

7 Suppose f is an entire function, $f(\sqrt{n}) = 0$ for $n = 1, 2, 3, \ldots$, and there is a positive constant α such that $|f(z)| < \exp(|z|^{\alpha})$ for all large enough $|z|$. For which α does it follow that $f(z) = 0$ for all z? [Consider $\sin(\pi z^2)$.]

8 Let $\{z_n\}$ be a sequence of distinct complex numbers, $z_n \neq 0$, such that $z_n \to \infty$ as $n \to \infty$, and let $\{m_n\}$ be a sequence of positive integers. Let g be a meromorphic function in the plane, which has a simple pole with residue m_n at each z_n and which has no other poles. If $z \notin \{z_n\}$, let $\gamma(z)$ be any path from 0 to z which passes through none of the points z_n, and define

$$f(z) = \exp\left\{\int_{\gamma(z)} g(\zeta)\, d\zeta\right\}.$$

Prove that $f(z)$ is independent of the choice of $\gamma(z)$ (although the integral itself is not), that f is holomorphic in the complement of $\{z_n\}$, that f has a removable singularity at each of the points z_n, and that the extension of f has a zero of order m_n at z_n.

The existence theorem contained in Theorem 15.9 can thus be deduced from the Mittag-Leffler theorem.

9 Suppose $0 < \alpha < 1$, $0 < \beta < 1$, $f \in H(U)$, $f(U) \subset U$, and $f(0) = \alpha$. How many zeros can f have in the disc $\bar{D}(0; \beta)$? What is the answer if (*a*) $\alpha = \frac{1}{2}$, $\beta = \frac{1}{2}$; (*b*) $\alpha = \frac{1}{4}$, $\beta = \frac{1}{2}$; (*c*) $\alpha = \frac{2}{3}$, $\beta = \frac{1}{3}$; (*d*) $\alpha = 1/1{,}000$, $\beta = 1/10$?

10 For $N = 1, 2, 3, \ldots$, define

$$g_N(z) = \prod_{n=N}^{\infty} \left(1 - \frac{z^2}{n^2}\right)$$

Prove that the ideal generated by $\{g_N\}$ in the ring of entire functions is not a principal ideal.

11 Under what conditions on a sequence of real numbers y_n does there exist a bounded holomorphic function in the open right half plane which is not identically zero but which has a zero at each point $1 + iy_n$? In particular, can this happen if (*a*) $y_n = \log n$, (*b*) $y_n = \sqrt{n}$, (*c*) $y_n = n$, (*d*) $y_n = n^2$?

12 Suppose $0 < |\alpha_n| < 1$, $\Sigma(1 - |\alpha_n|) < \infty$, and B is the Blaschke product with zeros at the points α_n. Let E be the set of all points $1/\bar{\alpha}_n$ and let Ω be the complement of the closure of E. Prove that the product actually converges uniformly on every compact subset of Ω, so that $B \in H(\Omega)$, and that B has a pole at each point of E. (This is of particular interest in those cases in which Ω is connected.)

13 Put $\alpha_n = 1 - n^{-2}$, for $n = 1, 2, 3, \ldots$, let B be the Blaschke product with zeros at these points α, and prove that $\lim_{r \to 1} B(r) = 0$. (It is understood that $0 < r < 1$.)

More precisely, show that the estimate

$$|B(r)| < \prod_{1}^{N-1} \frac{r - \alpha_n}{1 - \alpha_n r} < \prod_{1}^{N-1} \frac{\alpha_N - \alpha_n}{1 - \alpha_n} < 2e^{-N/3}$$

is valid if $\alpha_{N-1} < r < \alpha_N$.

14 Prove that there is a sequence $\{\alpha_n\}$ with $0 < \alpha_n < 1$, which tends to 1 so rapidly that the Blaschke product with zeros at the points α_n satisfies the condition

$$\limsup_{r \to 1} |B(r)| = 1.$$

Hence this B has no radial limit at $z = 1$.

15 Let φ be a linear fractional transformation which maps U onto U. For any $z \in U$ define the φ-orbit of z to be the set $\{\varphi_n(z)\}$, where $\varphi_0(z) = z$, $\varphi_n(z) = \varphi(\varphi_{n-1}(z))$, $n = 1, 2, 3, \ldots$. Ignore the case $\varphi(z) = z$.

(*a*) For which φ is it true that the φ-orbits satisfy the Blaschke condition $\Sigma(1 - |\varphi_n(z)|) < \infty$? [The answer depends in part on the location of the fixed points of φ. There may be one fixed point in U, or one fixed point on T, or two fixed points on T. In the last two cases it is advantageous to transfer the problem to (say) the upper half plane, and to consider transformations on it which either leave only ∞ fixed or leave 0 and ∞ fixed.]

(*b*) For which φ do there exist nonconstant functions $f \in H^\infty$ which are invariant under φ, i.e., which satisfy the relation $f(\varphi(z)) = f(z)$ for all $z \in U$?

16 Suppose $|\alpha_1| \leq |\alpha_2| \leq |\alpha_3| \leq \cdots < 1$, and let $n(r)$ be the number of terms in the sequence $\{\alpha_j\}$ such that $|\alpha_j| \leq r$. Prove that

$$\int_0^1 n(r)\, dr = \sum_{j=1}^{\infty} (1 - |\alpha_j|).$$

17 If $B(z) = \Sigma c_k z^k$ is a Blaschke product with at least one zero off the origin, is it possible to have $c_k \geq 0$ for $k = 0, 1, 2, \ldots$?

18 Suppose B is a Blaschke product all of whose zeros lie on the segment (0, 1) and

$$f(z) = (z - 1)^2 B(z).$$

Prove that the derivative of f is bounded in U.

19 Put $f(z) = \exp[-(1 + z)/(1 - z)]$. Using the notation of Theorem 15.19, show that

$$\lim_{r \to 1} \mu_r(f) < \mu^*(f),$$

although $f \in H^\infty$. Note the contrast with Theorem 15.24.

20 Suppose $\lambda_1 > \lambda_2 > \cdots$, and $\lambda_n \to 0$ in the Müntz-Szasz theorem. What is the concl theorem, under these conditions?

21 Prove an analogue of the Müntz-Szasz theorem, with $L^2(I)$ in place of $C(I)$.

22 Put $f_n(t) = t^n e^{-t}$ $(0 \leq t < \infty, n = 0, 1, 2, \ldots)$ and prove that the set of all finite linear c of the functions f_n is dense in $L^2(0, \infty)$. *Hint*: If $g \in L^2(0, \infty)$ is orthogonal to each f_n and if

$$F(z) = \int_0^\infty e^{-tz} \overline{g(t)}\, dt \qquad (\text{Re } z > 0),$$

then all derivatives of F are 0 at $z = 1$. Consider $F(1 + iy)$.

23 Suppose $\Omega \supset \bar{U}$, $f \in H(\Omega)$, $|f(e^{i\theta})| \geq 3$ for all real θ, $f(0) = 0$, and $\lambda_1, \lambda_2, \ldots, \lambda_N$ are $1 - f$ in U, counted according to their multiplicities. Prove that

$$|\lambda_1 \lambda_2 \cdots \lambda_N| < \tfrac{1}{2}.$$

Suggestion: Look at $B/(1 - f)$, where B is a certain Blaschke product.

CHAPTER

SIXTEEN

ANALYTIC CONTINUATION

In this chapter we shall be concerned with questions which arise because functions which are defined and holomorphic in some region can frequently be extended to holomorphic functions in some larger region. Theorem 10.18 shows that these extensions are uniquely determined by the given functions. The extension process is called *analytic continuation.* It leads in a very natural way to the consideration of functions which are defined on Riemann surfaces rather than in plane regions. This device makes it possible to replace "multiple-valued functions" (such as the square-root function or the logarithm) by functions. A systematic treatment of Riemann surfaces would take us too far afield, however, and we shall restrict the discussion to plane regions.

Regular Points and Singular Points

16.1 Definition Let D be an open circular disc, suppose $f \in H(D)$, and let β be a boundary point of D. We call β a *regular point* of f if there exists a disc D_1 with center at β and a function $g \in H(D_1)$ such that $g(z) = f(z)$ for all $z \in D \cap D_1$. Any boundary point of D which is not a regular point of f is called a *singular point* of f.

It is clear from the definition that the set of all regular points of f is an open (possibly empty) subset of the boundary of D.

In the following theorems we shall take the unit disc U for D, without any loss of generality.

16.2 Theorem *Suppose $f \in H(U)$, and the power series*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \qquad (z \in U) \tag{1}$$

has radius of convergence 1. *Then f has at least one singular point on the unit circle T.*

PROOF Suppose, on the contrary, that every point of T is a regular point of f. The compactness of T implies then that there are open discs $D_1, \ldots, D_n$ and functions $g_j \in H(D_j)$ such that the center of each D_j is on T, such that $T \subset D_1 \cup \cdots \cup D_n$, and such that $g_j(z) = f(z)$ in $D_j \cap U$, for $j = 1, \ldots, n$.

If $D_i \cap D_j \neq \varnothing$ and $V_{ij} = D_i \cap D_j \cap U$, then $V_{ij} \neq \varnothing$ (since the centers of the D_j are on T), and $g_i = f = g_j$ in V_{ij}. Since $D_i \cap D_j$ is connected, it follows from Theorem 10.18 that $g_i = g_j$ in $D_i \cap D_j$. Hence we may define a function h in $\Omega = U \cup D_1 \cup \cdots \cup D_n$ by

$$h(z) = \begin{cases} f(z) & (z \in U), \\ g_i(z) & (z \in D_i), \end{cases} \tag{2}$$

Since $\Omega \supset \bar{U}$ and Ω is open, there exists an $\epsilon > 0$ such that the disc $D(0; 1 + \epsilon) \subset \Omega$. But $h \in H(\Omega)$, $h(z)$ is given by (1) in U, and now Theorem 10.16 implies that the radius of convergence of (1) is at least $1 + \epsilon$, contrary to our assumption. ////

16.3 Definition If $f \in H(U)$ and if every point of T is a singular point of f, then T is said to be the *natural boundary* of f. In this case, f has no holomorphic extension to any region which properly contains U.

16.4 Remark It is very easy to see that there exist $f \in H(U)$ for which T is a natural boundary. In fact, if Ω is *any* region, it is easy to find an $f \in H(\Omega)$ which has no holomorphic extension to any larger region. To see this, let A be any countable set in Ω which has no limit point in Ω but such that every boundary point of Ω is a limit point of A. Apply Theorem 15.11 to get a function $f \in H(\Omega)$ which is 0 at every point of A but is not identically 0. If $g \in H(\Omega_1)$, where Ω_1 is a region which properly contains Ω, and if $g = f$ in Ω, the zeros of g would have a limit point in Ω_1, and we have a contradiction.

A simple explicit example is furnished by

$$f(z) = \sum_{n=0}^{\infty} z^{2^n} = z + z^2 + z^4 + z^8 + \cdots \qquad (z \in U). \tag{1}$$

This f satisfies the functional equation

$$f(z^2) = f(z) - z, \tag{2}$$

from which it follows (we leave the details to the reader) that f is unbounded on every radius of U which ends at $\exp\{2\pi ik/2^n\}$, where k and n are positive

integers. These points form a dense subset of T; and since the set of all singular points of f is closed, f has T as its natural boundary.

That this example is a power series with large gaps (i.e., with many zero coefficients) is no accident. The example is merely a special case of Theorem 16.6, due to Hadamard, which we shall derive from the following theorem of Ostrowski:

16.5 Theorem *Suppose λ, p_k, and q_k are positive integers,*

$$p_1 < p_2 < p_3 < \cdots,$$

and

$$\lambda q_k > (\lambda + 1)p_k \qquad (k = 1, 2, 3, \ldots). \tag{1}$$

Suppose

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \tag{2}$$

has radius of convergence 1, and $a_n = 0$ whenever $p_k < n < q_k$ for some k. If $s_p(z)$ is the pth partial sum of (2), and if β is a regular point of f on T, then the sequence $\{s_{p_k}(z)\}$ converges in some neighborhood of β.

Note that the full sequence $\{s_p(z)\}$ cannot converge at any point outside $\bar{U}$. The gap condition (1) ensures the existence of a subsequence which converges in a neighborhood of β, hence at some points outside $\bar{U}$. This phenomenon is called *overconvergence.*

PROOF If $g(z) = f(\beta z)$, then g also satisfies the gap condition. Hence we may assume, without loss of generality, that $\beta = 1$. Then f has a holomorphic extension to a region Ω which contains $U \cup \{1\}$. Put

$$\varphi(w) = \tfrac{1}{2}(w^\lambda + w^{\lambda+1}) \tag{3}$$

and define $F(w) = f(\varphi(w))$ for all w such that $\varphi(w) \in \Omega$. If $|w| \leq 1$ but $w \neq 1$, then $|\varphi(w)| < 1$, since $|1 + w| < 2$. Also, $\varphi(1) = 1$. It follows that there exists an $\epsilon > 0$ such that $\varphi(D(0; 1 + \epsilon)) \subset \Omega$. Note that the region $\varphi(D(0; 1 + \epsilon))$ contains the point 1. The series

$$F(w) = \sum_{m=0}^{\infty} b_m w^m \tag{4}$$

converges if $|w| < 1 + \epsilon$.

The highest and lowest powers of w in $[\varphi(w)]^n$ have exponents $(\lambda + 1)n$ and λn. Hence the highest exponent in $[\varphi(w)]^{p_k}$ is less than the lowest exponent in $[\varphi(w)]^{q_k}$, by (1). Since

$$F(w) = \sum_{n=0}^{\infty} a_n[\varphi(w)]^n \qquad (|w| < 1), \tag{5}$$

the gap condition satisfied by $\{a_n\}$ now implies that

$$\sum_{n=0}^{p_k} a_n[\varphi(w)]^n = \sum_{m=0}^{(\lambda+1)p_k} b_m w^m \qquad (k = 1, 2, 3, \ldots). \tag{6}$$

The right side of (6) converges, as $k \to \infty$, whenever $|w| < 1 + \epsilon$. Hence $\{s_{p_k}(z)\}$ converges for all $z \in \varphi(D(0; 1 + \epsilon))$. This is the desired conclusion. ////

Note: Actually, $\{s_{p_k}(z)\}$ converges uniformly in some neighborhood of β. We leave it to the reader to verify this by a more careful examination of the preceding proof.

16.6 Theorem *Suppose λ is a positive integer, $\{p_k\}$ is a sequence of positive integers such that*

$$p_{k+1} > \left(1 + \frac{1}{\lambda}\right)p_k \qquad (k = 1, 2, 3, \ldots), \tag{1}$$

and the power series

$$f(z) = \sum_{k=1}^{\infty} c_k z^{p_k} \tag{2}$$

has radius of convergence 1. *Then f has T as its natural boundary.*

PROOF The subsequence $\{s_{p_k}\}$ of Theorem 16.5 is now the same (except for repetitions) as the full sequence of partial sums of (2). The latter cannot converge at any point outside $\bar{U}$; hence Theorem 16.5 implies that no point of T can be a regular point of f. ////

16.7 Example Put $a_n = 1$ if n is a power of 2, put $a_n = 0$ otherwise, put $\eta_n = \exp(-\sqrt{n})$, and define

$$f(z) = \sum_{n=0}^{\infty} a_n \eta_n z^n. \tag{1}$$

Since

$$\limsup_{n \to \infty} |a_n \eta_n|^{1/n} = 1, \tag{2}$$

the radius of convergence of (1) is 1. By Hadamard's theorem, f has T as its natural boundary. Nevertheless, the power series of each derivative of f,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n \eta_n z^{n-k}, \tag{3}$$

converges uniformly on the closed unit disc. Each $f^{(k)}$ is therefore uniformly continuous on $\bar{U}$, and the restriction of f to T is infinitely differentiable, as a function of θ, in spite of the fact that T is the natural boundary of f.

The example demonstrates rather strikingly that the presence of singularities, in the sense of Definition 16.1, does not imply the presence of discontinuities or (stated less precisely) of any lack of smoothness.

This seems to be the natural place to insert a theorem in which continuity does preclude the existence of singularities:

16.8 Theorem *Suppose Ω is a region, L is a straight line or a circular arc, $\Omega - L$ is the union of two regions Ω_1 and Ω_2, f is continuous in Ω, and f is holomorphic in Ω_1 and in Ω_2. Then f is holomorphic in Ω.*

PROOF The use of linear fractional transformations shows that the general case follows if we prove the theorem for straight lines L. By Morera's theorem, it is enough to show that the integral of f over the boundary $\partial\Delta$ is 0 for every triangle Δ in Ω. The Cauchy theorem implies that the integral of f vanishes over every closed path γ in $\Delta \cap \Omega_1$ or in $\Delta \cap \Omega_2$. The continuity of f shows that this is still true if part of γ is in L, and the integral over $\partial\Delta$ is the sum of at most two terms of this sort. ////

Continuation along Curves

16.9 Definitions A *function element* is an ordered pair (f, D), where D is an open circular disc and $f \in H(D)$. Two function elements (f_0, D_0) and (f_1, D_1) are *direct continuations* of each other if two conditions hold: $D_0 \cap D_1 \neq \varnothing$, and $f_0(z) = f_1(z)$ for all $z \in D_0 \cap D_1$. In this case we write

$$(f_0, D_0) \sim (f_1, D_1). \tag{1}$$

A *chain* is a finite sequence $\mathscr{C}$ of discs, say $\mathscr{C} = \{D_0, D_1, \ldots, D_n\}$, such that $D_{i-1} \cap D_i \neq \varnothing$ for $i = 1, \ldots, n$. If (f_0, D_0) is given and if there exist elements (f_i, D_i) such that $(f_{i-1}, D_{i-1}) \sim (f_i, D_i)$ for $i = 1, \ldots, n$, then (f_n, D_n) is said to be the *analytic continuation of* (f_0, D_0) *along* $\mathscr{C}$. Note that f_n is uniquely determined by f_0 and $\mathscr{C}$ (if it exists at all). To see this, suppose (1) holds, and suppose (1) also holds with g_1 in place of f_1. Then $g_1 = f_0 = f_1$ in $D_0 \cap D_1$; and since D_1 is connected, we have $g_1 = f_1$ in D_1. The uniqueness of f_n now follows by induction on the number of terms in $\mathscr{C}$.

If (f_n, D_n) is the continuation of (f_0, D_0) along $\mathscr{C}$, and if $D_n \cap D_0 \neq \varnothing$, it need not be true that $(f_0, D_0) \sim (f_n, D_n)$; in other words, the relation $\sim$ is not transitive. The simplest example of this is furnished by the square-root function: Let D_0, D_1, and D_2 be discs of radius 1, with centers 1, ω, and ω^2, where $\omega^3 = 1$, choose $f_j \in H(D_j)$ so that $f_j^2(z) = z$ and so that $(f_0, D_0) \sim (f_1, D_1), (f_1, D_1) \sim (f_2, D_2)$. In $D_0 \cap D_2$ we have $f_2 = -f_0 \neq f_0$.

A chain $\mathscr{C} = \{D_0, \ldots, D_n\}$ is said to *cover* a curve γ with parameter interval $[0, 1]$ if there are numbers $0 = s_0 < s_1 < \cdots < s_n = 1$ such that $\gamma(0)$ is the center of D_0, $\gamma(1)$ is the center of D_n, and

$$\gamma([s_i, s_{i+1}]) \subset D_i \qquad (i = 0, \ldots, n-1). \tag{2}$$

If (f_0, D_0) can be continued along this $\mathscr{C}$ to (f_n, D_n), we call (f_n, D_n) an *analytic continuation of* (f_0, D_0) *along* γ (uniqueness will be proved in Theorem 16.11); (f_0, D_0) is then said to *admit* an analytic continuation along γ.

Although the relation (1) is not transitive, a restricted form of transitivity does hold. It supplies the key to the proof of Theorem 16.11.

16.10 Proposition *Suppose that* $D_0 \cap D_1 \cap D_2 \neq \varnothing$, $(D_0, f_0) \sim (D_1, f_1)$, *and* $(D_1, f_1) \sim (D_2, f_2)$. *Then* $(D_0, f_0) \sim (D_2, f_2)$.

PROOF By assumption, $f_0 = f_1$ in $D_0 \cap D_1$ and $f_1 = f_2$ in $D_1 \cap D_2$. Hence $f_0 = f_2$ in the nonempty open set $D_0 \cap D_1 \cap D_2$. Since f_0 and f_2 are holomorphic in $D_0 \cap D_2$ and $D_0 \cap D_2$ is connected, it follows that $f_0 = f_2$ in $D_0 \cap D_2$. ////

16.11 Theorem *If* (f, D) *is a function element and if* γ *is a curve which starts at the center of* D, *then* (f, D) *admits at most one analytic continuation along* γ.

Here is a more explicit statement of what the theorem asserts: If γ is covered by chains $\mathscr{C}_1 = \{A_0, A_1, \ldots, A_m\}$ and $\mathscr{C}_2 = \{B_0, B_1, \ldots B_n\}$, where $A_0 = B_0 = D$, if (f, D) can be analytically continued along $\mathscr{C}_1$ to a function element (g_m, A_m), and if (f, D) can be analytically continued along $\mathscr{C}_2$ to (h_n, B_n), then $g_m = h_n$ in $A_m \cap B_n$.

Since A_m and B_n are, by assumption, discs with the same center $\gamma(1)$, it follows that g_m and h_n have the same expansion in powers of $z - \gamma(1)$, and we may as well replace A_m and B_n by whichever is the larger one of the two. With this agreement, the conclusion is that $g_m = h_n$.

PROOF Let $\mathscr{C}_1$ and $\mathscr{C}_2$ be as above. There are numbers

$$0 = s_0 < s_1 < \cdots < s_m = 1 = s_{m+1}$$

and $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_n = 1 = \sigma_{n+1}$ such that

$$\gamma([s_i, s_{i+1}]) \subset A_i, \qquad \gamma([\sigma_j, \sigma_{j+1}]) \subset B_j \qquad (0 \leq i \leq m,\ 0 \leq j \leq n). \tag{1}$$

There are function elements $(g_i, A_i) \sim (g_{i+1}, A_{i+1})$ and $(h_j, B_j) \sim (h_{j+1}, B_{j+1})$, for $0 \le i \le m-1$ and $0 \le j \le n-1$. Here $g_0 = h_0 = f$.

We claim that if $0 \le i \le m$ and $0 \le j \le n$, and if $[s_i, s_{i+1}]$ intersects $[\sigma_j, \sigma_{j+1}]$, then $(g_i, A_i) \sim (h_j, B_j)$.

Assume there are pairs (i, j) for which this is wrong. Among them there is one for which $i + j$ is minimal. It is clear that then $i + j > 0$. Suppose $s_i \ge \sigma_j$. Then $i \ge 1$, and since $[s_i, s_{i+1}]$ intersects $[\sigma_j, \sigma_{j+1}]$, we see that

$$\gamma(s_i) \in A_{i-1} \cap A_i \cap B_j. \tag{2}$$

The minimality of $i + j$ shows that $(g_{i-1}, A_{i-1}) \sim (h_j, B_j)$; and since $(g_{i-1}, A_{i-1}) \sim (g_i, A_i)$, Proposition 16.10 implies that $(g_i, A_i) \sim (h_j, B_j)$. This contradicts our assumption. The possibility $s_i \le \sigma_j$ is ruled out in the same way.

So our claim is established. In particular, it holds for the pair (m, n), and this is what we had to prove. ////

16.12 Definition Suppose α and β are points in a topological space X and φ is a continuous mapping of the unit square $I^2 = I \times I$ (where $I = [0, 1]$) into X such that $\varphi(0, t) = \alpha$ and $\varphi(1, t) = \beta$ for all $t \in I$. The curves γ_t defined by

$$\gamma_t(s) = \varphi(s, t) \qquad (s \in I, t \in I) \tag{1}$$

are then said to form a *one-parameter family* $\{\gamma_t\}$ *of curves from* α *to* β *in* X.

We now come to a very important property of analytic continuation:

16.13 Theorem *Suppose* $\{\gamma_t\}$ $(0 \le t \le 1)$ *is a one-parameter family of curves from* α *to* β *in the plane,* D *is a disc with center at* α, *and the function element* (f, D) *admits analytic continuation along each* γ_t, *to an element* (g_t, D_t). *Then* $g_1 = g_0$.

The last equality is to be interpreted as in Theorem 16.11:

$$(g_1, D_1) \sim (g_0, D_0),$$

and D_0 and D_1 are discs with the same center, namely, β.

PROOF Fix $t \in I$. There is a chain $\mathscr{C} = \{A_0, \ldots, A_n\}$ which covers γ_t, with $A_0 = D$, such that (g_t, D_t) is obtained by continuation of (f, D) along $\mathscr{C}$. There are numbers $0 = s_0 < \cdots < s_n = 1$ such that

$$E_i = \gamma_t([s_i, s_{i+1}]) \subset A_i \qquad (i = 0, 1, \ldots, n-1). \tag{1}$$

There exists an $\epsilon > 0$ which is less than the distance from any of the compact sets E_i to the complement of the corresponding open disc A_i. The uniform continuity of φ on I^2 (see Definition 16.12) shows that there exists a $\delta > 0$ such that

$$|\gamma_t(s) - \gamma_u(s)| < \epsilon \qquad \text{if } s \in I, u \in I, |u - t| < \delta. \tag{2}$$

Suppose u satisfies these conditions. Then (2) shows that $\mathscr{C}$ covers γ_u, and therefore Theorem 16.11 shows that both g_t and g_u are obtained by continuation of (f, D) along this same chain $\mathscr{C}$. Hence $g_t = g_u$.

Thus each $t \in I$ is covered by a segment J_t such that $g_u = g_t$ for all $u \in I \cap J_t$. Since I is compact, I is covered by finitely many J_t; and since I is connected, we see in a finite number of steps that $g_1 = g_0$. ////

Our next item is an intuitively obvious topological fact.

16.14 Theorem *Suppose Γ_0 and Γ_1 are curves in a topological space X, with common initial point α and common end point β. If X is simply connected, then there exists a one-parameter family $\{\gamma_t\}$ $(0 \le t \le 1)$ of curves from α to β in X, such that $\gamma_0 = \Gamma_0$ and $\gamma_1 = \Gamma_1$.*

PROOF Let $[0, \pi]$ be the parameter interval of Γ_0 and Γ_1. Then

$$\Gamma(s) = \begin{cases} \Gamma_0(s) & (0 \le s \le \pi) \\ \Gamma_1(2\pi - s) & (\pi \le s \le 2\pi) \end{cases} \tag{1}$$

defines a *closed* curve in X. Since X is simply connected, Γ is null-homotopic in X. Hence there is a continuous H: $[0, 2\pi] \times [0, 1] \to X$ such that

$$H(s, 0) = \Gamma(s), \qquad H(s, 1) = c \in X, \qquad H(0, t) = H(2\pi, t). \tag{2}$$

If Φ: $\bar{U} \to X$ is defined by

$$\Phi(re^{i\theta}) = H(\theta, 1 - r) \qquad (0 \le r \le 1, 0 \le \theta \le 2\pi),$$

(2) implies that Φ is continuous. Put

$$\gamma_t(\theta) = \Phi[(1 - t)e^{i\theta} + te^{-i\theta}] \qquad (0 \le \theta \le \pi, 0 \le t \le 1).$$

Since $\Phi(e^{i\theta}) = H(\theta, 0) = \Gamma(\theta)$, it follows that

$$\gamma_t(0) = \Phi(1) = \Gamma(0) = \alpha \qquad (0 \le t \le 1),$$

$$\gamma_t(\pi) = \Phi(-1) = \Gamma(\pi) = \beta \qquad (0 \le t \le 1),$$

$$\gamma_0(\theta) = \Phi(e^{i\theta}) = \Gamma(\theta) = \Gamma_0(\theta) \qquad (0 \le \theta \le \pi)$$

and

$$\gamma_1(\theta) = \Phi(e^{-i\theta}) = \Phi(e^{i(2\pi - \theta)}) = \Gamma(2\pi - \theta) = \Gamma_1(\theta) \qquad (0 \le \theta \le \pi).$$

This completes the proof. ////

The Monodromy Theorem

The preceding considerations have essentially proved the following important theorem.

16.15 Theorem *Suppose Ω is a simply connected region, (f, D) is a function element, $D \subset \Omega$, and (f, D) can be analytically continued along every curve in Ω that starts at the center of D. Then there exists $g \in H(\Omega)$ such that $g(z) = f(z)$ for all $z \in D$.*

PROOF Let Γ_0 and Γ_1 be two curves in Ω from the center α of D to some point $\beta \in \Omega$. It follows from Theorems 16.13 and 16.14 that the analytic continuations of (f, D) along Γ_0 and Γ_1 lead to the same element (g_β, D_β), where D_β is a disc with center at β. If D_{β_1} intersects D_β, then $(g_{\beta_1}, D_{\beta_1})$ can be obtained by first continuing (f, D) to β, then along the straight line from β to β_1. This shows that $g_{\beta_1} = g_\beta$ in $D_{\beta_1} \cap D_\beta$.

The definition

$$g(z) = g_\beta(z) \qquad (z \in D_\beta)$$

is therefore consistent and gives the desired holomorphic extension of f. ////

16.16 Remark Let Ω be a plane region, fix $w \notin \Omega$, let D be a disc in Ω. Since D is simply connected, there exists $f \in H(D)$ such that $\exp[f(z)] = z - w$. Note that $f'(z) = (z - w)^{-1}$ in D, and that the latter function is holomorphic in all of Ω. This implies that (f, D) can be analytically continued along every path γ in Ω that starts at the center α of D: If γ goes from α to β, if $D_\beta = D(\beta; r) \subset \Omega$, if

$$\Gamma_z = \gamma \dotplus [\beta, z] \qquad (z \in D_\beta) \tag{1}$$

and if

$$g_\beta(z) = \int_{\Gamma_z} (\zeta - w)^{-1}\, d\zeta + f(\alpha) \qquad (z \in D_\beta), \tag{2}$$

then (g_β, D_β) is the continuation of (f, D) along γ.

Note that $g_\beta'(z) = (z - w)^{-1}$ in D_β.

Assume now that there exists $g \in H(\Omega)$ such that $g(z) = f(z)$ in D. Then $g'(z) = (z - w)^{-1}$ for all $z \in \Omega$. If Γ is a *closed* path in Ω, it follows that

$$\operatorname{Ind}_\Gamma(w) = \frac{1}{2\pi i} \int_\Gamma g'(z)\, dz = 0. \tag{3}$$

We conclude (with the aid of Theorem 13.11) *that the monodromy theorem fails in every plane region that is not simply connected.*

Construction of a Modular Function

16.17 The Modular Group This is the set G of all linear fractional transformations φ of the form

$$\varphi(z) = \frac{az + b}{cz + d} \tag{1}$$

where a, b, c, and d are integers and $ad - bc = 1$.

Since a, b, c, and d are real, each $\varphi \in G$ maps the real axis onto itself (except for ∞). The imaginary part of $\varphi(i)$ is $(c^2 + d^2)^{-1} > 0$. Hence

$$\varphi(\Pi^+) = \Pi^+ \quad (\varphi \in G), \tag{2}$$

where Π^+ is the open upper half plane. If φ is given by (1), then

$$\varphi^{-1}(w) = \frac{dw - b}{-cw + a} \tag{3}$$

so that $\varphi^{-1} \in G$. Also $\varphi \circ \psi \in G$ if $\varphi \in G$ and $\psi \in G$.

Thus G is a group, with composition as group operation. In view of (2) it is customary to regard G as a group of transformations on Π^+.

The transformations $z \to z + 1$ $(a = b = d = 1, \quad c = 0)$ and $z \to -1/z$ $(a = d = 0, b = -1, c = 1)$ belong to G. In fact, they generate G (i.e., there is no proper subgroup of G which contains these two transformations). This can be proved by the same method which will be used in Theorem 16.19(c).

A *modular function* is a holomorphic (or meromorphic) function f on Π^+ which is invariant under G or at least under some nontrivial subgroup Γ of G. This means that $f \circ \varphi = f$ for every $\varphi \in \Gamma$.

16.18 A Subgroup We shall take for Γ the group generated by σ and τ, where

$$\sigma(z) = \frac{z}{2z + 1}, \qquad \tau(z) = z + 2. \tag{1}$$

One of our objectives is the construction of a certain function λ which is invariant under Γ and which leads to a quick proof of the Picard theorem. Actually, it is the mapping properties of λ which are important in this proof, not its invariance, and a quicker construction (using just the Riemann mapping theorem and the reflection principle) can be given. But it is instructive to study the action of Γ on Π^+, in geometric terms, and we shall proceed along this route.

Let Q be the set of all z which satisfy the following four conditions, where $z = x + iy$:

$$y > 0, \qquad -1 \le x < 1, \qquad |2z + 1| \ge 1, \qquad |2z - 1| > 1. \tag{2}$$

Q is bounded by the vertical lines $x = -1$ and $x = 1$ and is bounded below by two semicircles of radius $\frac{1}{2}$, with centers at $-\frac{1}{2}$ and at $\frac{1}{2}$. Q contains those of its boundary points which lie in the left half of Π^+. Q contains no point of the real axis.

We claim that Q is a *fundamental domain of* Γ. This means that statements (*a*) and (*b*) of the following theorem are true.

16.19 Theorem *Let* Γ *and* Q *be as above.*

(*a*) *If* φ_1 *and* $\varphi_2 \in \Gamma$ *and* $\varphi_1 \neq \varphi_2$, *then* $\varphi_1(Q) \cap \varphi_2(Q) = \varnothing$.
(*b*) $\bigcup_{\varphi \in \Gamma} \varphi(Q) = \Pi^+$.
(*c*) Γ *contains all transformations* $\varphi \in G$ *of the form*

$$\varphi(z) = \frac{az + b}{cz + d} \tag{1}$$

for which a and d are odd integers, b and c are even.

PROOF Let Γ_1 be the set of all $\varphi \in G$ described in (*c*). It is easily verified that Γ_1 is a subgroup of G. Since $\sigma \in \Gamma_1$ and $\tau \in \Gamma_1$, it follows that $\Gamma \subset \Gamma_1$. To show that $\Gamma = \Gamma_1$, i.e., to prove (*c*), it is enough to prove that (*a'*) and (*b*) hold, where (*a'*) is the statement obtained from (*a*) by replacing Γ by Γ_1. For if (*a'*) and (*b*) hold, it is clear that Γ cannot be a proper subset of Γ_1.

We shall need the relation

$$\operatorname{Im} \varphi(z) = \frac{\operatorname{Im} z}{|cz + d|^2} \tag{2}$$

which is valid for every $\varphi \in G$ given by (1). The proof of (2) is a matter of straightforward computation, and depends on the relation $ad - bc = 1$.

We now prove (*a'*). Suppose φ_1 and $\varphi_2 \in \Gamma_1$, $\varphi_1 \neq \varphi_2$, and define $\varphi = \varphi_1^{-1} \circ \varphi_2$. If $z \in \varphi_1(Q) \cap \varphi_2(Q)$, then $\varphi_1^{-1}(z) \in Q \cap \varphi(Q)$. It is therefore enough to show that

$$Q \cap \varphi(Q) = \varnothing \tag{3}$$

if $\varphi \in \Gamma_1$ and φ is not the identity transformation.

The proof of (3) splits into three cases.

If $c = 0$ in (1), then $ad = 1$, and since a and d are integers, we have $a = d = \pm 1$. Hence $\varphi(z) = z + 2n$ for some integer $n \neq 0$, and the description of Q makes it evident that (3) holds.

If $c = 2d$, then $c = \pm 2$ and $d = \pm 1$ (since $ad - bc = 1$). Therefore $\varphi(z) = \sigma(z) + 2m$, where m is an integer. Since $\sigma(Q) \subset \bar{D}(\frac{1}{2}; \frac{1}{2})$, (3) holds.

If $c \neq 0$ and $c \neq 2d$, we claim that $|cz + d| > 1$ for all $z \in Q$. Otherwise, the disc $\bar{D}(-d/c; 1/|c|)$ would intersect Q. The description of Q shows that if $\alpha \neq -\frac{1}{2}$ is a real number and if $\bar{D}(\alpha; r)$ intersects Q, then at least one of the points -1, 0, 1 lies in $D(\alpha; r)$. Hence $|cw + d| < 1$, for $w = -1$ or 0 or 1. But for these w, $cw + d$ is an odd integer whose absolute value cannot be less than 1. So $|cz + d| > 1$, and it now follows from (2) that $\operatorname{Im} \varphi(z) < \operatorname{Im} z$ for

every $z \in Q$. If it were true for some $z \in Q$ that $\varphi(z) \in Q$, the same argument would apply to φ^{-1} and would show that

$$\text{Im } z = \text{Im } \varphi^{-1}(\varphi(z)) \leq \text{Im } \varphi(z). \tag{4}$$

This contradiction shows that (3) holds.

Hence (a') is proved.

To prove (b), let Σ be the union of the sets $\varphi(Q)$, for $\varphi \in \Gamma$. It is clear that $\Sigma \subset \Pi^+$. Also, Σ contains the sets $\tau^n(Q)$, for $n = 0, \pm 1, \pm 2, \ldots$, where $\tau^n(z) = z + 2n$. Since σ maps the circle $|2z + 1| = 1$ onto the circle $|2z - 1| = 1$, we see that Σ contains every $z \in \Pi^+$ which satisfies all inequalities

$$|2z - (2m + 1)| \geq 1 \qquad (m = 0, \pm 1, \pm 2, \ldots). \tag{5}$$

Fix $w \in \Pi^+$. Since Im $w > 0$, there are only finitely many pairs of integers c and d such that $|cw + d|$ lies below any given bound, and we can choose $\varphi_0 \in \Gamma$ so that $|cw + d|$ is minimized. By (2), this means that

$$\text{Im } \varphi(w) \leq \text{Im } \varphi_0(w) \qquad (\varphi \in \Gamma). \tag{6}$$

Put $z = \varphi_0(w)$. Then (6) becomes

$$\text{Im } \varphi(z) \leq \text{Im } z \qquad (\varphi \in \Gamma). \tag{7}$$

Apply (7) to $\varphi = \sigma\tau^{-n}$ and to $\varphi = \sigma^{-1}\tau^{-n}$. Since

$$(\sigma\tau^{-n})(z) = \frac{z - 2n}{2z - 4n + 1}, \qquad (\sigma^{-1}\tau^{-n})(z) = \frac{z - 2n}{-2z + 4n + 1}, \tag{8}$$

it follows from (2) and (7) that

$$|2z - 4n + 1| \geq 1, \qquad |2z - 4n - 1| \geq 1 \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{9}$$

Thus z satisfies (5), hence $z \in \Sigma$; and since $w = \varphi_0^{-1}(z)$ and $\varphi_0^{-1} \in \Gamma$, we have $w \in \Sigma$.

This completes the proof. ////

The following theorem summarizes some of the properties of the modular function λ which was mentioned in Sec. 16.18 and which will be used in Theorem 16.22.

16.20 Theorem *If Γ and Q are as described in Sec.* 16.18, *there exists a function $\lambda \in H(\Pi^+)$ such that*

(*a*) $\lambda \circ \varphi = \lambda$ *for every* $\varphi \in \Gamma$.
(*b*) λ *is one-to-one on* Q.
(*c*) *The range* Ω *of* λ [*which is the same as* $\lambda(Q)$, *by* (*a*)], *is the region consisting of all complex numbers different from* 0 *and* 1.
(*d*) λ *has the real axis as its natural boundary.*

PROOF Let Q_0 be the right half of Q. More precisely, Q_0 consists of all $z \in \Pi^+$ such that

$$0 < \operatorname{Re} z < 1, \qquad |2z - 1| > 1. \tag{1}$$

By Theorem 14.19 (and Remarks 14.20) there is a continuous function h on $\bar{Q}_0$ which is one-to-one on $\bar{Q}_0$ and holomorphic in Q_0, such that $h(Q_0) = \Pi^+$, $h(0) = 0$, $h(1) = 1$, and $h(\infty) = \infty$. The reflection principle (Theorem 11.14) shows that the formula

$$h(-x + iy) = \overline{h(x + iy)} \tag{2}$$

extends h to a continuous function on the closure $\bar{Q}$ of Q which is a conformal mapping of the interior of Q onto the complex plane minus the nonnegative real axis. We also see that h is one-to-one on Q, that $h(Q)$ is the region Ω described in (c), that

$$h(-1 + iy) = h(1 + iy) = h(\tau(-1 + iy)) \qquad (0 < y < \infty), \tag{3}$$

and that

$$h(-\tfrac{1}{2} + \tfrac{1}{2}e^{i\theta}) = h(\tfrac{1}{2} + \tfrac{1}{2}e^{i(\pi - \theta)}) = h(\sigma(-\tfrac{1}{2} + \tfrac{1}{2}e^{i\theta})) \qquad (0 < \theta < \pi). \tag{4}$$

Since h is real on the boundary of Q, (3) and (4) follow from (2) and the definitions of σ and τ.

We now define the function λ:

$$\lambda(z) = h(\varphi^{-1}(z)) \qquad (z \in \varphi(Q),\ \varphi \in \Gamma). \tag{5}$$

By Theorem 16.19, each $z \in \Pi^+$ lies in $\varphi(Q)$ for one and only one $\varphi \in \Gamma$. Thus (5) defines $\lambda(z)$ for $z \in \Pi^+$, and we see immediately that λ has properties (a) to (c) and that λ is holomorphic in the interior of each of the sets $\varphi(Q)$.

It follows from (3) and (4) that λ is continuous on

$$Q \cup \tau^{-1}(Q) \cup \sigma^{-1}(Q),$$

hence on an open set V which contains Q. Theorem 16.8 now shows that λ is holomorphic in V. Since Π^+ is covered by the union of the sets $\varphi(V)$, $\varphi \in \Gamma$, and since $\lambda \circ \varphi = \lambda$, we conclude that $\lambda \in H(\Pi^+)$.

Finally, the set of all numbers $\varphi(0) = b/d$ is dense on the real axis. If λ could be analytically continued to a region which properly contains Π^+, the zeros of λ would have a limit point in this region, which is impossible since λ is not constant. ////

The Picard Theorem

16.21 The so-called "little Picard theorem" asserts that every nonconstant entire function attains each value, with one possible exception. This is the theorem which is proved below. There is a stronger version: *Every entire function which is not a polynomial attains each value infinitely many times, again with one possible exception.* That one exception can occur is shown by $f(z) = e^z$, which omits the

value 0. The latter theorem is actually true in a local situation: *If f has an isolated singularity at a point z_0 and if f omits two values in some neighborhood of z_0, then z_0 is a removable singularity or a pole of f.* This so-called "big Picard theorem" is a remarkable strengthening of the theorem of Weierstrass (Theorem 10.21) which merely asserts that the image of every neighborhood of z_0 is dense in the plane if f has an essential singularity at z_0. We shall not prove it here.

16.22 Theorem *If f is an entire function and if there are two distinct complex numbers α and β which are not in the range of f, then f is constant.*

PROOF Without loss of generality we assume that $\alpha = 0$ and $\beta = 1$; if not, replace f by $(f - \alpha)/(\beta - \alpha)$. Then f maps the plane into the region Ω described in Theorem 16.20.

With each disc $D_1 \subset \Omega$ there is associated a region $V_1 \subset \Pi^+$ (in fact, there are infinitely many such V_1, one for each $\varphi \in \Gamma$) such that λ is one-to-one on V_1 and $\lambda(V_1) = D_1$; each such V_1 intersects at most two of the domains $\varphi(Q)$. Corresponding to each choice of V_1 there is a function $\psi_1 \in H(D_1)$ such that $\psi_1(\lambda(z)) = z$ for all $z \in V_1$.

If D_2 is another disc in Ω and if $D_1 \cap D_2 \neq \varnothing$, we can choose a corresponding V_2 so that $V_1 \cap V_2 \neq \varnothing$. The function elements (ψ_1, D_1) and (ψ_2, D_2) will then be direct analytic continuations of each other. Note that $\psi_i(D_i) \subset \Pi^+$.

Since the range of f is in Ω, there is a disc A_0 with center at 0 so that $f(A_0)$ lies in a disc D_0 in Ω. Choose $\psi_0 \in H(D_0)$, as above, put $g(z) = \psi_0(f(z))$ for $z \in A_0$, and let γ be any curve in the plane which starts at 0. The range of $f \circ \gamma$ is a compact subset of Ω. Hence γ can be covered by a chain of discs, $A_0, \ldots, A_n$, so that each $f(A_i)$ lies in a disc D_i in Ω, and we can choose $\psi_i \in H(D_i)$ so that (ψ_i, D_i) is a direct analytic continuation of (ψ_{i-1}, D_{i-1}), for $i = 1, \ldots n$. This gives an analytic continuation of the function element (g, A_0) along the chain $\{A_0, \ldots, A_n\}$; note that $\psi_n \circ f$ has positive imaginary part.

Since (g, A_0) can be analytically continued along every curve in the plane and since the plane is simply connected, the monodromy theorem implies that g extends to an entire function. Also, the range of g is in Π^+, hence $(g - i)/(g + i)$ is bounded, hence constant, by Liouville's theorem. This shows that g is constant, and since ψ_0 was one-to-one on $f(A_0)$ and A_0 was a non-empty open set, we conclude that f is constant. ////

Exercises

1 Suppose $f(z) = \Sigma a_n z^n$, $a_n \geq 0$, and the radius of convergence of the series is 1. Prove that f has a singularity at $z = 1$. *Hint*: Expand f in powers of $z - \frac{1}{2}$. If 1 were a regular point of f, the new series would converge at some $x > 1$. What would this imply about the original series?

2 Suppose (f, D) and (g, D) are function elements, P is a polynomial in two variables, and $P(f, g) = 0$ in D. Suppose f and g can be analytically continued along a curve γ, to f_1 and g_1. Prove that

$P(f_1, g_1) = 0$. Extend this to more than two functions. Is there such a theorem for some class of functions P which is larger than the polynomials?

3 Suppose Ω is a simply connected region, and u is a real harmonic function in Ω. Prove that there exists an $f \in H(\Omega)$ such that $u = \operatorname{Re} f$. Show that this fails in every region which is not simply connected.

4 Suppose X is the closed unit square in the plane, f is a continuous complex function on X, and f has no zero in X. Prove that there is a continuous function g on X such that $f = e^g$. For what class of spaces X (other than the above square) is this also true?

5 Prove that the transformations $z \to z + 1$ and $z \to -1/z$ generate the full modular group G. Let R consist of all $z = x + iy$ such that $|x| < \frac{1}{2}$, $y > 0$, and $|z| > 1$, plus those limit points which have $x \leq 0$. Prove that R is a fundamental domain of G.

6 Prove that G is also generated by the transformations φ and ψ, where

$$\varphi(z) = -\frac{1}{z}, \qquad \psi(z) = \frac{z-1}{z}.$$

Show that φ has period 2, ψ has period 3.

7 Find the relation between composition of linear fractional transformations and matrix multiplication. Try to use this to construct an algebraic proof of Theorem 16.19(c) or of the first part of Exercise 5.

8 Let E be a compact set on the real axis, of positive Lebesgue measure, let Ω be the complement of E, relative to the plane, and define

$$f(z) = \int_E \frac{dt}{t - z} \qquad (z \in \Omega).$$

Answer the following questions:

(*a*) Is f constant?

(*b*) Can f be extended to an entire function?

(*c*) Does $\lim zf(z)$ exist as $z \to \infty$? If so, what is it?

(*d*) Does f have a holomorphic square root in Ω?

(*e*) Is the real part of f bounded in Ω?

(*f*) Is the imaginary part of f bounded in Ω?

[If "yes" in (*e*) or (*f*), give a bound.]

(*g*) What is $\int_\gamma f(z)\,dz$ if γ is a positively oriented circle which has E in its interior?

(*h*) Does there exist a bounded holomorphic function φ in Ω which is not constant?

9 Check your answers in Exercise 8 against the special case

$$E = [-1, 1].$$

10 Call a compact set E in the plane *removable* if there are no nonconstant bounded holomorphic functions in the complement of E.

(*a*) Prove that every countable compact set is removable.

(*b*) If E is a compact subset of the real axis, and $m(E) = 0$, prove that E is removable. *Hint*: E can be surrounded by curves of arbitrarily small total length. Apply Cauchy's formula, as in Exercise 25, Chap 10.

(*c*) Suppose E is removable, Ω is a region, $E \subset \Omega$, $f \in H(\Omega - E)$, and f is bounded. Prove that f can be extended to a holomorphic function in Ω.

(*d*) Formulate and prove an analogue of part (*b*) for sets E which are not necessarily on the real axis.

(*e*) Prove that no compact connected subset of the plane (with more than one point) is removable.

11 For each positive number α, let Γ_α be the path with parameter interval $(-\infty, \infty)$ defined by

$$\Gamma_\alpha(t) = \begin{cases} -t - \pi i & (-\infty < t \leq -\alpha), \\ \alpha + \dfrac{\pi i t}{\alpha} & (-\alpha \leq t \leq \alpha), \\ t + \pi i & (\alpha \leq t < \infty). \end{cases}$$

Let Ω_α be the component of the complement of Γ_α^* which contains the origin, and define

$$f_\alpha(z) = \frac{1}{2\pi i} \int_{\Gamma_\alpha} \frac{\exp(e^w)}{w - z}\, dw \qquad (z \in \Omega_\alpha).$$

Prove that f_β is an analytic continuation of f_α if $\alpha < \beta$. Prove that therefore there is an entire function f whose restriction to Ω_α is f_α. Prove that

$$\lim_{r \to \infty} f(re^{i\theta}) = 0$$

for every $e^{i\theta} \neq 1$. (Here r is positive and θ is real, as usual.) Prove that f is not constant. [*Hint*: Look at $f(r)$.] If

$$g = f \exp(-f),$$

prove that

$$\lim_{r \to \infty} g(re^{i\theta}) = 0$$

for every $e^{i\theta}$.

Show that there exists an entire function h such that

$$\lim_{n \to \infty} h(nz) = \begin{cases} 1 & \text{if } z = 0, \\ 0 & \text{if } z \neq 0. \end{cases}$$

12 Suppose

$$f(z) = \sum_{k=1}^{\infty} \left(\frac{z - z^2}{2} \right)^{3^k} = \sum_{n=1}^{\infty} a_n z^n.$$

Find the regions in which the two series converge. Show that this illustrates Theorem 16.5. Find the singular point of f which is nearest to the origin.

13 Let $\Omega = \{z: \frac{1}{2} < |z| < 2\}$. For $n = 1, 2, 3, \ldots$ let X_n be the set of all $f \in H(\Omega)$ that are nth derivatives of some $g \in H(\Omega)$. [In other words, X_n is the range of the differential operator D^n with domain $H(\Omega)$.]

(*a*) Show that $f \in X_1$ if and only if $\int_\gamma f(z)\, dz = 0$, where γ is the positively oriented unit circle.

(*b*) Show that $f \in X_n$ for every n if and only if f extends to a holomorphic function in $D(0; 2)$.

14 Suppose Ω is a region, $p \in \Omega$, $R < \infty$. Let $\mathscr{F}$ be the class of all $f \in H(\Omega)$ such that $|f(p)| \leq R$ and $f(\Omega)$ contains neither 0 nor 1. Prove that $\mathscr{F}$ is a normal family.

15 Show that Theorem 16.2 leads to a very simple proof of the special case of the monodromy theorem (16.15) in which Ω and D are concentric discs. Combine this special case with the Riemann mapping theorem to prove Theorem 16.15 in the generality in which it is stated.

CHAPTER

SEVENTEEN

H^p-SPACES

This chapter is devoted to the study of certain subspaces of $H(U)$ which are defined by certain growth conditions; in fact, they are all contained in the class N defined in Chap. 15. These so-called H^p-spaces (named for G. H. Hardy) have a large number of interesting properties concerning factorizations, boundary values, and Cauchy-type representations in terms of measures on the boundary of U. We shall merely give some of the highlights, such as the theorem of F. and M. Riesz on measures μ whose Fourier coefficients $\hat{\mu}(n)$ are 0 for all $n < 0$, Beurling's classification of the invariant subspaces of H^2, and M. Riesz's theorem on conjugate functions.

A convenient approach to the subject is via subharmonic functions, and we begin with a brief outline of their properties.

Subharmonic Functions

17.1 Definition A function u defined in an open set Ω in the plane is said to be *subharmonic* if it has the following four properties.

(*a*) $-\infty \leq u(z) < \infty$ for all $z \in \Omega$.

(*b*) u is upper semicontinuous in Ω.

(*c*) Whenever $\bar{D}(a; r) \subset \Omega$, then

$$u(a) \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} u(a + re^{i\theta})\, d\theta.$$

(*d*) None of the integrals in (*c*) is $-\infty$.

Note that the integrals in (*c*) always exist and are not $+\infty$, since (*a*) and (*b*) imply that u is bounded above on every compact $K \subset \Omega$. [*Proof*: If K_n is the set of all $z \in K$ at which $u(z) \geq n$, then $K \supset K_1 \supset K_2 \cdots$, so either $K_n = \varnothing$ for some n, or $\bigcap K_n \neq \varnothing$, in which case $u(z) = \infty$ for some $z \in K$.] Hence (*d*) says that the integrands in (*c*) belong to $L^1(T)$.

Every real harmonic function is obviously subharmonic.

17.2 Theorem *If u is subharmonic in Ω, and if φ is a monotonically increasing convex function on R^1, then $\varphi \circ u$ is subharmonic.*

[To have $\varphi \circ u$ defined at all points of Ω, we put $\varphi(-\infty) = \lim \varphi(x)$ as $x \to -\infty$.]

PROOF First, $\varphi \circ u$ is upper semicontinuous, since φ is increasing and continuous. Next, if $\bar{D}(a; r) \subset \Omega$, we have

$$\varphi(u(a)) \leq \varphi\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} u(a + re^{i\theta})\, d\theta\right) \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(u(a + re^{i\theta}))\, d\theta.$$

The first of these inequalities holds since φ is increasing and u is subharmonic; the second follows from the convexity of φ, by Theorem 3.3. ////

17.3 Theorem *If Ω is a region, $f \in H(\Omega)$, and f is not identically* 0, *then* $\log |f|$ *is subharmonic in Ω, and so are* $\log^+ |f|$ *and* $|f|^p$ $(0 < p < \infty)$.

PROOF It is understood that $\log |f(z)| = -\infty$ if $f(z) = 0$. Then $\log |f|$ is upper semicontinuous in Ω, and Theorem 15.19 implies that $\log |f|$ is subharmonic.

The other assertions follow if we apply Theorem 17.2 to $\log |f|$ in place of u, with

$$\varphi(t) = \max(0, t) \quad \text{and} \quad \varphi(t) = e^{pt}. \qquad ////$$

17.4 Theorem *Suppose u is a continuous subharmonic function in Ω, K is a compact subset of Ω, h is a continuous real function on K which is harmonic in the interior V of K, and $u(z) \leq h(z)$ at all boundary points of K. Then $u(z) \leq h(z)$ for all $z \in K$.*

This theorem accounts for the term "subharmonic." Continuity of u is not necessary here, but we shall not need the general case and leave it as an exercise.

PROOF Put $u_1 = u - h$, and assume, to get a contradiction, that $u_1(z) > 0$ for some $z \in V$. Since u_1 is continuous on K, u_1 attains its maximum m on K; and since $u_1 \leq 0$ on the boundary of K, the set $E = \{z \in K: u_1(z) = m\}$ is a nonempty compact subset of V. Let z_0 be a boundary point of E. Then for

some $r > 0$ we have $\bar{D}(z_0; r) \subset V$, but some subarc of the boundary of $\bar{D}(z_0; r)$ lies in the complement of E. Hence

$$u_1(z_0) = m > \frac{1}{2\pi} \int_{-\pi}^{\pi} u_1(z_0 + re^{i\theta})\, d\theta,$$

and this means that u_1 is not subharmonic in V. But if u is subharmonic, so is $u - h$, by the mean value property of harmonic functions, and we have our contradiction. ////

17.5 Theorem *Suppose u is a continuous subharmonic function in U, and*

$$m(r) = \frac{1}{2\pi} \int_{-\pi}^{\pi} u(re^{i\theta})\, d\theta \qquad (0 \le r < 1). \tag{1}$$

If $r_1 < r_2$, then $m(r_1) \le m(r_2)$.

PROOF Let h be the continuous function on $\bar{D}(0; r_2)$ which coincides with u on the boundary of $\bar{D}(0; r_2)$ and which is harmonic in $D(0; r_2)$. By Theorem 17.4, $u \le h$ in $D(0; r_2)$. Hence

$$m(r_1) \le \frac{1}{2\pi} \int_{-\pi}^{\pi} h(r_1 e^{i\theta})\, d\theta = h(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(r_2 e^{i\theta})\, d\theta = m(r_2). \qquad ////$$

The Spaces H^p and N

17.6 Notation As in Secs 11.15 and 11.19, we define f_r on T by

$$f_r(e^{i\theta}) = f(re^{i\theta}) \qquad (0 \le r < 1) \tag{1}$$

if f is any continuous function with domain U, and we let σ denote Lebesgue measure on T, so normalized that $\sigma(T) = 1$. Accordingly, L^p-norms will refer to $L^p(\sigma)$. In particular,

$$\|f_r\|_p = \left\{ \int_T |f_r|^p\, d\sigma \right\}^{1/p} \qquad (0 < p < \infty), \tag{2}$$

$$\|f_r\|_\infty = \sup_\theta |f(re^{i\theta})|, \tag{3}$$

and we also introduce

$$\|f_r\|_0 = \exp \int_T \log^+ |f_r|\, d\sigma. \tag{4}$$

17.7 Definition If $f \in H(U)$ and $0 \le p \le \infty$, we put

$$\|f\|_p = \sup \{\|f_r\|_p : 0 \le r < 1\}. \tag{1}$$

If $0 < p \leq \infty$, H^p is defined to be the class of all $f \in H(U)$ for which $\|f\|_p < \infty$. (Note that this coincides with our previously introduced terminology in the case $p = \infty$.)

The class N consists of all $f \in H(U)$ for which $\|f\|_0 < \infty$.

It is clear that $H^\infty \subset H^p \subset H^s \subset N$ if $0 < s < p < \infty$.

17.8 Remarks (*a*) When $p < \infty$, Theorems 17.3 and 17.5 show that $\|f_r\|_p$ is a nondecreasing function of r, for every $f \in H(U)$; when $p = \infty$, the same follows from the maximum modulus theorem. Hence

$$\|f\|_p = \lim_{r \to 1} \|f_r\|_p. \tag{1}$$

(*b*) For $1 \leq p \leq \infty$, $\|f\|_p$ satisfies the triangle inequality, so that H^p is a normed linear space. To see this, note that the Minkowski inequality gives

$$\|(f+g)_r\|_p = \|f_r + g_r\|_p \leq \|f_r\|_p + \|g_r\|_p \tag{2}$$

if $0 < r < 1$. As $r \to 1$, we obtain

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p. \tag{3}$$

(*c*) Actually, H^p is a *Banach space*, if $1 \leq p \leq \infty$: To prove completeness, suppose $\{f_n\}$ is a Cauchy sequence in H^p, $|z| \leq r < R < 1$, and apply the Cauchy formula to $f_n - f_m$, integrating around the circle of radius R, center 0. This leads to the inequalities

$$(R-r)|f_n(z) - f_m(z)| \leq \|(f_n - f_m)_R\|_1 \leq \|(f_n - f_m)_R\|_p \leq \|f_n - f_m\|_p$$

from which we conclude that $\{f_n\}$ converges uniformly on compact subsets of U to a function $f \in H(U)$. Given $\epsilon > 0$, there is an m such that $\|f_n - f_m\|_p < \epsilon$ for all $n > m$, and then, for every $r < 1$,

$$\|(f - f_m)_r\|_p = \lim_{n \to \infty} \|(f_n - f_m)_r\|_p \leq \epsilon. \tag{4}$$

This gives $\|f - f_m\|_p \to 0$ as $m \to \infty$.

(*d*) For $p < 1$, H^p is still a vector space, but the triangle inequality is no longer satisfied by $\|f\|_p$.

We saw in Theorem 15.23 that the zeros of any $f \in N$ satisfy the Blaschke condition $\Sigma(1 - |\alpha_n|) < \infty$. Hence the same is true in every H^p. It is interesting that the zeros of any $f \in H^p$ can be divided out without increasing the norm:

17.9 Theorem *Suppose $f \in N$, $f \not\equiv 0$, and B is the Blaschke product formed with the zeros of f. Put $g = f/B$. Then $g \in N$ and $\|g\|_0 = \|f\|_0$.*

Moreover, if $f \in H^p$, then $g \in H^p$ and $\|g\|_p = \|f\|_p$ $(0 < p \leq \infty)$.

PROOF Note first that

$$|g(z)| \geq |f(z)| \qquad (z \in U). \tag{1}$$

In fact, strict inequality holds for every $z \in U$, unless f has no zeros in U, in which case $B = 1$ and $g = f$.

If s and t are nonnegative real numbers, the inequality

$$\log^+ (st) \leq \log^+ s + \log^+ t \tag{2}$$

holds since the left side is 0 if $st < 1$ and is $\log s + \log t$ if $st \geq 1$. Since $|g| = |f|/|B|$, (2) gives

$$\log^+ |g| \leq \log^+ |f| + \log \frac{1}{|B|}. \tag{3}$$

By Theorem 15.24, (3) implies that $\|g\|_0 \leq \|f\|_0$, and since (1) holds, we actually have $\|g\|_0 = \|f\|_0$.

Now suppose $f \in H^p$ for some $p > 0$. Let B_n be the finite Blaschke product formed with the first n zeros of f (we arrange these zeros in some sequence, taking multiplicities into account). Put $g_n = f/B_n$. For each n, $|B_n(re^{i\theta})| \to 1$ uniformly, as $r \to 1$. Hence $\|g_n\|_p = \|f\|_p$. As $n \to \infty$, $|g_n|$ increases to $|g|$, so that

$$\|g_r\|_p = \lim_{n \to \infty} \|(g_n)_r\|_p \qquad (0 < r < 1), \tag{4}$$

by the monotone convergence theorem. The right side of (4) is at most $\|f\|_p$, for all $r < 1$. If we let $r \to 1$, we obtain $\|g\|_p \leq \|f\|_p$. Equality follows now from (1), as before. ////

17.10 Theorem *Suppose* $0 < p < \infty$, $f \in H^p$, $f \not\equiv 0$, *and* B *is the Blaschke product formed with the zeros of* f. *Then there is a zero-free function* $h \in H^2$ *such that*

$$f = B \cdot h^{2/p}. \tag{1}$$

In particular, every $f \in H^1$ *is a product*

$$f = gh \tag{2}$$

in which both factors are in H^2.

PROOF By Theorem 17.9, $f/B \in H^p$; in fact, $\|f/B\|_p = \|f\|_p$. Since f/B has no zero in U and U is simply connected, there exists $\varphi \in H(U)$ so that $\exp(\varphi) = f/B$ (Theorem 13.11). Put $h = \exp(p\varphi/2)$. Then $h \in H(U)$ and $|h|^2 = |f/B|^p$, hence $h \in H^2$, and (1) holds.

In fact, $\|h\|_2^2 = \|f\|_p^p$.

To obtain (2), write (1) in the form $f = (Bh) \cdot h$. ////

We can now easily prove some of the most important properties of the H^p-spaces.

17.11 Theorem *If $0 < p < \infty$ and $f \in H^p$, then*

(*a*) *the nontangential maximal functions $N_\alpha f$ are in $L^p(T)$, for all $\alpha < 1$;*
(*b*) *the nontangential limits $f^*(e^{i\theta})$ exist a.e. on T, and $f^* \in L^p(T)$;*
(*c*) $\lim_{r\to 1} \|f^* - f_r\|_p = 0$, *and*
(*d*) $\|f^*\|_p = \|f\|_p$.

If $f \in H^1$ then f is the Cauchy integral as well as the Poisson integral of f^*.

PROOF We begin by proving (*a*) and (*b*) for the case $p > 1$. Since holomorphic functions are harmonic, Theorem 11.30(*b*) shows that every $f \in H^p$ is then the Poisson integral of a function (call it f^*) in $L^p(T)$. Hence $N_\alpha f \in L^p(T)$, by Theorem 11.25(*b*), and $f^*(e^{i\theta})$ is the nontangential limit of f at almost every $e^{i\theta} \in T$, by Theorem 11.23.

If $0 < p \leq 1$ and $f \in H^p$, use the factorization

$$f = Bh^{2/p} \tag{1}$$

given by Theorem 17.10, where B is a Blaschke product, $h \in H^2$, and h has no zero in U. Since $|f| \leq |h|^{2/p}$ in U, it follows that

$$(N_\alpha f)^p \leq (N_\alpha h)^2, \tag{2}$$

so that $N_\alpha f \in L^p(T)$, because $N_\alpha h \in L^2(T)$.

Similarly, the existence of B^* and h^* a.e. on T implies that the nontangential limits of f (call them f^*) exist a.e. Obviously, $|f^*| \leq N_\alpha f$ wherever f^* exists. Hence $f^* \in L^p(T)$.

This proves (*a*) and (*b*), for $0 < p < \infty$.

Since $f_r \to f^*$ a.e. and $|f_r| < N_\alpha f$, the dominated convergence theorem gives (*c*).

If $p \geq 1$, (*d*) follows from (*c*), by the triangle inequality. If $p < 1$, use Exercise 24, Chap. 3, to deduce (*d*) from (*c*).

Finally, if $f \in H^1$, $r < 1$, and $f_r(z) = f(rz)$, then $f_r \in H(D(0, 1/r))$, and therefore f_r can be represented in U by the Cauchy formula

$$f_r(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f_r(e^{it})}{1 - e^{-it}z}\, dt \tag{3}$$

and by the Poisson formula

$$f_r(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(z, e^{it}) f_r(e^{it})\, dt. \tag{4}$$

For each $z \in U$, $|1 - e^{-it}z|$ and $P(z, e^{it})$ are bounded functions on T. The case $p = 1$ of (c) leads therefore from (3) and (4) to

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{f^*(e^{it})}{1 - e^{-it}z}\, dt \tag{5}$$

and

$$f(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} P(z, e^{it}) f^*(e^{it})\, dt. \tag{6}$$

////

The space H^2 has a particularly simple characterization in terms of power series coefficients:

17.12 Theorem *Suppose* $f \in H(U)$ *and*

$$f(z) = \sum_0^\infty a_n z^n.$$

Then $f \in H^2$ *if and only if* $\sum_0^\infty |a_n|^2 < \infty$.

PROOF By Parseval's theorem, applied to f_r with $r < 1$,

$$\sum_0^\infty |a_n|^2 = \lim_{r\to 1} \sum_0^\infty |a_n|^2 r^{2n} = \lim_{r\to 1} \int_T |f_r|^2\, d\sigma = \|f\|_2^2. \qquad ////$$

The Theorem of F. and M. Riesz

17.13 Theorem *If* μ *is a complex Borel measure on the unit circle* T *and*

$$\int_T e^{-int}\, d\mu(t) = 0 \tag{1}$$

for $n = -1, -2, -3, \ldots$, *then* μ *is absolutely continuous with respect to Lebesgue measure.*

PROOF Put $f = P[d\mu]$. Then f satisfies

$$\|f_r\|_1 \le \|\mu\| \qquad (0 \le r < 1). \tag{2}$$

(See Sec. 11.17.) Since, setting $z = re^{i\theta}$,

$$P(z, e^{it}) = \sum_{-\infty}^{\infty} r^{|n|} e^{in\theta} e^{-int}, \tag{3}$$

as in Sec. 11.5, the assumption (1), which amounts to saying that the Fourier coefficients $\hat{\mu}(n)$ are 0 for all $n < 0$, leads to the power series

$$f(z) = \sum_0^\infty \hat{\mu}(n) z^n \qquad (z \in U). \tag{4}$$

By (4) and (2), $f \in H^1$. Hence $f = P[f^*]$, by Theorem 17.11, where $f^* \in L^1(T)$. The uniqueness of the Poisson integral representation (Theorem 11.30) shows now that $d\mu = f^*\, d\sigma$. ////

The remarkable feature of this theorem is that it derives the absolute continuity of a measure from an apparently unrelated condition, namely, the vanishing of one-half of its Fourier coefficients. In recent years the theorem has been extended to various other situations.

Factorization Theorems

We already know from Theorem 17.9 that every $f \in H^p$ (except $f = 0$) can be factored into a Blaschke product and a function $g \in H^p$ which has no zeros in U. There is also a factorization of g which is of a more subtle nature. It concerns, roughly speaking, the rapidity with which g tends to 0 along certain radii.

17.14 Definition An *inner function* is a function $M \in H^\infty$ for which $|M^*| = 1$ a.e. on T. (As usual, M^* denotes the radial limits of M.)

If φ is a positive measurable function on T such that $\log \varphi \in L^1(T)$, and if

$$Q(z) = c \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \log \varphi(e^{it})\, dt\right\} \tag{1}$$

for $z \in U$, then Q is called an *outer function*. Here c is a constant, $|c| = 1$.

Theorem 15.24 shows that every Blaschke product is an inner function, but there are others. They can be described as follows.

17.15 Theorem *Suppose c is a constant, $|c| = 1$, B is a Blaschke product, μ is a finite positive Borel measure on T which is singular with respect to Lebesgue measure, and*

$$M(z) = cB(z) \exp\left\{-\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t)\right\} \qquad (z \in U). \tag{1}$$

Then M is an inner function, and every inner function is of this form.

PROOF If (1) holds and $g = M/B$, then $\log|g|$ is the Poisson integral of $-d\mu$, hence $\log|g| \leq 0$, so that $g \in H^\infty$, and the same is true of M. Also $D\mu = 0$ a.e., since μ is singular (Theorem 7.13), and therefore the radial limits of

$\log |g|$ are 0 a.e. (Theorem 11.22). Since $|B^*| = 1$ a.e., we see that M is an inner function.

Conversely, let B be the Blaschke product formed with the zeros of a given inner function M and put $g = M/B$. Then $\log |g|$ is harmonic in U. Theorems 15.24 and 17.9 show that $|g| \leq 1$ in U and that $|g^*| = 1$ a.e. on T. Thus $\log |g| \leq 0$. We conclude from Theorem 11.30 that $\log |g|$ is the Poisson integral of $-d\mu$, for some positive measure μ on T. Since $\log |g^*| = 0$ a.e. on T, we have $D\mu = 0$ a.e. on T, so μ is singular. Finally, $\log |g|$ is the real part of

$$h(z) = -\int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z}\, d\mu(t),$$

and this implies that $g = c \exp(h)$ for some constant c with $|c| = 1$. Thus M is of the form (1).

This completes the proof. ////

The simplest example of an inner function which is not a Blaschke product is the following: Take $c = 1$ and $B = 1$, and let μ be the unit mass at $t = 0$. Then

$$M(z) = \exp\left\{\frac{z+1}{z-1}\right\},$$

which tends to 0 very rapidly along the radius which ends at $z = 1$.

17.16 Theorem *Suppose Q is the outer function related to φ as in Definition 17.14. Then*

(*a*) $\log |Q|$ *is the Poisson integral of* $\log \varphi$.
(*b*) $\lim_{r \to 1} |Q(re^{i\theta})| = \varphi(e^{i\theta})$ *a.e. on* T.
(*c*) $Q \in H^p$ *if and only if* $\varphi \in L^p(T)$. *In this case,* $\|Q\|_p = \|\varphi\|_p$.

PROOF (*a*) is clear by inspection and (*a*) implies that the radial limits of $\log |Q|$ are equal to $\log \varphi$ a.e. on T, which proves (*b*). If $Q \in H^p$, Fatou's lemma implies that $\|Q^*\|_p \leq \|Q\|_p$, so $\|\varphi\|_p \leq \|Q\|_p$, by (*b*). Conversely, if $\varphi \in L^p(T)$, then

$$|Q(re^{i\theta})|^p = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) \log \varphi^p(e^{it})\, dt\right\}$$

$$\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\varphi^p(e^{it})\, dt,$$

by the inequality between the geometric and arithmetic means (Theorem 3.3), and if we integrate the last inequality with respect to θ we find that $\|Q\|_p \leq \|\varphi\|_p$ if $p < \infty$. The case $p = \infty$ is trivial. ////

17.17 Theorem *Suppose* $0 < p \leq \infty$, $f \in H^p$, *and* f *is not identically* 0. *Then* $\log |f^*| \in L^1(T)$, *the outer function*

$$Q_f(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{it}+z}{e^{it}-z} \log |f^*(e^{it})|\, dt\right\} \tag{1}$$

is in H^p, *and there is an inner function* M_f *such that*

$$f = M_f Q_f. \tag{2}$$

Furthermore,

$$\log |f(0)| \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} \log |f^*(e^{it})|\, dt. \tag{3}$$

Equality holds in (3) *if and only if* M_f *is constant.*

The functions M_f and Q_f are called the *inner* and *outer factors* of f, respectively; Q_f depends only on the boundary values of $|f|$.

PROOF We assume first that $f \in H^1$. If B is the Blaschke product formed with the zeros of f and if $g = f/B$, Theorem 17.9 shows that $g \in H^1$; and since $|g^*| = |f^*|$ a.e. on T, it suffices to prove the theorem with g in place of f.

So let us assume that f has no zero in U and that $f(0) = 1$. Then $\log |f|$ is harmonic in U, $\log |f(0)| = 0$, and since $\log = \log^+ - \log^-$, the mean value property of harmonic functions implies that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log^- |f(re^{i\theta})|\, d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})\, d\theta \leq \|f\|_0 \leq \|f\|_1 \tag{4}$$

for $0 < r < 1$. It now follows from Fatou's lemma that both $\log^+ |f^*|$ and $\log^- |f^*|$ are in $L^1(T)$, hence so is $\log |f^*|$.

This shows that the definition (1) makes sense. By Theorem 17.16, $Q_f \in H^1$. Also, $|Q_f^*| = |f^*| \neq 0$ a.e., since $\log |f^*| \in L^1(T)$. If we can prove that

$$|f(z)| \leq |Q_f(z)| \qquad (z \in U), \tag{5}$$

then f/Q_f will be an inner function, and we obtain the factorization (2).

Since $\log |Q_f|$ is the Poisson integral of $\log |f^*|$, (5) is equivalent to the inequality

$$\log |f| \leq P[\log |f^*|], \tag{6}$$

which we shall now prove. Our notation is as in Chap. 11: $P[h]$ is the Poisson integral of the function $h \in L^1(T)$.

For $|z| \leq 1$ and $0 < R < 1$, put $f_R(z) = f(Rz)$. Fix $z \in U$. Then

$$\log |f_R(z)| = P[\log^+ |f_R|](z) - P[\log^- |f_R|](z). \tag{7}$$

Since $|\log^+ u - \log^+ v| \leq |u - v|$ for all real numbers u and v, and since $\|f_R - f^*\|_1 \to 0$ as $R \to 1$ (Theorem 17.11), the the first Poisson integral in (7) converges to $P[\log^+ |f^*|]$, as $R \to 1$. Hence Fatou's lemma gives

$$P[\log^- |f^*|] \leq \liminf_{R \to 1} P[\log^- |f_R|] = P[\log^+ |f^*|] - \log |f|, \quad (8)$$

which is the same as (6).

We have now established the factorization (2). If we put $z = 0$ in (5) we obtain (3); equality holds in (3) if and only if $|f(0)| = |Q_f(0)|$, i.e., if and only if $|M_f(0)| = 1$; and since $\|M_f\|_\infty = 1$, this happens only when M_f is a constant.

This completes the proof for the case $p = 1$.

If $1 < p \leq \infty$, then $H^p \subset H^1$, hence all that remains to be proved is that $Q_f \in H^p$. But if $f \in H^p$, then $|f^*| \in L^p(T)$, by Fatou's lemma; hence $Q_f \in H^p$, by Theorem 17.16(*c*).

Theorem 17.10 reduces the case $p < 1$ to the case $p = 2$. ////

The fact that $\log |f^*| \in L^1(T)$ has a consequence which we have already used in the proof but which is important enough to be stated separately:

17.18 Theorem *If $0 < p \leq \infty$, $f \in H^p$, and f is not identically 0, then at almost all points of T we have $f^*(e^{i\theta}) \neq 0$.*

PROOF If $f^* = 0$ then $\log |f^*| = -\infty$, and if this happens on a set of positive measure, then

$$\int_{-\pi}^{\pi} \log |f^*(e^{it})| \, dt = -\infty. \qquad ////$$

Observe that Theorem 17.18 imposes a quantitative restriction on the location of the zeros of the radial limits of an $f \in H^p$. Inside U the zeros are also quantitatively restricted, by the Blaschke condition.

As usual, we can rephrase the above result about zeros as a uniqueness theorem:

If $f \in H^p$, $g \in H^p$, and $f^*(e^{i\theta}) = g^*(e^{i\theta})$ on some subset of T whose Lebesgue measure is positive, then $f(z) = g(z)$ for all $z \in U$.

17.19 Let us take a quick look at the class N, with the purpose of determining how much of Theorems 17.17 and 17.18 is true here. If $f \in N$ and $f \not\equiv 0$, we can divide by a Blaschke product and get a quotient g which has no zero in U and which is in N (Theorem 17.9). Then $\log |g|$ is harmonic, and since

$$|\log |g|| = 2 \log^+ |g| - \log |g| \quad (1)$$

and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|g(re^{i\theta})|\,d\theta = \log|g(0)|, \tag{2}$$

we see that $\log|g|$ satisfies the hypotheses of Theorem 11.30 and is therefore the Poisson integral of a real measure μ. Thus

$$f(z) = cB(z)\exp\left\{\int_T \frac{e^{it}+z}{e^{it}-z}\,d\mu(t)\right\}, \tag{3}$$

where c is a constant, $|c| = 1$, and B is a Blaschke product.

Observe how the assumption that the integrals of $\log^+|g|$ are bounded (which is a quantitative formulation of the statement that $|g|$ does not get too close to ∞) implies the boundedness of the integrals of $\log^-|g|$ (which says that $|g|$ does not get too close to 0 at too many places).

If μ is a negative measure, the exponential factor in (3) is in H^∞. Apply the Jordan decomposition to μ. This shows:

To every $f \in N$ there correspond two functions b_1 and $b_2 \in H^\infty$ such that b_2 has no zero in U and $f = b_1/b_2$.

Since $b_2^* \neq 0$ a.e., it follows that f has finite radial limits a.e. Also, $f^* \neq 0$ a.e.

Is $\log|f^*| \in L^1(T)$? Yes, and the proof is identical to the one given in Theorem 17.17.

However, the inequality (3) of Theorem 17.17 need no longer hold. For example, if

$$f(z) = \exp\left\{\frac{1+z}{1-z}\right\}, \tag{4}$$

then $\|f\|_0 = e$, $|f^*| = 1$ a.e., and

$$\log|f(0)| = 1 > 0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} \log|f^*(e^{it})|\,dt. \tag{5}$$

The Shift Operator

17.20 Invariant Subspaces Consider a bounded linear operator S on a Banach space X; that is to say, S is a bounded linear transformation of X into X. If a closed subspace Y of X has the property that $S(Y) \subset Y$, we call Y an *S-invariant subspace.* Thus the S-invariant subspaces of X are exactly those which are mapped into themselves by S.

The knowledge of the invariant subspaces of an operator S helps us to visualize its action. (This is a very general—and hence rather vague—principle: In studying any transformation of any kind, it helps to know what the transformation leaves fixed.) For instance, if S is a linear operator on an n-dimensional vector space X and if S has n linearly independent characteristic vectors $x_1, \ldots,$

x_n, the one-dimensional spaces spanned by any of these x_i are S-invariant, and we obtain a very simple description of S if we take $\{x_1, \ldots, x_n\}$ as a basis of X.

We shall describe the invariant subspaces of the so-called "shift operator" S on ℓ^2. Here ℓ^2 is the space of all complex sequences

$$x = \{\xi_0, \xi_1, \xi_2, \xi_3, \ldots\} \tag{1}$$

for which

$$\|x\| = \left\{\sum_{n=0}^{\infty} |\xi_n|^2\right\}^{1/2} < \infty, \tag{2}$$

and S takes the element $x \in \ell^2$ given by (1) to

$$Sx = \{0, \xi_0, \xi_1, \xi_2, \ldots\}. \tag{3}$$

It is clear that S is a bounded linear operator on ℓ^2 and that $\|S\| = 1$.

A few S-invariant subspaces are immediately apparent: If Y_k is the set of all $x \in \ell^2$ whose first k coordinates are 0, then Y_k is S-invariant.

To find others we make use of a Hilbert space isomorphism between ℓ^2 and H^2 which converts the shift operator S to a multiplication operator on H^2. The point is that this multiplication operator is easier to analyze (because of the richer structure of H^2 as a space of holomorphic functions) than is the case in the original setting of the sequence space ℓ^2.

We associate with each $x \in \ell^2$, given by (1), the function

$$f(z) = \sum_{n=0}^{\infty} \xi_n z^n \qquad (z \in U). \tag{4}$$

By Theorem 17.12, this defines a linear one-to-one mapping of ℓ^2 onto H^2. If

$$y = \{\eta_n\}, \qquad g(z) = \sum_{n=0}^{\infty} \eta_n z^n \tag{5}$$

and if the inner product in H^2 is defined by

$$(f, g) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f^*(e^{i\theta})\overline{g^*(e^{i\theta})}\, d\theta, \tag{6}$$

the Parseval theorem shows that $(f, g) = (x, y)$. Thus we have a Hilbert space isomorphism of ℓ^2 onto H^2, and the shift operator S has turned into a multiplication operator (which we still denote by S) on H^2:

$$(Sf)(z) = zf(z) \qquad (f \in H^2, z \in U). \tag{7}$$

The previously mentioned invariant subspaces Y_k are now seen to consist of all $f \in H^2$ which have a zero of order at least k at the origin. This gives a clue: For any finite set $\{\alpha_1, \ldots, \alpha_k\} \subset U$, the space Y of all $f \in H^2$ such that $f(\alpha_1) = \cdots = f(\alpha_k) = 0$ is S-invariant. If B is the finite Blaschke product with zeros at $\alpha_1, \ldots, \alpha_k$, then $f \in Y$ if and only if $f/B \in H^2$. Thus $Y = BH^2$.

This suggests that *infinite* Blaschke products may also give rise to S-invariant subspaces and, more generally, that Blaschke products might be replaced by arbitrary inner functions φ. It is not hard to see that each φH^2 is a closed S-invariant subspace of H^2, but that *every* closed S-invariant subspace of H^2 is of this form is a deeper result.

17.21 Beurling's Theorem

(*a*) *For each inner function* φ *the space*

$$\varphi H^2 = \{\varphi f : f \in H^2\} \tag{1}$$

is a closed S-invariant subspace of H^2.

(*b*) *If* φ_1 *and* φ_2 *are inner functions and if* $\varphi_1 H^2 = \varphi_2 H^2$, *then* φ_1/φ_2 *is constant.*

(*c*) *Every closed S-invariant subspace* Y *of* H^2, *other than* $\{0\}$, *contains an inner function* φ *such that* $Y = \varphi H^2$.

PROOF H^2 is a Hilbert space, relative to the norm

$$\|f\|_2 = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |f^*(e^{i\theta})|^2\, d\theta\right\}^{1/2}. \tag{2}$$

If φ is an inner function, then $|\varphi^*| = 1$ a.e. The mapping $f \to \varphi f$ is therefore an isometry of H^2 into H^2; being an isometry, its range φH^2 is a closed subspace of H^2. [*Proof*: If $\varphi f_n \to g$ in H^2, then $\{\varphi f_n\}$ is a Cauchy sequence, hence so is $\{f_n\}$, hence $f_n \to f \in H^2$, so $g = \varphi f \in \varphi H^2$.] The S-invariance of φH^2 is also trivial, since $z \cdot \varphi f = \varphi \cdot zf$. Hence (*a*) holds.

If $\varphi_1 H^2 = \varphi_2 H^2$, then $\varphi_1 = \varphi_2 f$ for some $f \in H^2$, hence $\varphi_1/\varphi_2 \in H^2$. Similarly, $\varphi_2/\varphi_1 \in H^2$. Put $\varphi = \varphi_1/\varphi_2$ and $h = \varphi + (1/\varphi)$. Then $h \in H^2$, and since $|\varphi^*| = 1$ a.e. on T, h^* is real a.e. on T. Since h is the Poisson integral of h^*, it follows that h is real in U, hence h is constant. Then φ must be constant, and (*b*) is proved.

The proof of (*c*) will use a method originated by Helson and Lowdenslager. Suppose Y is a closed S-invariant subspace of H^2 which does not consist of 0 alone. Then there is a smallest integer k such that Y contains a function f of the form

$$f(z) = \sum_{n=k}^{\infty} c_n z^n, \qquad c_k = 1. \tag{3}$$

Then $f \notin zY$, where we write zY for the set of all g of the form $g(z) = zf(z), f \in Y$. It follows that zY is a *proper* closed subspace of Y [closed by the argument used in the proof of (*a*)], so Y contains a nonzero vector which is orthogonal to zY (Theorem 4.11).

So there exists a $\varphi \in Y$ such that $\|\varphi\|_2 = 1$ and $\varphi \perp zY$. Then $\varphi \perp z^n\varphi$, for $n = 1, 2, 3, \ldots$. By the definition of the inner product in H^2 [see 17.20(6)] this means that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\varphi^*(e^{i\theta})|^2 e^{-in\theta}\, d\theta = 0 \qquad (n = 1, 2, 3, \ldots). \tag{4}$$

These equations are preserved if we replace the left sides by their complex conjugates, i.e., if we replace n by $-n$. Thus all Fourier coefficients of the function $|\varphi^*|^2 \in L^1(T)$ are 0, except the one corresponding to $n = 0$, which is 1. Since L^1-functions are determined by their Fourier coefficients (Theorem 5.15), it follows that $|\varphi^*| = 1$ a.e. on T. But $\varphi \in H^2$, so φ is the Poisson integral of φ^*, and hence $|\varphi| \leq 1$. We conclude that φ is an inner function.

Since $\varphi \in Y$ and Y is S-invariant, we have $\varphi z^n \in Y$ for all $n \geq 0$, hence $\varphi P \in Y$ for every polynomial P. The polynomials are dense in H^2 (the partial sums of the power series of any $f \in H^2$ converge to f in the H^2-norm, by Parseval's theorem), and since Y is closed and $|\varphi| \leq 1$, it follows that $\varphi H^2 \subset Y$. We have to prove that this inclusion is *not* proper. Since φH^2 is closed, it is enough to show that the assumptions $h \in Y$ and $h \perp \varphi H^2$ imply $h = 0$.

If $h \perp \varphi H^2$, then $h \perp \varphi z^n$ for $n = 0, 1, 2, \ldots$, or

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h^*(e^{i\theta})\overline{\varphi^*(e^{i\theta})}e^{-in\theta}\, d\theta = 0 \qquad (n = 0, 1, 2, \ldots). \tag{5}$$

If $h \in Y$, then $z^n h \in zY$ if $n = 1, 2, 3, \ldots$, and our choice of φ shows that $z^n h \perp \varphi$, or

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} h^*(e^{i\theta})\overline{\varphi^*(e^{i\theta})}e^{-in\theta}\, d\theta = 0 \qquad (n = -1, -2, -3, \ldots). \tag{6}$$

Thus all Fourier coefficients of $h^*\overline{\varphi^*}$ are 0, hence $h^*\overline{\varphi^*} = 0$ a.e. on T; and since $|\varphi^*| = 1$ a.e., we have $h^* = 0$ a.e. Therefore $h = 0$, and the proof is complete. ////

17.22 Remark If we combine Theorems 17.15 and 17.21, we see that the S-invariant subspaces of H^2 are characterized by the following data: a sequence of complex numbers $\{\alpha_n\}$ (possibly finite, or even empty) such that $|\alpha_n| < 1$ and $\Sigma(1 - |\alpha_n|) < \infty$, and a positive Borel measure μ on T, singular with respect to Lebesgue measure (so $D\mu = 0$ a.e.). It is easy (we leave this as an exercise) to find conditions, in terms of $\{\alpha_n\}$ and μ, which ensure that one S-invariant subspace of H^2 contains another. The partially ordered set of all S-invariant subspaces is thus seen to have an extremely complicated structure, much more complicated than one might have expected from the simple definition of the shift operator on ℓ^2.

We conclude the section with an easy consequence of Theorem 17.21 which depends on the factorization described in Theorem 17.17.

17.23 Theorem *Suppose M_f is the inner factor of a function $f \in H^2$, and Y is the smallest closed S-invariant subspace of H^2 which contains f. Then*

$$Y = M_f H^2. \tag{1}$$

In particular, $Y = H^2$ if and only if f is an outer function.

PROOF Let $f = M_f Q_f$ be the factorization of f into its inner and outer factors. It is clear that $f \in M_f H^2$; and since $M_f H^2$ is closed and S-invariant, we have $Y \subset M_f H^2$.

On the other hand, Theorem 17.21 shows that there is an inner function φ such that $Y = \varphi H^2$. Since $f \in Y$, there exists an $h = M_h Q_h \in H^2$ such that

$$M_f Q_f = \varphi M_h Q_h. \tag{2}$$

Since inner functions have absolute value 1 a.e. on T, (2) implies that $Q_f = Q_h$, hence $M_f = \varphi M_h \in Y$, and therefore Y must contain the smallest S-invariant closed subspace which contains M_f. Thus $M_f H^2 \subset Y$, and the proof is complete. ////

It may be of interest to summarize these results in terms of two questions to which they furnish answers.

If $f \in H^2$, which functions $g \in H^2$ can be approximated in the H^2-norm by functions of the form fP, where P runs through the polynomials? *Answer*: Precisely those g for which $g/M_f \in H^2$.

For which $f \in H^2$ is it true that the set $\{fP\}$ is dense in H^2? *Answer*: Precisely for those f for which

$$\log |f(0)| = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log |f^*(e^{it})| \, dt.$$

Conjugate Functions

17.24 Formulation of the Problem Every real harmonic function u in the unit disc U is the real part of one and only one $f \in H(U)$ such that $f(0) = u(0)$. If $f = u + iv$, the last requirement can also be stated in the form $v(0) = 0$. The function v is called the *harmonic conjugate* of u, or the *conjugate function* of u.

Suppose now that u satisfies

$$\sup_{r<1} \|u_r\|_p < \infty \tag{1}$$

for some p. Does it follow that (1) holds then with v in place of u?

Equivalently, does it follow that $f \in H^p$?

The answer (given by M. Riesz) is affirmative if $1 < p < \infty$. (For $p = 1$ and $p = \infty$ it is negative; see Exercise 24.) The precise statement is given by Theorem 17.26.

Let us recall that every harmonic u that satisfies (1) is the Poisson integral of a function $u^* \in L^p(T)$ (Theorem 11.30) if $1 < p < \infty$. Theorem 11.11 suggests therefore another restatement of the problem:

If $1 < p < \infty$, *and if we associate to each* $h \in L^p(T)$ *the holomorphic function*

$$(\psi h)(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} h(e^{it})\, dt \qquad (z \in U), \tag{2}$$

do all of these functions ψh *lie in* H^p?

Exercise 25 deals with some other aspects of this problem.

17.25 Lemma *If* $1 < p \leq 2$, $\delta = \pi/(1 + p)$, $\alpha = (\cos \delta)^{-1}$, *and* $\beta = \alpha^p(1 + \alpha)$, *then*

$$1 \leq \beta(\cos \varphi)^p - \alpha \cos p\varphi \qquad \left(-\frac{\pi}{2} \leq \varphi \leq \frac{\pi}{2}\right). \tag{1}$$

PROOF If $\delta \leq |\varphi| \leq \pi/2$, then the right side of (1) is not less than

$$-\alpha \cos p\varphi \geq -\alpha \cos p\delta = \alpha \cos \delta = 1,$$

and it exceeds $\beta(\cos \delta)^p - \alpha = 1$ if $|\varphi| \leq \delta$. ////

17.26 Theorem *If* $1 < p < \infty$, *then there is a constant* $A_p < \infty$ *such that the inequality*

$$\|\psi h\|_p \leq A_p \|h\|_p \tag{1}$$

holds for every $h \in L^p(T)$.

More explicitly, the conclusion is that ψh (defined in Sec. 17.24) is in H^p, and that

$$\int_T |(\psi h)_r|^p\, d\sigma \leq A_p^p \int_T |h|^p\, d\sigma \qquad (0 \leq r < 1) \tag{2}$$

where $d\sigma = d\theta/2\pi$ is the normalized Lebesgue measure on T.

Note that h is not required to be a real function in this theorem, which asserts that $\psi: L^p \to H^p$ is a bounded linear operator.

PROOF Assume first that $1 < p \leq 2$, that $h \in L^p(T)$, $h \geq 0$, $h \not\equiv 0$, and let u be the real part of $f = \psi h$. Formula 11.5(2) shows that $u = P[h]$, hence $u > 0$ in U. Since U is simply connected and f has no zero in U, there is a $g \in H(U)$ such that $g = f^p$, $g(0) > 0$. Also, $u = |f| \cos \varphi$, where φ is a real function with domain U that satisfies $|\varphi| < \pi/2$.

If $\alpha = \alpha_p$ and $\beta = \beta_p$ are chosen as in Lemma 17.25, it follows that

$$\int_T |f_r|^p\, d\sigma \leq \beta \int_T (u_r)^p\, d\sigma - \alpha \int_T |f_r|^p \cos(p\varphi_r)\, d\sigma \tag{3}$$

for $0 \leq r < 1$.

Note that $|f|^p \cos p\varphi = \operatorname{Re} g$. The mean value property of harmonic functions shows therefore that the last integral in (3) is equal to $\operatorname{Re} g(0) > 0$. Hence

$$\int_T |f_r|^p \, d\sigma \leq \beta \int_T h^p \, d\sigma \qquad (0 \leq r < 1) \tag{4}$$

because $u = P[h]$ implies $\|u_r\|_p \leq \|h\|_p$. Thus

$$\|\psi h\|_p \leq \beta^{1/p} \|h\|_p \tag{5}$$

if $h \in L^p(T)$, $h \geq 0$.

If h is an arbitrary (complex) function in $L^p(T)$, the preceding result applies to the positive and negative parts of the real and imaginary parts of h.

This proves (2), for $1 < p \leq 2$, with $A_p = 4\beta^{1/p}$.

To complete the proof, consider the case $2 < p < \infty$. Let $w \in L^q(T)$, where q is the exponent conjugate to p. Put $\tilde{w}(e^{i\theta}) = w(e^{-i\theta})$. A simple computation, using Fubini's theorem, shows for any $h \in L^p(T)$ that

$$\int_T (\psi h)_r \tilde{w} \, d\sigma = \int_T (\psi w)_r \tilde{h} \, d\sigma \qquad (0 \leq r < 1). \tag{6}$$

Since $q < 2$, (2) holds with w and q in place of h and p, so that (6) leads to

$$\left| \int_T (\psi h)_r \tilde{w} \, d\sigma \right| \leq A_q \|w\|_q \|h\|_p. \tag{7}$$

Now let w range over the unit ball of $L^q(T)$ and take the supremum on the left side of (7). The result is

$$\left\{ \int_T |(\psi h)_r|^p \, d\sigma \right\}^{1/p} \leq A_q \left\{ \int_T |h|^p \, d\sigma \right\}^{1/p} \qquad (0 \leq r < 1). \tag{8}$$

Hence (2) holds again, with $A_p \leq A_q$. ////

(If we take the smallest admissible values for A_p and A_q, the last calculation can be reversed, and shows that $A_p = A_q$.)

Exercises

1 Prove Theorems 17.4 and 17.5 for upper semicontinuous subharmonic functions.

2 Assume $f \in H(\Omega)$ and prove that $\log(1 + |f|)$ is subharmonic in Ω.

3 Suppose $0 < p \leq \infty$ and $f \in H(U)$. Prove that $f \in H^p$ if and only if there is a harmonic function u in U such that $|f(z)|^p \leq u(z)$ for all $z \in U$. Prove that if there is one such *harmonic majorant* u of $|f|^p$, then there is at least one, say u_f. (Explicitly, $|f|^p \leq u_f$ and u_f is harmonic; and if $|f|^p \leq u$ and u is harmonic, then $u_f \leq u$.) Prove that $\|f\|_p = u_f(0)^{1/p}$. *Hint*: Consider the harmonic functions in $D(0; R)$, $R < 1$, with boundary values $|f|^p$, and let $R \to 1$.

4 Prove likewise that $f \in N$ if and only if $\log^+ |f|$ has a harmonic majorant in U.

5 Suppose $f \in H^p$, $\varphi \in H(U)$, and $\varphi(U) \subset U$. Does it follow that $f \circ \varphi \in H^p$? Answer the same question with N in place of H^p.

6 If $0 < r < s \leq \infty$, show that H^s is a *proper* subclass of H^r.

7 Show that H^∞ is a *proper* subclass of the intersection of all H^p with $p < \infty$.

8 If $f \in H^1$ and $f^* \in L^p(T)$, prove that $f \in H^p$.

9 Suppose $f \in H(U)$ and $f(U)$ is not dense in the plane. Prove that f has finite radial limits at almost all points of T.

10 Fix $\alpha \in U$. Prove that the mapping $f \to f(\alpha)$ is a bounded linear functional on H^2. Since H^2 is a Hilbert space, this functional can be represented as an inner product with some $g \in H^2$. Find this g.

11 Fix $\alpha \in U$. How large can $|f'(\alpha)|$ be if $\|f\|_2 \leq 1$? Find the extremal functions. Do the same for $f^{(n)}(\alpha)$.

12 Suppose $p \geq 1$, $f \in H^p$, and f^* is real a.e. on T. Prove that f is then constant. Show that this result is false for every $p < 1$.

13 Suppose $f \in H(U)$, and suppose there exists an $M < \infty$ such that f maps every circle of radius $r < 1$ and center 0 onto a curve γ_r whose length is at most M. Prove that f has a continuous extension to $\bar{U}$ and that the restriction of f to T is absolutely continuous.

14 Suppose μ is a complex Borel measure on T such that

$$\int_T e^{int}\, d\mu(t) = 0 \qquad (n = 1, 2, 3, \ldots).$$

Prove that then either $\mu = 0$ or the support of μ is all of T.

15 Suppose K is a *proper* compact subset of the unit circle T. Prove that every continuous function on K can be uniformly approximated on K by polynomials. *Hint*: Use Exercise 14.

16 Complete the proof of Theorem 17.17 for the case $0 < p < 1$.

17 Let φ be a nonconstant inner function with no zero in U.

(*a*) Prove that $1/\varphi \notin H^p$ if $p > 0$.

(*b*) Prove that there is at least one $e^{i\theta} \in T$ such that $\lim_{r \to 1} \varphi(re^{i\theta}) = 0$.

Hint: $\log |\varphi|$ is a negative harmonic function.

18 Suppose φ is a nonconstant inner function, $|\alpha| < 1$, and $\alpha \notin \varphi(U)$. Prove that $\lim_{r \to 1} \varphi(re^{i\theta}) = \alpha$ for at least one $e^{i\theta} \in T$.

19 Suppose $f \in H^1$ and $1/f \in H^1$. Prove that f is then an outer function.

20 Suppose $f \in H^1$ and $\operatorname{Re}[f(z)] > 0$ for all $z \in U$. Prove that f is an outer function.

21 Prove that $f \in N$ if and only if $f = g/h$, where g and $h \in H^\infty$ and h has no zero in U.

22 Prove the following converse of Theorem 15.24:

If $f \in H(U)$ and if

$$\lim_{r \to 1} \int_{-\pi}^{\pi} |\log |f(re^{i\theta})||\, d\theta = 0, \qquad (*)$$

then f is a Blaschke product. *Hint*: (*) implies

$$\lim_{r \to 1} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})|\, d\theta = 0.$$

Since $\log^+ |f| \geq 0$, it follows from Theorems 17.3 and 17.5 that $\log^+ |f| = 0$, so $|f| \leq 1$. Now $f = Bg$, g has no zeros, $|g| \leq 1$, and (*) holds with $1/g$ in place of f. By the first argument, $|1/g| \leq 1$. Hence $|g| = 1$.

23 Find the conditions mentioned in Sec. 17.22.

24 The conformal mapping of U onto a vertical strip shows that M. Riesz's theorem on conjugate functions cannot be extended to $p = \infty$. Deduce that it cannot be extended to $p = 1$ either.

25 Suppose $1 < p < \infty$, and associate with each $f \in L^p(T)$ its Fourier coefficients

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{it}) e^{-int}\, dt \qquad (n = 0, \pm 1, \pm 2, \ldots).$$

Deduce the following statements from Theorem 17.26:

(*a*) To each $f \in L^p(T)$ there corresponds a function $g \in L^p(T)$ such that $\hat{g}(n) = \hat{f}(n)$ for $n \geq 0$ but $\hat{g}(n) = 0$ for all $n < 0$. In fact, there is a constant C, depending only on p, such that

$$\|g\|_p \leq C \|f\|_p.$$

The mapping $f \to g$ is thus a bounded linear projection of $L^p(T)$ into $L^p(T)$. The Fourier series of g is obtained from that of f by deleting the terms with $n < 0$.

(*b*) Show that the same is true if we delete the terms with $n < k$, where k is any given integer.

(*c*) Deduce from (*b*) that the partial sums s_n of the Fourier series of any $f \in L^p(T)$ form a bounded sequence in $L^p(T)$. Conclude further that we actually have

$$\lim_{n \to \infty} \|f - s_n\|_p = 0.$$

(*d*) If $f \in L^p(T)$ and if

$$F(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n,$$

then $F \in H^p$, and every $F \in H^p$ is so obtained. Thus the projection mentioned in (*a*) may be regarded as a mapping of $L^p(T)$ onto H^p.

26 Show that there is a much simpler proof of Theorem 17.26 if $p = 2$, and find the best value of A_2.

27 Suppose $f(z) = \sum_0^\infty a_n z^n$ in U and $\sum |a_n| < \infty$. Prove that

$$\int_0^1 |f'(re^{i\theta})|\, dr < \infty$$

for all θ.

28 Prove that the following statements are correct if $\{n_k\}$ is a sequence of positive integers which tends to ∞ sufficiently rapidly. If

$$f(z) = \sum_{k=1}^{\infty} \frac{z^{n_k}}{k}$$

then $|f'(z)| > n_k/(10k)$ for all z such that

$$1 - \frac{1}{n_k} < |z| < 1 - \frac{1}{2n_k}.$$

Hence

$$\int_0^1 |f'(re^{i\theta})|\, dr = \infty$$

for every θ, although

$$\lim_{R \to 1} \int_0^R f'(re^{i\theta})\, dr$$

exists (and is finite) for almost all θ. Interpret this geometrically, in terms of the lengths of the images under f of the radii in U.

29 Use Theorem 17.11 to obtain the following characterization of the boundary values of H^p-functions, for $1 \leq p \leq \infty$:

A function $g \in L^p(T)$ is f^* (a.e.) for some $f \in H^p$ if and only if

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{it}) e^{-int}\, dt = 0$$

for all negative integers n.

CHAPTER
EIGHTEEN

ELEMENTARY THEORY OF BANACH ALGEBRAS

Introduction

18.1 Definitions A *complex algebra* is a vector space A over the complex field in which an associative and distributive multiplication is defined, i.e.,

$$x(yz) = (xy)z, \qquad (x + y)z = xz + yz, \qquad x(y + z) = xy + xz \tag{1}$$

for x, y, and $z \in A$, and which is related to scalar multiplication so that

$$\alpha(xy) = x(\alpha y) = (\alpha x)y \tag{2}$$

for x and $y \in A$, α a scalar.

If there is a norm defined in A which makes A into a normed linear space and which satisfies the multiplicative inequality

$$\|xy\| \leq \|x\|\|y\| \qquad (x \text{ and } y \in A), \tag{3}$$

then A is a *normed complex algebra.* If, in addition, A is a complete metric space relative to this norm, i.e., if A is a Banach space, then we call A a *Banach algebra.*

The inequality (3) makes multiplication a continuous operation. This means that if $x_n \to x$ and $y_n \to y$, then $x_n y_n \to xy$, which follows from (3) and the identity

$$x_n y_n - xy = (x_n - x)y_n + x(y_n - y). \tag{4}$$

Note that we have not required that A be commutative, i.e., that $xy = yx$ for all x and $y \in A$, and we shall not do so except when explicitly stated.

However, we *shall* assume that A has a *unit*. This is an element e such that

$$xe = ex = x \qquad (x \in A). \tag{5}$$

It is easily seen that there is at most one such e ($e' = e'e = e$) and that $\|e\| \geq 1$, by (3). We shall make the additional assumption that

$$\|e\| = 1. \tag{6}$$

An element $x \in A$ will be called *invertible* if x has an *inverse* in A, i.e., if there exists an element $x^{-1} \in A$ such that

$$x^{-1}x = xx^{-1} = e. \tag{7}$$

Again, it is easily seen that no $x \in A$ has more than one inverse.

If x and y are invertible in A, so are x^{-1} and xy, since $(xy)^{-1} = y^{-1}x^{-1}$. The invertible elements therefore form a group with respect to multiplication.

The *spectrum* of an element $x \in A$ is the set of all complex numbers λ such that $x - \lambda e$ is *not* invertible. We shall denote the spectrum of x by $\sigma(x)$.

18.2 The theory of Banach algebras contains a great deal of interplay between algebraic properties on the one hand and topological ones on the other. We already saw an example of this in Theorem 9.21, and shall see others. There are also close relations between Banach algebras and holomorphic functions: The easiest proof of the fundamental fact that $\sigma(x)$ is never empty depends on Liouville's theorem concerning entire functions, and the spectral radius formula follows naturally from theorems about power series. This is one reason for restricting our attention to *complex* Banach algebras. The theory of real Banach algebras (we omit the definition, which should be obvious) is not so satisfactory.

The Invertible Elements

In this section, A will be a complex Banach algebra with unit e, and G will be the set of all invertible elements of A.

18.3 Theorem *If $x \in A$ and $\|x\| < 1$, then $e + x \in G$,*

$$(e + x)^{-1} = \sum_{n=0}^{\infty} (-1)^n x^n, \tag{1}$$

and

$$\|(e + x)^{-1} - e + x\| \leq \frac{\|x\|^2}{1 - \|x\|}. \tag{2}$$

PROOF The multiplicative inequality 18.1(3) shows that $\|x^n\| \leq \|x\|^n$. If

$$s_N = e - x + x^2 - \cdots + (-1)^N x^N, \tag{3}$$

it follows that $\{s_N\}$ is a Cauchy sequence in A, hence the series in (1) converges (with respect to the norm of A) to an element $y \in A$. Since multiplication is continuous and

$$(e + x)s_N = e + (-1)^N x^{N+1} = s_N(e + x), \tag{4}$$

we see that $(e + x)y = e = y(e + x)$. This gives (1), and (2) follows from

$$\left\| \sum_{n=2}^{\infty} (-1)^n x^n \right\| \leq \sum_{n=2}^{\infty} \|x^n\| \leq \sum_{n=2}^{\infty} \|x\|^n = \frac{\|x\|^2}{1 - \|x\|}. \tag{5}$$

////

18.4 Theorem *Suppose* $x \in G$, $\|x^{-1}\| = 1/\alpha$, $h \in A$, *and* $\|h\| = \beta < \alpha$. *Then* $x + h \in G$, *and*

$$\|(x + h)^{-1} - x^{-1} + x^{-1}hx^{-1}\| \leq \frac{\beta^2}{\alpha^2(\alpha - \beta)}. \tag{1}$$

PROOF $\|x^{-1}h\| \leq \beta/\alpha < 1$, hence $e + x^{-1}h \in G$, by Theorem 18.3; and since $x + h = x(e + x^{-1}h)$, we have $x + h \in G$ and

$$(x + h)^{-1} = (e + x^{-1}h)^{-1}x^{-1}. \tag{2}$$

Thus

$$(x + h)^{-1} - x^{-1} + x^{-1}hx^{-1} = [(e + x^{-1}h)^{-1} - e + x^{-1}h]x^{-1}, \tag{3}$$

and the inequality (1) follows from Theorem 18.3, with $x^{-1}h$ in place of x. ////

Corollary 1 *G is an open set, and the mapping* $x \to x^{-1}$ *is a homeomorphism of G onto G.*

For if $x \in G$ and $\|h\| \to 0$, (1) implies that $\|(x + h)^{-1} - x^{-1}\| \to 0$. Thus $x \to x^{-1}$ is continuous; it clearly maps G onto G, and since it is its own inverse, it is a homeomorphism.

Corollary 2 *The mapping* $x \to x^{-1}$ *is differentiable. Its differential at any* $x \in G$ *is the linear operator which takes* $h \in A$ *to* $-x^{-1}hx^{-1}$.

This can also be read off from (1). Note that the notion of the differential of a transformation makes sense in any normed linear space, not just in R^k, as in Definition 7.22. If A is commutative, the above differential takes h to $-x^{-2}h$, which agrees with the fact that the derivative of the holomorphic function z^{-1} is $-z^{-2}$.

Corollary 3 *For every* $x \in A$, $\sigma(x)$ *is compact, and* $|\lambda| \leq \|x\|$ *if* $\lambda \in \sigma(x)$.

For if $|\lambda| > \|x\|$, then $e - \lambda^{-1}x \in G$, by Theorem 18.3, and the same is true of $x - \lambda e = -\lambda(e - \lambda^{-1}x)$; hence $\lambda \notin \sigma(x)$. To prove that $\sigma(x)$ is closed, observe (*a*) $\lambda \in \sigma(x)$ if and only if $x - \lambda e \notin G$; (*b*) the complement of G is a closed subset of A, by Corollary 1; and (*c*) the mapping $\lambda \to x - \lambda e$ is a continuous mapping of the complex plane into A.

18.5 Theorem *Let Φ be a bounded linear functional on A, fix $x \in A$, and define*

$$f(\lambda) = \Phi[(x - \lambda e)^{-1}] \qquad (\lambda \notin \sigma(x)). \tag{1}$$

Then f is holomorphic in the complement of $\sigma(x)$, and $f(\lambda) \to 0$ as $\lambda \to \infty$.

PROOF Fix $\lambda \notin \sigma(x)$ and apply Theorem 18.4 with $x - \lambda e$ in place of x and with $(\lambda - \mu)e$ in place of h. We see that there is a constant C, depending on x and λ, such that

$$\|(x - \mu e)^{-1} - (x - \lambda e)^{-1} + (\lambda - \mu)(x - \lambda e)^{-2}\| \leq C|\mu - \lambda|^2 \tag{2}$$

for all μ which are close enough to λ. Thus

$$\frac{(x - \mu e)^{-1} - (x - \lambda e)^{-1}}{\mu - \lambda} \to (x - \lambda e)^{-2} \tag{3}$$

as $\mu \to \lambda$, and if we apply Φ to both sides of (3), the continuity and linearity of Φ show that

$$\frac{f(\mu) - f(\lambda)}{\mu - \lambda} \to \Phi[(x - \lambda e)^{-2}]. \tag{4}$$

So f is differentiable and hence holomorphic outside $\sigma(x)$. Finally, as $\lambda \to \infty$ we have

$$\lambda f(\lambda) = \Phi[\lambda(x - \lambda e)^{-1}] = \Phi\left[\left(\frac{x}{\lambda} - e\right)^{-1}\right] \to \Phi(-e), \tag{5}$$

by the continuity of the inversion mapping in G. ////

18.6 Theorem *For every $x \in A$, $\sigma(x)$ is compact and not empty.*

PROOF We already know that $\sigma(x)$ is compact. Fix $x \in A$, and fix $\lambda_0 \notin \sigma(x)$. Then $(x - \lambda_0 e)^{-1} \neq 0$, and the Hahn-Banach theorem implies the existence of a bounded linear functional Φ on A such that $f(\lambda_0) \neq 0$, where f is defined as in Theorem 18.5. If $\sigma(x)$ were empty, Theorem 18.5 would imply that f is an entire function which tends to 0 at ∞, hence $f(\lambda) = 0$ for every λ, by Liouville's theorem, and this contradicts $f(\lambda_0) \neq 0$. So $\sigma(x)$ is not empty. ////

18.7 Theorem (Gelfand-Mazur) *If A is a complex Banach algebra with unit in which each nonzero element is invertible, then A is (isometrically isomorphic to) the complex field.*

An algebra in which each nonzero element is invertible is called a *division algebra*. Note that the commutativity of A is not part of the hypothesis; it is part of the conclusion.

PROOF If $x \in A$ and $\lambda_1 \neq \lambda_2$, at least one of the elements $x - \lambda_1 e$ and $x - \lambda_2 e$ must be invertible, since they cannot both be 0. It now follows from Theorem 18.6 that $\sigma(x)$ consists of exactly one point, say $\lambda(x)$, for each $x \in A$. Since $x - \lambda(x)e$ is not invertible, it must be 0, hence $x = \lambda(x)e$. The mapping $x \to \lambda(x)$ is therefore an isomorphism of A onto the complex field, which is also an isometry, since $|\lambda(x)| = \|\lambda(x)e\| = \|x\|$ for all $x \in A$. ///

18.8 Definition For any $x \in A$, the *spectral radius* $\rho(x)$ of x is the radius of the smallest closed disc with center at the origin which contains $\sigma(x)$ (sometimes this is also called the *spectral norm* of x; see Exercise 14):

$$\rho(x) = \sup \{|\lambda| : \lambda \in \sigma(x)\}.$$

18.9 Theorem (Spectral Radius Formula) *For every $x \in A$,*

$$\lim_{n \to \infty} \|x^n\|^{1/n} = \rho(x). \tag{1}$$

(This existence of the limit is part of the conclusion.)

PROOF Fix $x \in A$, let n be a positive integer, λ a complex number, and assume $\lambda^n \notin \sigma(x^n)$. We have

$$(x^n - \lambda^n e) = (x - \lambda e)(x^{n-1} + \lambda x^{n-2} + \cdots + \lambda^{n-1} e). \tag{2}$$

Multiply both sides of (2) by $(x^n - \lambda^n e)^{-1}$. This shows that $x - \lambda e$ is invertible, hence $\lambda \notin \sigma(x)$.

So if $\lambda \in \sigma(x)$, then $\lambda^n \in \sigma(x^n)$ for $n = 1, 2, 3, \ldots$. Corollary 3 to Theorem 18.4 shows that $|\lambda^n| \leq \|x^n\|$, and therefore $|\lambda| \leq \|x^n\|^{1/n}$. This gives

$$\rho(x) \leq \liminf_{n \to \infty} \|x^n\|^{1/n}. \tag{3}$$

Now if $|\lambda| > \|x\|$, it is easy to verify that

$$(\lambda e - x) \sum_{n=0}^{\infty} \lambda^{-n-1} x^n = e. \tag{4}$$

The above series is therefore $-(x - \lambda e)^{-1}$. Let Φ be a bounded linear functional on A and define f as in Theorem 18.5. By (4), the expansion

$$f(\lambda) = -\sum_{n=0}^{\infty} \Phi(x^n) \lambda^{-n-1} \tag{5}$$

is valid for all λ such that $|\lambda| > \|x\|$. By Theorem 18.5, f is holomorphic outside $\sigma(x)$, hence in the set $\{\lambda: |\lambda| > \rho(x)\}$. It follows that the power series (5) converges if $|\lambda| > \rho(x)$. In particular,

$$\sup_n |\Phi(\lambda^{-n}x^n)| < \infty \qquad (|\lambda| > \rho(x)) \tag{6}$$

for every bounded linear functional Φ on A.

It is a consequence of the Hahn-Banach theorem (Sec. 5.21) that the norm of any element of A is the same as its norm as a linear functional on the dual space of A. Since (6) holds for every Φ, we can now apply the Banach-Steinhaus theorem and conclude that to each λ with $|\lambda| > \rho(x)$ there corresponds a real number $C(\lambda)$ such that

$$\|\lambda^{-n}x^n\| \leq C(\lambda) \qquad (n = 1, 2, 3, \ldots). \tag{7}$$

Multiply (7) by $|\lambda|^n$ and take nth roots. This gives

$$\|x^n\|^{1/n} \leq |\lambda|\,[C(\lambda)]^{1/n} \qquad (n = 1, 2, 3, \ldots) \tag{8}$$

if $|\lambda| > \rho(x)$, and hence

$$\limsup_{n\to\infty} \|x^n\|^{1/n} \leq \rho(x). \tag{9}$$

The theorem follows from (3) and (9). ////

18.10 Remarks

(*a*) Whether an element of A is or is not invertible in A is a purely algebraic property. Thus the spectrum of x, and likewise the spectral radius $\rho(x)$, are defined in terms of the algebraic structure of A, regardless of any metric (or topological) considerations. The limit in the statement of Theorem 18.9, on the other hand, depends on metric properties of A. This is one of the remarkable features of the theorem: It asserts the equality of two quantities which arise in entirely different ways.

(*b*) Our algebra may be a subalgebra of a larger Banach algebra B (an example follows), and then it may very well happen that some $x \in A$ is not invertible in A but is invertible in B. The spectrum of x therefore depends on the algebra; using the obvious notation, we have $\sigma_A(x) \supset \sigma_B(x)$, and the inclusion may be proper. The *spectral radius* of x, however, is unaffected by this, since Theorem 18.9 shows that it can be expressed in terms of metric properties of powers of x, and these are independent of anything that happens outside A.

18.11 Example Let $C(T)$ be the algebra of all continuous complex functions on the unit circle T (with pointwise addition and multiplication and the supremum norm), and let A be the set of all $f \in C(T)$ which can be extended to a continuous function F on the closure of the unit disc U, such that F is holomorphic in U. It is easily seen that A is a subalgebra of $C(T)$. If $f_n \in A$

and $\{f_n\}$ converges uniformly on T, the maximum modulus theorem forces the associated sequence $\{F_n\}$ to converge uniformly on the closure of U. This shows that A is a *closed* subalgebra of $C(T)$, and so A is itself a Banach algebra.

Define the function f_0 by $f_0(e^{i\theta}) = e^{i\theta}$. Then $F_0(z) = z$. The spectrum of f_0 as an element of A consists of the closed unit disc; with respect to $C(T)$, the spectrum of f_0 consists only of the unit circle. In accordance with Theorem 18.9, the two spectral radii coincide.

Ideals and Homomorphisms

From now on we shall deal only with *commutative* algebras.

18.12 Definition A subset I of a commutative complex algebra A is said to be an *ideal* if (*a*) I is a subspace of A (in the vector space sense) and (*b*) $xy \in I$ whenever $x \in A$ and $y \in I$. If $I \neq A$, I is a *proper* ideal. *Maximal ideals* are proper ideals which are not contained in any larger proper ideals. Note that no proper ideal contains an invertible element.

If B is another complex algebra, a mapping φ of A into B is called a *homomorphism* if φ is a linear mapping which also preserves multiplication: $\varphi(x)\varphi(y) = \varphi(xy)$ for all x and $y \in A$. The *kernel* (or null space) of φ is the set of all $x \in A$ such that $\varphi(x) = 0$. It is trivial to verify that the kernel of a homomorphism is an ideal. For the converse, see Sec. 18.14.

18.13 Theorem *If A is a commutative complex algebra with unit, every proper ideal of A is contained in a maximal ideal. If, in addition, A is a Banach algebra, every maximal ideal of A is closed.*

PROOF The first part is an almost immediate consequence of the Hausdorff maximality principle (and holds in any commutative ring with unit). Let I be a proper ideal of A. Partially order the collection $\mathscr{P}$ of all proper ideals of A which contain I (by set inclusion), and let M be the union of the ideals in some maximal linearly ordered subcollection $\mathscr{Q}$ of $\mathscr{P}$. Then M is an ideal (being the union of a *linearly* ordered collection of ideals), $I \subset M$, and $M \neq A$, since no member of $\mathscr{P}$ contains the unit of A. The maximality of $\mathscr{Q}$ implies that M is a maximal ideal of A.

If A is a Banach algebra, the closure $\overline{M}$ of M is also an ideal (we leave the details of the proof of this statement to the reader). Since M contains no invertible element of A and since the set of all invertible elements is open, we have $\overline{M} \neq A$, and the maximality of M therefore shows that $\overline{M} = M$. ////

18.14 Quotient Spaces and Quotient Algebras Suppose J is a subspace of a vector space A, and associate with each $x \in A$ the coset

$$\varphi(x) = x + J = \{x + y: y \in J\}. \tag{1}$$

If $x_1 - x_2 \in J$, then $\varphi(x_1) = \varphi(x_2)$. If $x_1 - x_2 \notin J$, $\varphi(x_1) \cap \varphi(x_2) = \varnothing$. The set of all cosets of J is denoted by A/J; it is a vector space if we define

$$\varphi(x) + \varphi(y) = \varphi(x + y), \qquad \lambda\varphi(x) = \varphi(\lambda x) \tag{2}$$

for x and $y \in A$ and scalars λ. Since J is a vector space, the operations (2) are well-defined; this means that if $\varphi(x) = \varphi(x')$ and $\varphi(y) = \varphi(y')$, then

$$\varphi(x) + \varphi(y) = \varphi(x') + \varphi(y'), \qquad \lambda\varphi(x) = \lambda\varphi(x'). \tag{3}$$

Also, φ is clearly a linear mapping of A onto A/J; the zero element of A/J is $\varphi(0) = J$.

Suppose next that A is not merely a vector space but a commutative algebra and that J is a proper ideal of A. If $x' - x \in J$ and $y' - y \in J$, the identity

$$x'y' - xy = (x' - x)y' + x(y' - y) \tag{4}$$

shows that $x'y' - xy \in J$. Therefore multiplication can be defined in A/J in a consistent manner:

$$\varphi(x)\varphi(y) = \varphi(xy) \qquad (x \text{ and } y \in A). \tag{5}$$

It is then easily verified that A/J is an algebra, and φ is a homomorphism of A onto A/J whose kernel is J.

If A has a unit element e, then $\varphi(e)$ is the unit of A/J, and *A/J is a field if and only if J is a maximal ideal.*

To see this, suppose $x \in A$ and $x \notin J$, and put

$$I = \{ax + y : a \in A, y \in J\}. \tag{6}$$

Then I is an ideal in A which contains J properly, since $x \in I$. If J is maximal, $I = A$, hence $ax + y = e$ for some $a \in A$ and $y \in J$, hence $\varphi(a)\varphi(x) = \varphi(e)$; and this says that every nonzero element of A/J is invertible, so that A/J is a field. If J is not maximal, we can choose x as above so that $I \neq A$, hence $e \notin I$, and then $\varphi(x)$ is not invertible in A/J.

18.15 Quotient Norms Suppose A is a normed linear space, J is a *closed* subspace of A, and $\varphi(x) = x + J$, as above. Define

$$\|\varphi(x)\| = \inf \{\|x + y\| : y \in J\}. \tag{1}$$

Note that $\|\varphi(x)\|$ is the greatest lower bound of the norms of those elements which lie in the coset $\varphi(x)$; this is the same as the distance from x to J. We call the norm defined in A/J by (1) the *quotient norm* of A/J. It has the following properties:

(*a*) *A/J is a normed linear space.*
(*b*) *If A is a Banach space, so is A/J.*
(*c*) *If A is a commutative Banach algebra and J is a proper closed ideal, then A/J is a commutative Banach algebra.*

These are easily verified:

If $x \in J$, $\|\varphi(x)\| = 0$. If $x \notin J$, the fact that J is closed implies that $\|\varphi(x)\| > 0$. It is clear that $\|\lambda\varphi(x)\| = |\lambda|\,\|\varphi(x)\|$. If x_1 and $x_2 \in A$ and $\epsilon > 0$, there exist y_1 and $y_2 \in J$ so that

$$\|x_i + y_i\| < \|\varphi(x_i)\| + \epsilon \qquad (i = 1, 2). \tag{2}$$

Hence

$$\|\varphi(x_1 + x_2)\| \leq \|x_1 + x_2 + y_1 + y_2\| < \|\varphi(x_1)\| + \|\varphi(x_2)\| + 2\epsilon, \tag{3}$$

which gives the triangle inequality and proves (*a*).

Suppose A is complete and $\{\varphi(x_n)\}$ is a Cauchy sequence in A/J. There is a subsequence for which

$$\|\varphi(x_{n_i}) - \varphi(x_{n_{i+1}})\| < 2^{-i} \qquad (i = 1, 2, 3, \ldots), \tag{4}$$

and there exist elements z_i so that $z_i - x_{n_i} \in J$ and $\|z_i - z_{i+1}\| < 2^{-i}$. Thus $\{z_i\}$ is a Cauchy sequence in A; and since A is complete, there exists $z \in A$ such that $\|z_i - z\| \to 0$. It follows that $\varphi(x_{n_i})$ converges to $\varphi(z)$ in A/J. But if a Cauchy sequence has a convergent subsequence, then the full sequence converges. Thus A/J is complete, and we have proved (*b*).

To prove (*c*), choose x_1 and $x_2 \in A$ and $\epsilon > 0$, and choose y_1 and $y_2 \in J$ so that (2) holds. Note that $(x_1 + y_1)(x_2 + y_2) \in x_1x_2 + J$, so that

$$\|\varphi(x_1x_2)\| \leq \|(x_1 + y_1)(x_2 + y_2)\| \leq \|x_1 + y_1\|\,\|x_2 + y_2\|. \tag{5}$$

Now (2) implies

$$\|\varphi(x_1x_2)\| \leq \|\varphi(x_1)\|\,\|\varphi(x_2)\|. \tag{6}$$

Finally, if e is the unit element of A, take $x_1 \notin J$ and $x_2 = e$ in (6); this gives $\|\varphi(e)\| \geq 1$. But $e \in \varphi(e)$, and the definition of the quotient norm shows that $\|\varphi(e)\| \leq \|e\| = 1$. So $\|\varphi(e)\| = 1$, and the proof is complete.

18.16 Having dealt with these preliminaries, we are now in a position to derive some of the key facts concerning commutative Banach algebras.

Suppose, as before, that A is a commutative complex Banach algebra with unit element e. We associate with A the set Δ of all complex homomorphisms of A; these are the homomorphisms of A onto the complex field, or, in different terminology, the *multiplicative linear functionals* on A which are not identically 0. As before, $\sigma(x)$ denotes the spectrum of the element $x \in A$, and $\rho(x)$ is the spectral radius of x.

Then the following relations hold:

18.17 Theorem

(*a*) *Every maximal ideal M of A is the kernel of some $h \in \Delta$.*
(*b*) *$\lambda \in \sigma(x)$ if and only if $h(x) = \lambda$ for some $h \in \Delta$.*
(*c*) *x is invertible in A if and only if $h(x) \neq 0$ for every $h \in \Delta$.*
(*d*) *$h(x) \in \sigma(x)$ for every $x \in A$ and $h \in \Delta$.*
(*e*) *$|h(x)| \leq \rho(x) \leq \|x\|$ for every $x \in A$ and $h \in \Delta$.*

PROOF If M is a maximal ideal of A, then A/M is a field; and since M is closed (Theorem 18.13), A/M is a Banach algebra. By Theorem 18.7 there is an isomorphism j of A/M onto the complex field. If $h = j \circ \varphi$, where φ is the homomorphism of A onto A/M whose kernel is M, then $h \in \Delta$ and the kernel of h is M. This proves (*a*).

If $\lambda \in \sigma(x)$, then $x - \lambda e$ is not invertible; hence the set of all elements $(x - \lambda e)y$, where $y \in A$, is a proper ideal of A, which lies in a maximal ideal (by Theorem 18.13), and (*a*) shows that there exists an $h \in \Delta$ such that $h(x - \lambda e) = 0$. Since $h(e) = 1$, this gives $h(x) = \lambda$.

On the other hand, if $\lambda \notin \sigma(x)$, then $(x - \lambda e)y = e$ for some $y \in A$. It follows that $h(x - \lambda e)h(y) = 1$ for every $h \in \Delta$, so that $h(x - \lambda e) \neq 0$, or $h(x) \neq \lambda$. This proves (*b*).

Since x is invertible if and only if $0 \notin \sigma(x)$, (*c*) follows from (*b*).

Finally, (*d*) and (*e*) are immediate consequences of (*b*). ////

Note that (*e*) implies that the norm of h, as a linear functional, is at most 1. In particular, each $h \in \Delta$ is continuous. This was already proved earlier (Theorem 9.21).

Applications

We now give some examples of theorems whose statements involve no algebraic concepts but which can be proved by Banach algebra techniques.

18.18 Theorem *Let $A(U)$ be the set of all continuous functions on the closure $\bar{U}$ of the open unit disc U whose restrictions to U are holomorphic. Suppose $f_1, \ldots, f_n$ are members of $A(U)$, such that*

$$|f_1(z)| + \cdots + |f_n(z)| > 0 \tag{1}$$

for every $z \in \bar{U}$. Then there exist $g_1, \ldots, g_n \in A(U)$ such that

$$\sum_{i=1}^{n} f_i(z)g_i(z) = 1 \qquad (z \in \bar{U}). \tag{2}$$

PROOF Since sums, products, and uniform limits of holomorphic functions are holomorphic, $A(U)$ is a Banach algebra, with the supremum norm. The set J of all functions $\Sigma f_i g_i$, where the g_i are arbitrary members of $A(U)$, is an ideal of $A(U)$. We have to prove that J contains the unit element 1 of $A(U)$. By Theorem 18.13 this happens if and only if J lies in no maximal ideal of $A(U)$. By Theorem 18.17(*a*) it is therefore enough to prove that there is no homomorphism h of $A(U)$ onto the complex field such that $h(f_i) = 0$ for every i $(1 \leq i \leq n)$.

Before we determine these homomorphisms, let us note that the polynomials form a dense subset of $A(U)$. To see this, suppose $f \in A(U)$ and $\epsilon > 0$; since f is uniformly continuous on $\bar{U}$, there exists an $r < 1$ such that $|f(z) - f(rz)| < \epsilon$ for all $z \in \bar{U}$; the expansion of $f(rz)$ in powers of z converges if $|rz| < 1$, hence converges to $f(rz)$ uniformly for $z \in \bar{U}$, and this gives the desired approximation.

Now let h be a complex homomorphism of $A(U)$. Put $f_0(z) = z$. Then $f_0 \in A(U)$. It is obvious that $\sigma(f_0) = \bar{U}$. By Theorem 18.17(*d*) there exists an $\alpha \in \bar{U}$ such that $h(f_0) = \alpha$. Hence $h(f_0^n) = \alpha^n = f_0^n(\alpha)$, for $n = 1, 2, 3, \ldots$, so $h(P) = P(\alpha)$ for every polynomial P. Since h is continuous and since the polynomials are dense in $A(U)$, it follows that $h(f) = f(\alpha)$ for every $f \in A(U)$.

Our hypothesis (1) implies that $|f_i(\alpha)| > 0$ for at least one index i, $1 \leq i \leq n$. Thus $h(f_i) \neq 0$.

We have proved that to each $h \in \Delta$ there corresponds at least one of the given functions f_i such that $h(f_i) \neq 0$, and this, as we noted above, is enough to prove the theorem. ////

Note: We have also determined all maximal ideals of $A(U)$, in the course of the preceding proof, since each is the kernel of some $h \in \Delta$: *If* $\alpha \in \bar{U}$ *and if* M_α *is the set of all* $f \in A(U)$ *such that* $f(\alpha) = 0$, *then* M_α *is a maximal ideal of* $A(U)$, *and all maximal ideals of* $A(U)$ *are obtained in this way.*

$A(U)$ is often called the *disc algebra.*

18.19 The restrictions of the members of $A(U)$ to the unit circle T form a closed subalgebra of $C(T)$. This is the algebra A discussed in Example 18.11. In fact, A is a *maximal* subalgebra of $C(T)$. More explicitly, *if* $A \subset B \subset C(T)$ *and* B *is a closed (relative to the supremum norm) subalgebra of* $C(T)$, *then either* $B = A$ *or* $B = C(T)$.

It is easy to see (compare with Exercise 29, Chap. 17) that A consists precisely of those $f \in C(T)$ for which

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta}) e^{-in\theta}\, d\theta = 0 \qquad (n = -1, -2, -3, \ldots). \tag{1}$$

Hence the above mentioned maximality theorem can be stated as an approximation theorem:

18.20 Theorem *Suppose* $g \in C(T)$ *and* $\hat{g}(n) \neq 0$ *for some* $n < 0$. *Then to every* $f \in C(T)$ *and to every* $\epsilon > 0$ *there correspond polynomials*

$$P_n(e^{i\theta}) = \sum_{k=0}^{m(n)} a_{n,k} e^{ik\theta} \qquad (n = 0, \ldots, N) \tag{1}$$

such that

$$|f(e^{i\theta}) - \sum_{n=0}^{N} P_n(e^{i\theta}) g^n(e^{i\theta})| < \epsilon \qquad (e^{i\theta} \in T). \tag{2}$$

PROOF Let B be the closure in $C = C(T)$ of the set of all functions of the form

$$\sum_{n=0}^{N} P_n g^n. \tag{3}$$

The theorem asserts that $B = C$. Let us assume $B \neq C$.

The set of all functions (3) (note that N is not fixed) is a complex algebra. Its closure B is a Banach algebra which contains the function f_0, where $f_0(e^{i\theta}) = e^{i\theta}$. Our assumption that $B \neq C$ implies that $1/f_0 \notin B$, for otherwise B would contain f_0^n for all integers n, hence all trigonometric polynomials would be in B; and since the trigonometric polynomials are dense in C (Theorem 4.25) we should have $B = C$.

So f_0 is not invertible in B. By Theorem 18.17 there is a complex homomorphism h of B such that $h(f_0) = 0$. Every homomorphism onto the complex field satisfies $h(1) = 1$; and since $h(f_0) = 0$, we also have

$$h(f_0^n) = [h(f_0)]^n = 0 \qquad (n = 1, 2, 3, \ldots). \tag{4}$$

We know that h is a linear functional on B, of norm at most 1. The Hahn-Banach theorem extends h to a linear functional on C (still denoted by h) of the same norm. Since $h(1) = 1$ and $\|h\| \leq 1$, the argument used in Sec. 5.22 shows that h is a *positive* linear functional on C. In particular, $h(f)$ is real for real f; hence $h(\bar{f}) = \overline{h(f)}$. Since f_0^{-n} is the complex conjugate of f_0^n, it follows that (4) also holds for $n = -1, -2, -3, \ldots$. Thus

$$h(f_0^n) = \begin{cases} 1 & \text{if } n = 0, \\ 0 & \text{if } n \neq 0. \end{cases} \tag{5}$$

Since the trigonometric polynomials are dense in C, there is only one bounded linear functional on C which satisfies (5). Hence h is given by the formula

$$h(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})\, d\theta \qquad (f \in C). \tag{6}$$

Now if n is a positive integer, $gf_0^n \in B$; and since h is multiplicative on B, (6) gives

$$\hat{g}(-n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(e^{i\theta}) e^{in\theta}\, d\theta = h(gf_0^n) = h(g)h(f_0^n) = 0, \tag{7}$$

by (5). This contradicts the hypothesis of the theorem. ////

We conclude with a theorem due to Wiener.

18.21 Theorem *Suppose*

$$f(e^{i\theta}) = \sum_{-\infty}^{\infty} c_n e^{in\theta}, \qquad \sum_{-\infty}^{\infty} |c_n| < \infty, \tag{1}$$

and $f(e^{i\theta}) \neq 0$ *for every real* θ. *Then*

$$\frac{1}{f(e^{i\theta})} = \sum_{-\infty}^{\infty} \gamma_n e^{in\theta} \quad \text{with} \quad \sum_{-\infty}^{\infty} |\gamma_n| < \infty. \tag{2}$$

PROOF We let A be the space of all complex functions f on the unit circle which satisfy (1), with the norm

$$\|f\| = \sum_{-\infty}^{\infty} |c_n|. \tag{3}$$

It is clear that A is a Banach space. In fact, A is isometrically isomorphic to ℓ^1, the space of all complex functions on the integers which are integrable with respect to the counting measure. But A is also a commutative Banach algebra, under pointwise multiplication. For if $g \in A$ and $g(e^{i\theta}) = \Sigma b_n e^{in\theta}$, then

$$f(e^{i\theta})g(e^{i\theta}) = \sum_n \left(\sum_k c_{n-k} b_k \right) e^{in\theta} \tag{4}$$

and hence

$$\|fg\| = \sum_n \left| \sum_k c_{n-k} b_k \right| \leq \sum_k |b_k| \sum_n |c_{n-k}| = \|f\| \cdot \|g\|. \tag{5}$$

Also, the function 1 is the unit of A, and $\|1\| = 1$.

Put $f_0(e^{i\theta}) = e^{i\theta}$, as before. Then $f_0 \in A$, $1/f_0 \in A$, and $\|f_0^n\| = 1$ for $n = 0, \pm 1, \pm 2, \ldots$. If h is any complex homomorphism of A and $h(f_0) = \lambda$, the fact that $\|h\| \leq 1$ implies that

$$|\lambda^n| = |h(f_0^n)| \leq \|f_0^n\| = 1 \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{6}$$

Hence $|\lambda| = 1$. In other words, to each h corresponds a point $e^{i\alpha} \in T$ such that $h(f_0) = e^{i\alpha}$, so

$$h(f_0^n) = e^{in\alpha} = f_0^n(e^{i\alpha}) \qquad (n = 0, \pm 1, \pm 2, \ldots). \tag{7}$$

If f is given by (1), then $f = \Sigma c_n f_0^n$. This series converges in A; and since h is a continuous linear functional on A, we conclude from (7) that

$$h(f) = f(e^{i\alpha}) \qquad (f \in A). \tag{8}$$

Our hypothesis that f vanishes at no point of T says therefore that f is not in the kernel of any complex homomorphism of A, and now Theorem 18.17 implies that f is invertible in A. But this is precisely what the theorem asserts. ////

Exercises

1 Suppose $B(X)$ is the algebra of all bounded linear operators on the Banach space X, with

$$(A_1 + A_2)(x) = A_1x + A_2x, \quad (A_1A_2)(x) = A_1(A_2x), \quad \|A\| = \sup \frac{\|Ax\|}{\|x\|},$$

if A, A_1, and $A_2 \in B(X)$. Prove that $B(X)$ is a Banach algebra.

2 Let n be a positive integer, let X be the space of all complex n-tuples (normed in any way, as long as the axioms for a normed linear space are satisfied), and let $B(X)$ be as in Exercise 1. Prove that the spectrum of each member of $B(X)$ consists of at most n complex numbers. What are they?

3 Take $X = L^2(-\infty, \infty)$, suppose $\varphi \in L^\infty(-\infty, \infty)$, and let M be the multiplication operator which takes $f \in L^2$ to φf. Show that M is a bounded linear operator on L^2 and that the spectrum of M is equal to the essential range of φ (Chap. 3, Exercise 19).

4 What is the spectrum of the shift operator on ℓ^2? (See Sec. 17.20 for the definition.)

5 Prove that the closure of an ideal in a Banach algebra is an ideal.

6 If X is a compact Hausdorff space, find all maximal ideals in $C(X)$.

7 Suppose A is a commutative Banach algebra with unit, which is generated by a single element x. This means that the polynomials in x are dense in A. Prove that the complement of $\sigma(x)$ is a connected subset of the plane. *Hint*: If $\lambda \notin \sigma(x)$, there are polynomials P_n such that $P_n(x) \to (x - \lambda e)^{-1}$ in A. Prove that $P_n(z) \to (z - \lambda)^{-1}$ uniformly for $z \in \sigma(x)$.

8 Suppose $\sum_0^\infty |c_n| < \infty$, $f(z) = \sum_0^\infty c_n z^n$, $|f(z)| > 0$ for every $z \in \bar{U}$, and $1/f(z) = \sum_0^\infty a_n z^n$. Prove that $\sum_0^\infty |a_n| < \infty$.

9 Prove that a closed linear subspace of the Banach algebra $L^1(R^1)$ (see Sec. 9.19) is translation invariant if and only if it is an ideal.

10 Show that $L^1(T)$ is a commutative Banach algebra (without unit) if multiplication is defined by

$$(f * g)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t - s)g(s)\, ds.$$

Find all complex homomorphisms of $L^1(T)$, as in Theorem 9.23. If E is a set of integers and if I_E is the set of all $f \in L^1(T)$ such that $\hat{f}(n) = 0$ for all $n \in E$, prove that I_E is a closed ideal in $L^1(T)$, and prove that every closed ideal in $L^1(T)$ is obtained in this manner.

11 The *resolvent* $R(\lambda, x)$ of an element x in a Banach algebra with unit is defined as

$$R(\lambda, x) = (\lambda e - x)^{-1}$$

for all complex λ for which this inverse exists. Prove the identity

$$R(\lambda, x) - R(\mu, x) = (\mu - \lambda)R(\lambda)R(\mu)$$

and use it to give an alternative proof of Theorem 18.5.

12 Let A be a commutative Banach algebra with unit. The *radical* of A is defined to be the intersection of all maximal ideals of A. Prove that the following three statements about an element $x \in A$ are equivalent:

(*a*) x is in the radical of A.

(*b*) $\lim_{n \to \infty} \|x^n\|^{1/n} = 0$.

(*c*) $h(x) = 0$ for every complex homomorphism of A.

13 Find an element x in a Banach algebra A (for instance, a bounded linear operator on a Hilbert space) such that $x^n \neq 0$ for all $n > 0$, but $\lim_{n \to 0} \|x^n\|^{1/n} = 0$.

14 Suppose A is a commutative Banach algebra with unit, and let Δ be the set of all complex homomorphisms of A, as in Sec. 18.16. Associate with each $x \in A$ a function $\hat{x}$ on Δ by the formula

$$\hat{x}(h) = h(x) \qquad (h \in \Delta).$$

$\hat{x}$ is called the Gelfand transform of x.

Prove that the mapping $x \to \hat{x}$ is a homomorphism of A onto an algebra $\hat{A}$ of complex functions on Δ, with pointwise multiplication. Under what condition on A is this homomorphism an isomorphism? (See Exercise 12.)

Prove that the spectral radius $\rho(x)$ is equal to

$$\|\hat{x}\|_\infty = \sup \{|\hat{x}(h)| : h \in \Delta\}.$$

Prove that the range of the function $\hat{x}$ is exactly the spectrum $\sigma(x)$.

15 If A is a commutative Banach algebra *without* unit, let A_1 be the algebra of all ordered pairs (x, λ), with $x \in A$ and λ a complex number; addition and multiplication are defined in the "obvious" way, and $\|(x, \lambda)\| = \|x\| + |\lambda|$. Prove that A_1 is a commutative Banach algebra with unit and that the mapping $x \to (x, 0)$ is an isometric isomorphism of A onto a maximal ideal of A_1. This is a standard embedding of an algebra without unit in one with unit.

16 Show that H^∞ is a commutative Banach algebra with unit, relative to the supremum norm and pointwise addition and multiplication. The mapping $f \to f(\alpha)$ is a complex homomorphism of H^∞, whenever $|\alpha| < 1$. Prove that there must be others.

17 Show that the set of all functions $(z - 1)^2 f$, where $f \in H^\infty$, is an ideal in H^∞ which is not closed. *Hint*:

$$|(1 - z)^2(1 + \epsilon - z)^{-1} - (1 - z)| < \epsilon \qquad \text{if } |z| < 1, \epsilon > 0.$$

18 Suppose φ is an inner function. Prove that $\{\varphi f : f \in H^\infty\}$ is a closed ideal in H^∞. In other words, prove that if $\{f_n\}$ is a sequence in H^∞ such that $\varphi f_n \to g$ uniformly in U, then $g/\varphi \in H^\infty$.

CHAPTER

NINETEEN

HOLOMORPHIC FOURIER TRANSFORMS

Introduction

19.1 In Chap. 9 the Fourier transform of a function f on R^1 was defined to be a function $\hat{f}$ on R^1. Frequently $\hat{f}$ can be extended to a function which is holomorphic in a certain region of the plane. For instance, if $f(t) = e^{-|t|}$, then $\hat{f}(x) = (1 + x^2)^{-1}$, a rational function. This should not be too surprising. For each real t, the kernel e^{itz} is an entire function of z, so one should expect that there are conditions on f under which $\hat{f}$ will be holomorphic in certain regions.

We shall describe two classes of holomorphic functions which arise in this manner.

For the first one, let F be any function in $L^2(-\infty, \infty)$ which vanishes on $(-\infty, 0)$ [i.e., take $F \in L^2(0, \infty)$] and define

$$f(z) = \int_0^\infty F(t)e^{itz}\, dt \qquad (z \in \Pi^+), \tag{1}$$

where Π^+ is the set of all $z = x + iy$ with $y > 0$. If $z \in \Pi^+$, then $|e^{itz}| = e^{-ty}$, which shows that the integral in (1) exists as a Lebesgue integral.

If $\operatorname{Im} z > \delta > 0$, $\operatorname{Im} z_n > \delta$, and $z_n \to z$, the dominated convergence theorem shows that

$$\lim_{n\to\infty} \int_0^\infty |\exp(itz_n) - \exp(itz)|^2\, dt = 0$$

because the integrand is bounded by the L^1-function $4 \exp(-2\delta t)$ and tends to 0 for every $t > 0$. The Schwarz inequality implies therefore that f is continuous in Π^+. The theorems of Fubini and Cauchy show that $\int_\gamma f(z)\, dz = 0$ for every closed path γ in Π^+. By Morera's theorem, $f \in H(\Pi^+)$.

Let us rewrite (1) in the form

$$f(x+iy) = \int_0^\infty F(t)e^{-ty}e^{itx}\,dt, \tag{2}$$

regard y as fixed, and apply Plancherel's theorem. We obtain

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x+iy)|^2\,dx = \int_0^\infty |F(t)|^2 e^{-2ty}\,dt \le \int_0^\infty |F(t)|^2\,dt \tag{3}$$

for every $y > 0$. [Note that our notation now differs from that in Chap. 9. There the underlying measure was Lebesgue measure divided by $\sqrt{2\pi}$. Here we just use Lebesgue measure. This accounts for the factor $1/(2\pi)$ in (3).] This shows:

(*a*) *If f is of the form* (1), *then f is holomorphic in Π^+ and its restrictions to horizontal lines in Π^+ form a bounded set in $L^2(-\infty, \infty)$.*

Our second class consists of all f of the form

$$f(z) = \int_{-A}^{A} F(t)e^{itz}\,dt \tag{4}$$

where $0 < A < \infty$ and $F \in L^2(-A, A)$. These functions f are entire (the proof is the same as above), and they satisfy a growth condition:

$$|f(z)| \le \int_{-A}^{A} |F(t)|\,e^{-ty}\,dt \le e^{A|y|}\int_{-A}^{A} |F(t)|\,dt. \tag{5}$$

If C is this last integral, then $C < \infty$, and (5) implies that

$$|f(z)| \le Ce^{A|z|}. \tag{6}$$

[Entire functions which satisfy (6) are said to be of *exponential type.*] Thus:

(*b*) *Every f of the form* (4) *is an entire function which satisfies* (6) *and whose restriction to the real axis lies in L^2* (by the Plancherel theorem).

It is a remarkable fact that the converses of (*a*) and (*b*) are true. This is the content of Theorems 19.2 and 19.3.

Two Theorems of Paley and Wiener

19.2 Theorem *Suppose $f \in H(\Pi^+)$ and*

$$\sup_{0<y<\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} |f(x+iy)|^2\,dx = C < \infty. \tag{1}$$

Then there exists an $F \in L^2(0, \infty)$ such that

$$f(z) = \int_0^\infty F(t)e^{itz}\,dt \qquad (z \in \Pi^+) \tag{2}$$

and

$$\int_0^\infty |F(t)|^2\, dt = C. \tag{3}$$

Note: The function F we are looking for is to have the property that $f(x + iy)$ is the Fourier transform of $F(t)e^{-yt}$ (we regard y as a positive constant). Let us apply the inversion formula (whether or not this is correct does not matter; we are trying to motivate the proof that follows): The desired F should be of the form

$$F(t) = e^{ty} \cdot \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x + iy)e^{-itx}\, dx = \frac{1}{2\pi} \int f(z)e^{-itz}\, dz. \tag{4}$$

The last integral is over a horizontal line in Π^+, and if this argument is correct at all, the integral will not depend on the particular line we happen to choose. This suggests that the Cauchy theorem should be invoked.

PROOF Fix y, $0 < y < \infty$. For each $\alpha > 0$ let Γ_α be the rectangular path with vertices at $\pm\alpha + i$ and $\pm\alpha + iy$. By Cauchy's theorem

$$\int_{\Gamma_\alpha} f(z)e^{-itz}\, dz = 0. \tag{5}$$

We consider only real values of t. Let $\Phi(\beta)$ be the integral of $f(z)e^{-itz}$ over the straight line interval from $\beta + i$ to $\beta + iy$ (β real). Put $I = [y, 1]$ if $y < 1$, $I = [1, y]$ if $1 < y$. Then

$$|\Phi(\beta)|^2 = \left| \int_I f(\beta + iu)e^{-it(\beta+iu)}\, du \right|^2 \le \int_I |f(\beta + iu)|^2\, du \int_I e^{2tu}\, du. \tag{6}$$

Put

$$\Lambda(\beta) = \int_I |f(\beta + iu)|^2\, du. \tag{7}$$

Then (1) shows, by Fubini's theorem, that

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \Lambda(\beta)\, d\beta \le Cm(I). \tag{8}$$

Hence there is a sequence $\{\alpha_j\}$ such that $\alpha_j \to \infty$ and

$$\Lambda(\alpha_j) + \Lambda(-\alpha_j) \to 0 \qquad (j \to \infty). \tag{9}$$

By (6), this implies that

$$\Phi(\alpha_j) \to 0, \qquad \Phi(-\alpha_j) \to 0 \qquad \text{as } j \to \infty. \tag{10}$$

Note that this holds for every t and that the sequence $\{\alpha_j\}$ does not depend on t.

Let us define

$$g_j(y, t) = \frac{1}{2\pi} \int_{-\alpha_j}^{\alpha_j} f(x + iy)e^{-itx}\, dx. \tag{11}$$

Then we deduce from (5) and (10) that

$$\lim_{j\to\infty} [e^{ty}g_j(y, t) - e^t g_j(1, t)] = 0 \qquad (-\infty < t < \infty). \tag{12}$$

Write $f_y(x)$ for $f(x + iy)$. Then $f_y \in L^2(-\infty, \infty)$, by hypothesis, and the Plancherel theorem asserts that

$$\lim_{j\to\infty} \int_{-\infty}^{\infty} |\hat{f}_y(t) - g_j(y, t)|^2\, dt = 0, \tag{13}$$

where $\hat{f}_y$ is the Fourier transform of f_y. A subsequence of $\{g_j(y, t)\}$ converges therefore pointwise to $\hat{f}_y(t)$, for almost all t (Theorem 3.12). If we define

$$F(t) = e^t \hat{f}_1(t), \tag{14}$$

it now follows from (12) that

$$F(t) = e^{ty}\hat{f}_y(t). \tag{15}$$

Note that (14) does not involve y and that (15) holds for every $y \in (0, \infty)$. Plancherel's theorem can be applied to (15):

$$\int_{-\infty}^{\infty} e^{-2ty} |F(t)|^2\, dt = \int_{-\infty}^{\infty} |\hat{f}_y(t)|^2\, dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |f_y(x)|^2\, dx \le C. \tag{16}$$

If we let $y \to \infty$, (16) shows that $F(t) = 0$ a.e. in $(-\infty, 0)$.

If we let $y \to 0$, (16) shows that

$$\int_0^{\infty} |F(t)|^2\, dt \le C. \tag{17}$$

It now follows from (15) that $\hat{f}_y \in L^1$ if $y > 0$. Hence Theorem 9.14 gives

$$f_y(x) = \int_{-\infty}^{\infty} \hat{f}_y(t)e^{itx}\, dt \tag{18}$$

or

$$f(z) = \int_0^{\infty} F(t)e^{-yt}e^{itx}\, dt = \int_0^{\infty} F(t)e^{itz}\, dt \qquad (z \in \Pi^+). \tag{19}$$

This is (2), and now (3) follows from (17) and formula 19.1(3). ////

19.3 Theorem *Suppose A and C are positive constants and f is an entire function such that*

$$|f(z)| \leq Ce^{A|z|} \tag{1}$$

for all z, and

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty. \tag{2}$$

Then there exists an $F \in L^2(-A, A)$ such that

$$f(z) = \int_{-A}^{A} F(t)e^{itz} \, dt \tag{3}$$

for all z.

PROOF Put $f_\epsilon(x) = f(x)e^{-\epsilon|x|}$, for $\epsilon > 0$ and x real. We shall show that

$$\lim_{\epsilon \to 0} \int_{-\infty}^{\infty} f_\epsilon(x)e^{-itx} \, dx = 0 \qquad (t \text{ real}, |t| > A). \tag{4}$$

Since $\|f_\epsilon - f\|_2 \to 0$ as $\epsilon \to 0$, the Plancherel theorem implies that the Fourier transforms of f_ϵ converge in L^2 to the Fourier transform F of f (more precisely, of the restriction of f to the real axis). Hence (4) will imply that F vanishes outside $[-A, A]$, and then Theorem 9.14 shows that (3) holds for almost every real z. Since each side of (3) is an entire function, it follows that (3) holds for every complex z.

Thus (4) implies the theorem.

For each real α, let Γ_α be the path defined by

$$\Gamma_\alpha(s) = se^{i\alpha} \qquad (0 \leq s < \infty), \tag{5}$$

put

$$\Pi_\alpha = \{w: \text{Re } (we^{i\alpha}) > A\}, \tag{6}$$

and if $w \in \Pi_\alpha$, define

$$\Phi_\alpha(w) = \int_{\Gamma_\alpha} f(z)e^{-wz} \, dz = e^{i\alpha} \int_0^{\infty} f(se^{i\alpha}) \exp(-wse^{i\alpha}) \, ds. \tag{7}$$

By (1) and (5), the absolute value of the integrand is at most

$$C \exp\{-[\text{Re } (we^{i\alpha}) - A]s\},$$

and it follows (as in Sec. 19.1) that Φ_α is holomorphic in the half plane Π_α.

However, more is true if $\alpha = 0$ and if $\alpha = \pi$: We have

$$\Phi_0(w) = \int_0^{\infty} f(x)e^{-wx} \, dx \qquad (\text{Re } w > 0), \tag{8}$$

$$\Phi_\pi(w) = -\int_{-\infty}^{0} f(x)e^{-wx}\,dx \qquad (\text{Re } w < 0). \tag{9}$$

Φ_0 and Φ_π are holomorphic in the indicated half planes because of (2).

The significance of the functions Φ_α to (4) lies in the easily verified relation

$$\int_{-\infty}^{\infty} f_\epsilon(x)e^{-itx}\,dx = \Phi_0(\epsilon + it) - \Phi_\pi(-\epsilon + it) \qquad (t \text{ real}). \tag{10}$$

Hence we have to prove that the right side of (10) tends to 0 as $\epsilon \to 0$, if $t > A$ and if $t < -A$.

We shall do this by showing that any two of our functions Φ_α agree in the intersection of their domains of definition, i.e., that they are analytic continuations of each other. Once this is done, we can replace Φ_0 and Φ_π by $\Phi_{\pi/2}$ in (10) if $t < -A$, and by $\Phi_{-\pi/2}$ if $t > A$, and it is then obvious that the difference tends to 0 as $\epsilon \to 0$.

So suppose $0 < \beta - \alpha < \pi$. Put

$$\gamma = \frac{\alpha + \beta}{2}, \qquad \eta = \cos\frac{\beta - \alpha}{2} > 0. \tag{11}$$

If $w = |w|e^{-i\gamma}$, then

$$\text{Re } (we^{i\alpha}) = \eta|w| = \text{Re } (we^{i\beta}) \tag{12}$$

so that $w \in \Pi_\alpha \cap \Pi_\beta$ as soon as $|w| > A/\eta$. Consider the integral

$$\int_\Gamma f(z)e^{-wz}\,dz \tag{13}$$

over the circular arc Γ given by $\Gamma(t) = re^{it}$, $\alpha \le t \le \beta$. Since

$$\text{Re } (-wz) = -|w|r\cos(t - \gamma) \le -|w|r\eta, \tag{14}$$

the absolute value of the integrand in (13) does not exceed

$$C \exp\{(A - |w|\eta)r\}.$$

If $|w| > A/\eta$ it follows that (13) tends to 0 as $r \to \infty$.

We now apply the Cauchy theorem. The integral of $f(z)e^{-wz}$ over the interval $[0, re^{i\beta}]$ is equal to the sum of (13) and the integral over $[0, re^{i\alpha}]$. Since (13) tends to 0 as $r \to \infty$, we conclude that $\Phi_\alpha(w) = \Phi_\beta(w)$ if $w = |w|e^{-i\gamma}$ and $|w| > A/\eta$, and then Theorem 10.18 shows that Φ_α and Φ_β coincide in the intersection of the half planes in which they were originally defined.

This completes the proof. ////

19.4 Remarks Each of the two preceding proofs depended on a typical application of Cauchy's theorem. In Theorem 19.2 we replaced integration over one horizontal line by integration over another to show that 19.2(15) was

independent of y. In Theorem 19.3, replacement of one ray by another was used to construct analytic continuations; the result actually was that the functions Φ_α are restrictions of one function Φ which is holomorphic in the complement of the interval $[-Ai, Ai]$.

The class of functions described in Theorem 19.2 is the half plane analogue of the class H^2 discussed in Chap. 17. Theorem 19.3 will be used in the proof of the Denjoy-Carleman theorem (Theorem 19.11).

Quasi-analytic Classes

19.5 If Ω is a region and if $z_0 \in \Omega$, every $f \in H(\Omega)$ is uniquely determined by the numbers $f(z_0), f'(z_0), f''(z_0), \ldots$. On the other hand, there exist infinitely differentiable functions on R^1 which are not identically 0 but which vanish on some interval. Thus we have here a uniqueness property which holomorphic functions possess but which does not hold in C^∞ (the class of all infinitely differentiable complex functions on R^1).

If $f \in H(\Omega)$, the growth of the sequence $\{|f^{(n)}(z_0)|\}$ is restricted by Theorem 10.26. It is therefore reasonable to ask whether the above uniqueness property holds in suitable subclasses of C^∞ in which the growth of the derivatives is subject to some restrictions. This motivates the following definitions; the answer to our question is given by Theorem 19.11.

19.6 The Classes $C\{M_n\}$ If $M_0, M_1, M_2, \ldots$ are positive numbers, we let $C\{M_n\}$ be the class of all $f \in C^\infty$ which satisfy inequalities of the form

$$\|D^n f\|_\infty \leq \beta_f B_f^n M_n \qquad (n = 0, 1, 2, \ldots). \tag{1}$$

Here $D^0 f = f$, $D^n f$ is the nth derivative of f if $n \geq 1$, the norm is the supremum norm over R^1, and β_f and B_f are positive constants (depending on f, but not on n).

If f satisfies (1), then

$$\limsup_{n\to\infty} \left\{\frac{\|D^n f\|_\infty}{M_n}\right\}^{1/n} \leq B_f. \tag{2}$$

This shows that B_f is a more significant quantity than β_f. However, if β_f were omitted in (1), the case $n = 0$ would imply $\|f\|_\infty \leq M_0$, an undesirable restriction. The inclusion of β_f makes $C\{M_n\}$ into a vector space.

Each $C\{M_n\}$ is invariant under affine transformations. More explicitly, suppose $f \in C\{M_n\}$ and $g(x) = f(ax + b)$. Then g satisfies (1), with $\beta_g = \beta_f$ and $B_g = aB_f$.

We shall make two standing assumptions on the sequences $\{M_n\}$ under consideration:

$$M_0 = 1. \tag{3}$$

$$M_n^2 \leq M_{n-1} M_{n+1} \qquad (n = 1, 2, 3, \ldots). \tag{4}$$

Assumption (4) can be expressed in the form: $\{\log M_n\}$ *is a convex sequence.*

These assumptions will simplify some of our work, and they involve no loss of generality. [One can prove, although we shall not do so, that every class $C\{M_n\}$ is equal to a class $C\{M_n\}$, where $\{M_n\}$ satisfies (3) and (4).]

The following result illustrates the utility of (3) and (4):

19.7 Theorem *Each $C\{M_n\}$ is an algebra, with respect to pointwise multiplication.*

PROOF Suppose f and $g \in C\{M_n\}$, and β_f, B_f, β_g, and B_g are the corresponding constants. The product rule for differentiation shows that

$$D^n(fg) = \sum_{j=0}^{n} \binom{n}{j} (D^j f) \cdot (D^{n-j} g). \tag{1}$$

Hence

$$|D^n(fg)| \leq \beta_f \beta_g \sum_{j=0}^{n} \binom{n}{j} B_f^j B_g^{n-j} M_j M_{n-j}. \tag{2}$$

The convexity of $\{\log M_n\}$, combined with $M_0 = 1$, shows that $M_j M_{n-j} \leq M_n$ for $0 \leq j \leq n$. Hence the binomial theorem leads from (2) to

$$\|D^n(fg)\|_\infty \leq \beta_f \beta_g (B_f + B_g)^n M_n \qquad (n = 0, 1, 2, \ldots), \tag{3}$$

so that $fg \in C\{M_n\}$. ////

19.8 Definition A class $C\{M_n\}$ is said to be *quasi-analytic* if the conditions

$$f \in C\{M_n\}, \qquad (D^n f)(0) = 0 \qquad (\text{for } n = 0, 1, 2, \ldots) \tag{1}$$

imply that $f(x) = 0$ for all $x \in R^1$.

The content of the definition is of course unchanged if $(D^n f)(0)$ is replaced by $(D^n f)(x_0)$, where x_0 is any given point.

The quasi-analytic classes are thus the ones which have the uniqueness property we mentioned in Sec. 19.6. One of these classes is very intimately related to holomorphic functions:

19.9 Theorem *The class $C\{n!\}$ consists of all f to which there corresponds a $\delta > 0$ such that f can be extended to a bounded holomorphic function in the strip defined by $|\operatorname{Im}(z)| < \delta$.*

Consequently $C\{n!\}$ is a quasi-analytic class.

PROOF Suppose $f \in H(\Omega)$ and $|f(z)| < \beta$ for all $z \in \Omega$, where Ω consists of all $z = x + iy$ with $|y| < \delta$. It follows from Theorem 10.26 that

$$|(D^n f)(x)| \leq \beta \delta^{-n} n! \qquad (n = 0, 1, 2, \ldots) \tag{1}$$

for all real x. The restriction of f to the real axis therefore belongs to $C\{n!\}$.

Conversely, suppose f is defined on the real axis and $f \in C\{n!\}$. In other words,

$$\|D^n f\|_\infty \leq \beta B^n n! \qquad (n = 0, 1, 2, \ldots). \tag{2}$$

We claim that the representation

$$f(x) = \sum_{n=0}^{\infty} \frac{(D^n f)(a)}{n!} (x - a)^n \tag{3}$$

is valid for all $a \in R^1$ if $a - B^{-1} < x < a + B^{-1}$. This follows from Taylor's formula

$$f(x) = \sum_{j=0}^{n-1} \frac{(D^j f)(a)}{j!} (x - a)^j + \frac{1}{(n-1)!} \int_a^x (x - t)^{n-1} (D^n f)(t)\, dt, \tag{4}$$

which one obtains by repeated integrations by part. By (2) the last term in (4) (the "remainder") is dominated by

$$n\beta B^n \left| \int_a^x (x - t)^{n-1}\, dt \right| = \beta\, |B(x - a)|^n. \tag{5}$$

If $|B(x - a)| < 1$, this tends to 0 as $n \to \infty$, and (3) follows.

We can now replace x in (3) by any complex number z such that $|z - a| < 1/B$. This defines a holomorphic function F_a in the disc with center at a and radius $1/B$, and $F_a(x) = f(x)$ if x is real and $|x - a| < 1/B$. The various functions F_a are therefore analytic continuations of each other; they form a holomorphic extension F of f in the strip $|y| < 1/B$.

If $0 < \delta < 1/B$ and $z = a + iy$, $|y| < \delta$, then

$$|F(z)| = |F_a(z)| = \left| \sum_{n=0}^{\infty} \frac{(D^n f)(a)}{n!} (iy)^n \right| \leq \beta \sum_{n=0}^{\infty} (B\delta)^n = \frac{\beta}{1 - B\delta}.$$

This shows that F is bounded in the strip $|y| < \delta$, and the proof is complete. ////

19.10 Theorem *The class $C\{M_n\}$ is quasi-analytic if and only if $C\{M_n\}$ contains no nontrivial function with compact support.*

PROOF If $C\{M_n\}$ is quasi-analytic, if $f \in C\{M_n\}$, and if f has compact support, then evidently f and all its derivatives vanish at some point, hence $f(x) = 0$ for all x.

Suppose $C\{M_n\}$ is not quasi-analytic. Then there exists an $f \in C\{M_n\}$ such that $(D^n f)(0) = 0$ for $n = 0, 1, 2, \ldots$, but $f(x_0) \neq 0$ for some x_0. We may assume $x_0 > 0$. If $g(x) = f(x)$ for $x \geq 0$ and $g(x) = 0$ for $x < 0$, then $g \in C\{M_n\}$. Put $h(x) = g(x)g(2x_0 - x)$. By Theorem 19.7, $h \in C\{M_n\}$. Also, $h(x) = 0$ if $x < 0$ and if $x > 2x_0$. But $h(x_0) = f^2(x_0) \neq 0$. Thus h is a nontrivial member of $C\{M_n\}$ with compact support. ////

We are now ready for the fundamental theorem about quasi-analytic clases.

The Denjoy-Carleman Theorem

19.11 Theorem *Suppose* $M_0 = 1$, $M_n^2 \leq M_{n-1}M_{n+1}$ *for* $n = 1, 2, 3, \ldots$, *and*

$$Q(x) = \sum_{n=0}^{\infty} \frac{x^n}{M_n}, \qquad q(x) = \sup_{n \geq 0} \frac{x^n}{M_n},$$

for $x > 0$. *Then each of the following five conditions implies the other four*:

(*a*) $C\{M_n\}$ *is not quasi-analytic.*

(*b*) $\displaystyle\int_0^{\infty} \log Q(x) \frac{dx}{1+x^2} < \infty.$

(*c*) $\displaystyle\int_0^{\infty} \log q(x) \frac{dx}{1+x^2} < \infty.$

(*d*) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{M_n}\right)^{1/n} < \infty.$

(*e*) $\displaystyle\sum_{n=1}^{\infty} \frac{M_{n-1}}{M_n} < \infty.$

Note: If $M_n \to \infty$ very rapidly as $n \to \infty$, then $Q(x)$ tends to infinity slowly as $x \to \infty$. Thus each of the five conditions says, in its own way, that $M_n \to \infty$ rapidly. Note also that $Q(x) \geq 1$ and $q(x) \geq 1$. The integrals in (*b*) and (*c*) are thus always defined. It may happen that $Q(x) = \infty$ for some $x < \infty$. In that case, the integral (*b*) is $+\infty$, and the theorem asserts that $C\{M_n\}$ is quasi-analytic.

If $M_n = n!$, then $M_{n-1}/M_n = 1/n$, hence (*e*) is violated, and the theorem asserts that $C\{n!\}$ is quasi-analytic, in accordance with Theorem 19.9.

PROOF THAT (*a*) IMPLIES (*b*) Assume that $C\{M_n\}$ is not quasi-analytic. Then $C\{M_n\}$ contains a nontrivial function with compact support (Theorem 19.10). An affine change of variable gives a function $F \in C\{M_n\}$, with support in some interval $[0, A]$, such that

$$\|D^n F\|_{\infty} \leq 2^{-n} M_n \qquad (n = 0, 1, 2, \ldots) \tag{1}$$

and such that F is not identically zero. Define

$$f(z) = \int_0^A F(t) e^{itz}\, dt \tag{2}$$

and

$$g(w) = f\left(\frac{i - iw}{1 + w}\right). \tag{3}$$

Then f is entire. If $\operatorname{Im} z > 0$, the absolute value of the integrand in (2) is at most $|F(t)|$. Hence f is bounded in the upper half plane; therefore g is bounded in U. Also, g is continuous on $\bar{U}$, except at the point $w = -1$. Since f is not identically 0 (by the uniqueness theorem for Fourier transforms) the same is true of g, and now Theorem 15.19 shows that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \log|g(e^{i\theta})|\,d\theta > -\infty. \tag{4}$$

If $x = i(1 - e^{i\theta})/(1 + e^{i\theta}) = \tan(\theta/2)$, then $d\theta = 2(1 + x^2)^{-1}\,dx$, so (4) is the same as

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \log|f(x)|\,\frac{dx}{1+x^2} > -\infty. \tag{5}$$

On the other hand, partial integration of (2) gives

$$f(z) = (iz)^{-n}\int_0^A (D^nF)(t)e^{itz}\,dt \qquad (z \neq 0) \tag{6}$$

since F and all its derivatives vanish at 0 and at A. It now follows from (1) and (6) that

$$|x^n f(x)| \leq 2^{-n}AM_n \qquad (x \text{ real}, n = 0, 1, 2, \ldots). \tag{7}$$

Hence

$$Q(x)|f(x)| = \sum_{n=0}^{\infty} \frac{x^n|f(x)|}{M_n} \leq 2A \qquad (x \geq 0), \tag{8}$$

and (5) and (8) imply that (*b*) holds. ////

PROOF THAT (*b*) IMPLIES (*c*) $q(x) \leq Q(x)$. ////

PROOF THAT (*c*) IMPLIES (*d*) Put $a_n = M_n^{1/n}$. Since $M_0 = 1$ and since $M_n^2 \leq M_{n-1}M_{n+1}$, it is easily verified that $a_n \leq a_{n+1}$, for $n > 0$. If $x \geq ea_n$, then $x^n/M_n \geq e^n$, so

$$\log q(x) \geq \log\frac{x^n}{M_n} \geq \log e^n = n. \tag{9}$$

Hence

$$\begin{aligned} e\int_{ea_1}^{\infty} \log q(x)\cdot\frac{dx}{x^2} &\geq e\sum_{n=1}^{N} n\int_{ea_n}^{ea_{n+1}} x^{-2}\,dx + e\int_{ea_{N+1}}^{\infty} (N+1)x^{-2}\,dx \\ &= \sum_{n=1}^{N} n\left(\frac{1}{a_n} - \frac{1}{a_{n+1}}\right) + \frac{N+1}{a_{N+1}} = \sum_{n=1}^{N+1}\frac{1}{a_n} \end{aligned} \tag{10}$$

for every N. This shows that (*c*) implies (*d*). ////

PROOF THAT (*d*) IMPLIES (*e*) Put

$$\lambda_n = \frac{M_{n-1}}{M_n} \qquad (n = 1, 2, 3, \ldots). \tag{11}$$

Then $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$, and if $a_n = M_n^{1/n}$, as above, we have

$$(a_n \lambda_n)^n \leq M_n \cdot \lambda_1 \lambda_2 \cdots \lambda_n = 1. \tag{12}$$

Thus $\lambda_n \leq 1/a_n$, and the convergence of $\Sigma(1/a_n)$ implies that of $\Sigma\lambda_n$. ////

PROOF THAT (*e*) IMPLIES (*a*) The assumption now is that $\Sigma\lambda_n < \infty$, where λ_n is given by (11). We claim that the function

$$f(z) = \left(\frac{\sin z}{z}\right)^2 \prod_{n=1}^{\infty} \frac{\sin \lambda_n z}{\lambda_n z} \tag{13}$$

is an entire function of exponential type, not identically zero, which satisfies the inequalities

$$|x^k f(x)| \leq M_k \left(\frac{\sin x}{x}\right)^2 \qquad (x \text{ real}, k = 0, 1, 2, \ldots). \tag{14}$$

Note first that $1 - z^{-1} \sin z$ has a zero at the origin. Hence there is a constant B such that

$$\left|1 - \frac{\sin z}{z}\right| \leq B|z| \qquad (|z| \leq 1). \tag{15}$$

It follows that

$$\left|1 - \frac{\sin \lambda_n z}{\lambda_n z}\right| \leq B\lambda_n |z| \qquad \left(|z| \leq \frac{1}{\lambda_n}\right), \tag{16}$$

so that the series

$$\sum_{n=1}^{\infty} \left|1 - \frac{\sin \lambda_n z}{\lambda_n z}\right| \tag{17}$$

converges uniformly on compact sets. (Note that $1/\lambda_n \to \infty$ as $n \to \infty$, since $\Sigma\lambda_n < \infty$.) The infinite product (13) therefore defines an entire function f which is not identically zero.

Next, the identity

$$\frac{\sin z}{z} = \frac{1}{2} \int_{-1}^{1} e^{itz}\, dt \tag{18}$$

shows that $|z^{-1} \sin z| \leq e^{|y|}$ if $z = x + iy$. Hence

$$|f(z)| \leq e^{A|z|}, \qquad \text{with } A = 2 + \sum_{n=1}^{\infty} \lambda_n. \tag{19}$$

For real x, we have $|\sin x| \leq |x|$ and $|\sin x| \leq 1$. Hence

$$|x^k f(x)| \leq |x^k| \left(\frac{\sin x}{x}\right)^2 \prod_{n=1}^{k} \left|\frac{\sin \lambda_n x}{\lambda_n x}\right|$$

$$\leq \left(\frac{\sin x}{x}\right)^2 (\lambda_1 \cdots \lambda_k)^{-1} = M_k \left(\frac{\sin x}{x}\right)^2. \tag{20}$$

This gives (14), and if we integrate (14) we obtain

$$\frac{1}{\pi} \int_{-\infty}^{\infty} |x^k f(x)|\, dx \leq M_k \qquad (k = 0, 1, 2, \ldots). \tag{21}$$

We have proved that f satisfies the hypotheses of Theorem 19.3. The Fourier transform of f,

$$F(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-itx}\, dx \qquad (t \text{ real}) \tag{22}$$

is therefore a function with compact support, not identically zero, and (21) shows that $F \in C^\infty$ and that

$$(D^k F)(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (-ix)^k f(x) e^{-itx}\, dx, \tag{23}$$

by repeated application of Theorem 9.2(f). Hence $\|D^k F\|_\infty \leq M_k$, by (21), which shows that $F \in C\{M_n\}$.

Hence $C\{M_n\}$ is not quasi-analytic, and the proof is complete. ////

Exercises

1 Suppose f is an entire function of exponential type and

$$\varphi(y) = \int_{-\infty}^{\infty} |f(x + iy)|^2\, dx.$$

Prove that either $\varphi(y) = \infty$ for all real y or $\varphi(y) < \infty$ for all real y. Prove that $f = 0$ if φ is a bounded function.

2 Suppose f is an entire function of exponential type such that the restriction of f to two nonparallel lines belongs to L^2. Prove that $f = 0$.

3 Suppose f is an entire function of exponential type whose restriction to two nonparallel lines is bounded. Prove that f is constant. (Apply Exercise 9 of Chap. 12.)

4 Suppose f is entire, $|f(z)| < C \exp(A|z|)$, and $f(z) = \Sigma a_n z^n$. Put

$$\Phi(w) = \sum_{n=0}^{\infty} \frac{n!\, a_n}{w^{n+1}}.$$

Prove that the series converges if $|w| > A$, that

$$f(z) = \frac{1}{2\pi i} \int_\Gamma \Phi(w) e^{wz}\, dw$$

if $\Gamma(t) = (A + \epsilon)e^{it}$, $0 \le t \le 2\pi$, and that Φ is the function which occurred in the proof of Theorem 19.3. (See also Sec. 19.4.)

5 Suppose f satisfies the hypothesis of Theorem 19.2. Prove that the Cauchy formula

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{f(\xi + i\epsilon)}{\xi + i\epsilon - z}\, d\xi \qquad (0 < \epsilon < y) \tag{*}$$

holds; here $z = x + iy$. Prove that

$$f^*(x) = \lim_{y \to 0} f(x + iy)$$

exists for almost all x. What is the relation between f^* and the function F which occurs in Theorem 19.2? Is (*) true with $\epsilon = 0$ and with f^* in place of f in the integrand?

6 Suppose $\varphi \in L^2(-\infty, \infty)$ and $\varphi > 0$. Prove that there exists an f with $|f| = \varphi$ such that the Fourier transform of f vanishes on a half line if and only if

$$\int_{-\infty}^{\infty} \log \varphi(x)\, \frac{dx}{1 + x^2} > -\infty.$$

Suggestion: Consider f^*, as in Exercise 5, where $f = \exp(u + iv)$ and

$$u(z) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x - t)^2 + y^2} \log \varphi(t)\, dt.$$

7 Let f be a complex function on a closed set E in the plane. Prove that the following two conditions on f are equivalent:

(*a*) There is an open set $\Omega \supset E$ and a function $F \in H(\Omega)$ such that $F(z) = f(z)$ for $z \in E$.

(*b*) To each $\alpha \in E$ there corresponds a neighborhood V_α of α and a function $F_\alpha \in H(V_\alpha)$ such that $F_\alpha(z) = f(z)$ in $V_\alpha \cap E$.

(A special case of this was proved in Theorem 19.9.)

8 Prove that $C\{n!\} = C\{n^n\}$.

9 Prove that there are quasi-analytic classes which are larger than $C\{n!\}$.

10 Put $\lambda_n = M_{n-1}/M_n$, as in the proof of Theorem 19.11. Pick $g_0 \in C_c(R^1)$, and define

$$g_n(x) = (2\lambda_n)^{-1}\int_{-\lambda_n}^{\lambda_n} g_{n-1}(x - t)\, dt \qquad (n = 1, 2, 3, \ldots).$$

Prove directly (without using Fourier transforms or holomorphic functions) that $g = \lim g_n$ is a function which demonstrates that (*e*) implies (*a*) in Theorem 19.11. (You may choose any g_0 that is convenient.)

11 Find an explicit formula for a function $\varphi \in C^\infty$, with support in $[-2, 2]$, such that $\varphi(x) = 1$ if $-1 \le x \le 1$.

12 Prove that to every sequence $\{\alpha_n\}$ of complex numbers there corresponds a function $f \in C^\infty$ such that $(D^n f)(0) = \alpha_n$ for $n = 0, 1, 2, \ldots$. *Suggestion*: If φ is as in Exercise 11, if $\beta_n = \alpha_n/n!$, if $g_n(x) = \beta_n x^n \varphi(x)$, and if

$$f_n(x) = \lambda_n^{-n} g_n(\lambda_n x) = \beta_n x^n \varphi(\lambda_n x),$$

then $\|D^k f_n\|_\infty < 2^{-n}$ for $k = 0, \ldots, n - 1$, provided that λ_n is large enough. Take $f = \Sigma f_n$.

13 Construct a function $f \in C^\infty$ such that the power series

$$\sum_{n=0}^{\infty} \frac{(D^n f)(a)}{n!}(x - a)^n$$

has radius of convergence 0 for every $a \in R^1$. *Suggestion*: Put

$$f(z) = \sum_{k=1}^{\infty} c_k e^{i\lambda_k x},$$

where $\{c_k\}$ and $\{\lambda_k\}$ are sequences of positive numbers, chosen so that $\Sigma c_k \lambda_k^n < \infty$ for $n = 0, 1, 2, \ldots$ and so that $c_n \lambda_n^n$ increases very rapidly and is much larger than the sum of all the other terms in the series $\Sigma c_k \lambda_k^n$.

For instance, put $c_k = \lambda_k^{1-k}$, and choose $\{\lambda_k\}$ so that

$$\lambda_k > 2 \sum_{j=1}^{k-1} c_j \lambda_j^k \quad \text{and} \quad \lambda_k > k^{2k}.$$

14 Suppose $C\{M_n\}$ is quasi-analytic, $f \in C\{M_n\}$, and $f(x) = 0$ for infinitely many $x \in [0, 1]$. What follows?

15 Let X be the vector space of all entire functions f that satisfy $|f(z)| \leq Ce^{\pi|z|}$ for some $C < \infty$, and whose restriction to the real axis is in L^2. Associate with each $f \in X$ its restriction to the integers. Prove that $f \to \{f(n)\}$ is a linear one-to-one mapping of X onto ℓ^2.

16 Assume f is a measurable function on $(-\infty, \infty)$ such that $|f(x)| < e^{-|x|}$ for all x. Prove that its Fourier transform $\hat{f}$ cannot have compact support, unless $f(x) = 0$ a.e.

CHAPTER

TWENTY

UNIFORM APROXIMATION BY POLYNOMIALS

Introduction

20.1 Let K^0 be the interior of a compact set K in the complex plane. (By definition, K^0 is the union of all open discs which are subsets of K; of course, K^0 may be empty even if K is not.) Let $P(K)$ denote the set of all functions on K which are uniform limits of polynomials in z.

Which functions belong to $P(K)$?

Two necessary conditions come to mind immediately: If $f \in P(K)$, then $f \in C(K)$ and $f \in H(K^0)$.

The question arises whether these necessary conditions are also sufficient. The answer is negative whenever K separates the plane (i.e., when the complement of K is not connected). We saw this in Sec. 13.8. On the other hand, if K is an interval on the real axis (in which case $K^0 = \varnothing$), the Weierstrass approximation theorem asserts that

$$P(K) = C(K).$$

So the answer is positive if K is an interval. Runge's theorem also points in this direction, since it states, for compact sets K which do not separate the plane, that $P(K)$ contains at least all those $f \in C(K)$ which have holomorphic extensions to some open set $\Omega \supset K$.

In this chapter we shall prove the theorem of Mergelyan which states, without any superfluous hypotheses, that the above-mentioned necessary conditions are also sufficient if K does not separate the plane.

The principal ingredients of the proof are: Tietze's extension theorem, a smoothing process invoving convolutions, Runge's theorem, and Lemma 20.2, whose proof depends on properties of the class $\mathscr{S}$ which was introduced in Chap. 14.

Some Lemmas

20.2 Lemma *Suppose D is an open disc of radius $r > 0$, $E \subset D$, E is compact and connected, $\Omega = S^2 - E$ is connected, and the diameter of E is at least r. Then there is a function $g \in H(\Omega)$ and a constant b, with the following property: If*

$$Q(\zeta, z) = g(z) + (\zeta - b)g^2(z), \tag{1}$$

the inequalities

$$|Q(\zeta, z)| < \frac{100}{r} \tag{2}$$

$$\left| Q(\zeta, z) - \frac{1}{z - \zeta} \right| < \frac{1{,}000r^2}{|z - \zeta|^3} \tag{3}$$

hold for all $z \in \Omega$ and for all $\zeta \in D$.

We recall that S^2 is the Riemann sphere and that the diameter of E is the supremum of the numbers $|z_1 - z_2|$, where $z_1 \in E$ and $z_2 \in E$.

PROOF We assume, without loss of generality, that the center of D is at the origin. So $D = D(0; r)$.

The implication $(d) \to (b)$ of Theorem 13.11 shows that Ω is simply connected. (Note that $\infty \in \Omega$.) By the Riemann mapping theorem there is therefore a conformal mapping F of U onto Ω such that $F(0) = \infty$. F has an expansion of the form

$$F(w) = \frac{a}{w} + \sum_{n=0}^{\infty} c_n w^n \qquad (w \in U). \tag{4}$$

We define

$$g(z) = \frac{1}{a} F^{-1}(z) \qquad (z \in \Omega), \tag{5}$$

where F^{-1} is the mapping of Ω onto U which inverts F, and we put

$$b = \frac{1}{2\pi i} \int_\Gamma z g(z)\, dz, \tag{6}$$

where Γ is the positively oriented circle with center 0 and radius r.

By (4), Theorem 14.15 can be applied to F/a. It asserts that the diameter of the complement of $(F/a)(U)$ is at most 4. Therefore diam $E \leq 4|a|$. Since diam $E \geq r$, it follows that

$$|a| \geq \frac{r}{4}. \tag{7}$$

Since g is a conformal mapping of Ω onto $D(0; 1/|a|)$, (7) shows that

$$|g(z)| < \frac{4}{r} \qquad (z \in \Omega) \tag{8}$$

and since Γ is a path in Ω, of length $2\pi r$, (6) gives

$$|b| < 4r. \tag{9}$$

If $\zeta \in D$, then $|\zeta| < r$, so (1), (8), and (9) imply

$$|Q| \leq \frac{4}{r} + 5r\left(\frac{16}{r^2}\right) < \frac{100}{r}. \tag{10}$$

This proves (2).

Fix $\zeta \in D$.

If $z = F(w)$, then $zg(z) = wF(w)/a$; and since $wF(w) \to a$ as $w \to 0$, we have $zg(z) \to 1$ as $z \to \infty$. Hence g has an expansion of the form

$$g(z) = \frac{1}{z-\zeta} + \frac{\lambda_2(\zeta)}{(z-\zeta)^2} + \frac{\lambda_3(\zeta)}{(z-\zeta)^3} + \cdots \qquad (|z-\zeta| > 2r). \tag{11}$$

Let Γ_0 be a large circle with center at 0; (11) gives (by Cauchy's theorem) that

$$\lambda_2(\zeta) = \frac{1}{2\pi i}\int_{\Gamma_0} (z-\zeta)g(z)\,dz = b - \zeta. \tag{12}$$

Substitute this value of $\lambda_2(\zeta)$ into (11). Then (1) shows that the function

$$\varphi(z) = \left[Q(\zeta, z) - \frac{1}{z-\zeta}\right](z-\zeta)^3 \tag{13}$$

is bounded as $z \to \infty$. Hence φ has a removable singularity at ∞. If $z \in \Omega \cap D$, then $|z-\zeta| < 2r$, so (2) and (13) give

$$|\varphi(z)| < 8r^3|Q(\zeta, z)| + 4r^2 < 1{,}000r^2. \tag{14}$$

By the maximum modulus theorem, (14) holds for all $z \in \Omega$. This proves (3).

////

20.3 Lemma *Suppose $f \in C_c'(R^2)$, the space of all continuously differentiable functions in the plane, with compact support. Put*

$$\bar{\partial} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right). \tag{1}$$

Then the following "Cauchy formula" holds:

$$f(z) = -\frac{1}{\pi}\iint_{R^2} \frac{(\bar{\partial}f)(\zeta)}{\zeta - z}\,d\xi\,d\eta \qquad (\zeta = \xi + i\eta). \tag{2}$$

PROOF This may be deduced from Green's theorem. However, here is a simple direct proof:

Put $\varphi(r, \theta) = f(z + re^{i\theta})$, $r > 0$, θ real. If $\zeta = z + re^{i\theta}$, the chain rule gives

$$(\bar{\partial} f)(\zeta) = \frac{1}{2} e^{i\theta} \left[\frac{\partial}{\partial r} + \frac{i}{r} \frac{\partial}{\partial \theta} \right] \varphi(r, \theta). \tag{3}$$

The right side of (2) is therefore equal to the limit, as $\epsilon \to 0$, of

$$-\frac{1}{2\pi} \int_\epsilon^\infty \int_0^{2\pi} \left(\frac{\partial \varphi}{\partial r} + \frac{i}{r} \frac{\partial \varphi}{\partial \theta} \right) d\theta \, dr. \tag{4}$$

For each $r > 0$, φ is periodic in θ, with period 2π. The integral of $\partial\varphi/\partial\theta$ is therefore 0, and (4) becomes

$$-\frac{1}{2\pi} \int_0^{2\pi} d\theta \int_\epsilon^\infty \frac{\partial \varphi}{\partial r} \, dr = \frac{1}{2\pi} \int_0^{2\pi} \varphi(\epsilon, \theta) \, d\theta. \tag{5}$$

As $\epsilon \to 0$, $\varphi(\epsilon, \theta) \to f(z)$ uniformly. This gives (2). ////

We shall establish Tietze's extension theorem in the same setting in which we proved Urysohn's lemma, since it is a fairly direct consequence of that lemma.

20.4 Tietze's Extension Theorem *Suppose K is a compact subset of a locally compact Hausdorff space X, and $f \in C(K)$. Then there exists an $F \in C_c(X)$ such that $F(x) = f(x)$ for all $x \in K$.*

(As in Lusin's theorem, we can also arrange it so that $\|F\|_X = \|f\|_K$.)

PROOF Assume f is real, $-1 \le f \le 1$. Let W be an open set with compact closure so that $K \subset W$. Put

$$K^+ = \{x \in K : f(x) \ge \tfrac{1}{3}\}, \qquad K^- = \{x \in K : f(x) \le -\tfrac{1}{3}\}. \tag{1}$$

Then K^+ and K^- are disjoint compact subsets of W. As a consequence of Urysohn's lemma there is a function $f_1 \in C_c(X)$ such that $f_1(x) = \frac{1}{3}$ on K^+, $f_1(x) = -\frac{1}{3}$ on K^-, $-\frac{1}{3} \le f_1(x) \le \frac{1}{3}$ for all $x \in X$, and the support of f_1 lies in W. Thus

$$|f - f_1| \le \tfrac{2}{3} \text{ on } K, \qquad |f_1| \le \tfrac{1}{3} \text{ on } X. \tag{2}$$

Repeat this construction with $f - f_1$ in place of f: There exists an $f_2 \in C_c(X)$, with support in W, so that

$$|f - f_1 - f_2| \le (\tfrac{2}{3})^2 \text{ on } K, \qquad |f_2| \le \tfrac{1}{3} \cdot \tfrac{2}{3} \text{ on } X. \tag{3}$$

In this way we obtain functions $f_n \in C_c(X)$, with supports in W, such that

$$|f - f_1 - \cdots - f_n| \le (\tfrac{2}{3})^n \text{ on } K, \qquad |f_n| \le \tfrac{1}{3} \cdot (\tfrac{2}{3})^{n-1} \text{ on } X. \tag{4}$$

Put $F = f_1 + f_2 + f_3 + \cdots$. By (4), the series converges to f on K, and it converges uniformly on X. Hence F is continuous. Also, the support of F lies in $\overline{W}$. ////

Mergelyan's Theorem

20.5 Theorem *If K is a compact set in the plane whose complement is connected, if f is a continuous complex function on K which is holomorphic in the interior of K, and if $\epsilon > 0$, then there exists a polynomial P such that $|f(z) - P(z)| < \epsilon$ for all $z \in K$.*

If the interior of K is empty, then part of the hypothesis is vacuously satisfied, and the conclusion holds for every $f \in C(K)$. Note that K need not be connected.

PROOF By Tietze's theorem, f can be extended to a continuous function in the plane, with compact support. We fix one such extension, and denote it again by f.

For any $\delta > 0$, let $\omega(\delta)$ be the supremum of the numbers

$$|f(z_2) - f(z_1)|$$

where z_1 and z_2 are subject to the condition $|z_2 - z_1| \leq \delta$. Since f is uniformly continuous, we have

$$\lim_{\delta \to 0} \omega(\delta) = 0. \tag{1}$$

From now on, δ will be fixed. We shall prove that there is a polynomial P such that

$$|f(z) - P(z)| < 10{,}000\omega(\delta) \qquad (z \in K). \tag{2}$$

By (1), this proves the theorem.

Our first objective is the construction of a function $\Phi \in C_c'(R^2)$, such that for all z

$$|f(z) - \Phi(z)| \leq \omega(\delta), \tag{3}$$

$$|(\bar{\partial}\Phi)(z)| < \frac{2\omega(\delta)}{\delta}, \tag{4}$$

and

$$\Phi(z) = -\frac{1}{\pi} \iint_X \frac{(\bar{\partial}\Phi)(\zeta)}{\zeta - z}\, d\xi\, d\eta \qquad (\zeta = \xi + i\eta), \tag{5}$$

where X is the set of all points in the support of Φ whose distance from the complement of K does not exceed δ. (Thus X contains no point which is "far within" K.)

We construct Φ as the convolution of f with a smoothing function A. Put $a(r) = 0$ if $r > \delta$, put

$$a(r) = \frac{3}{\pi\delta^2}\left(1 - \frac{r^2}{\delta^2}\right)^2 \qquad (0 \le r \le \delta), \tag{6}$$

and define

$$A(z) = a(|z|) \tag{7}$$

for all complex z. It is clear that $A \in C_c'(R^2)$. We claim that

$$\iint_{R^2} A = 1, \tag{8}$$

$$\iint_{R^2} \bar{\partial} A = 0, \tag{9}$$

$$\iint_{R^2} |\bar{\partial} A| = \frac{24}{15\delta} < \frac{2}{\delta}. \tag{10}$$

The constants are so adjusted in (6) that (8) holds. (Compute the integral in polar coordinates.) (9) holds simply because A has compact support. To compute (10), express $\bar{\partial} A$ in polar coordinates, as in the proof of Lemma 20.3, and note that $\partial A/\partial\theta = 0$, $|\partial A/\partial r| = -a'(r)$.

Now define

$$\Phi(z) = \iint_{R^2} f(z - \zeta)A(\zeta)\, d\xi\, d\eta = \iint_{R^2} A(z - \zeta)f(\zeta)\, d\xi\, d\eta. \tag{11}$$

Since f and A have compact support, so does Φ. Since

$$\Phi(z) - f(z) = \iint_{R^2} [f(z - \zeta) - f(z)]A(\zeta)\, d\xi\, d\eta \tag{12}$$

and $A(\zeta) = 0$ if $|\zeta| > \delta$, (3) follows from (8). The difference quotients of A converge boundedly to the corresponding partial derivatives of A, since

$A \in C_c'(R^2)$. Hence the last expression in (11) may be differentiated under the integral sign, and we obtain

$$(\bar{\partial}\Phi)(z) = \iint_{R^2} (\bar{\partial}A)(z - \zeta) f(\zeta)\, d\xi\, d\eta$$

$$= \iint_{R^2} f(z - \zeta)(\bar{\partial}A)(\zeta)\, d\xi\, d\eta$$

$$= \iint_{R^2} [f(z - \zeta) - f(z)](\bar{\partial}A)(\zeta)\, d\xi\, d\eta. \tag{13}$$

The last equality depends on (9). Now (10) and (13) give (4). If we write (13) with Φ_x and Φ_y in place of $\bar{\partial}\Phi$, we see that Φ has continuous partial derivatives. Hence Lemma 20.3 applies to Φ, and (5) will follow if we can show that $\bar{\partial}\Phi = 0$ in G, where G is the set of all $z \in K$ whose distance from the complement of K exceeds δ. We shall do this by showing that

$$\Phi(z) = f(z) \qquad (z \in G); \tag{14}$$

note that $\bar{\partial}f = 0$ in G, since f is holomorphic there. (We recall that $\bar{\partial}$ is the Cauchy-Riemann operator defined in Sec. 11.1.) Now if $z \in G$, then $z - \zeta$ is in the interior of K for all ζ with $|\zeta| < \delta$. The mean value property for harmonic functions therefore gives, by the first equation in (11),

$$\Phi(z) = \int_0^\delta a(r) r\, dr \int_0^{2\pi} f(z - re^{i\theta})\, d\theta$$

$$= 2\pi f(z) \int_0^\delta a(r) r\, dr = f(z) \iint_{R^2} A = f(z) \tag{15}$$

for all $z \in G$.

We have now proved (3), (4), and (5).

The definition of X shows that X is compact and that X can be covered by finitely many open discs $D_1, \ldots, D_n$, of radius 2δ, whose centers are *not* in K. Since $S^2 - K$ is connected, the center of each D_j can be joined to ∞ by a polygonal path in $S^2 - K$. It follows that each D_j contains a compact connected set E_j, of diameter at least 2δ, so that $S^2 - E_j$ is connected and so that $K \cap E_j = \varnothing$.

We now apply Lemma 20.2, with $r = 2\delta$. There exist functions

$g_j \in H(S^2 - E_j)$ and constants b_j so that the inequalities

$$|Q_j(\zeta, z)| < \frac{50}{\delta}, \tag{16}$$

$$\left| Q_j(\zeta, z) - \frac{1}{z - \zeta} \right| < \frac{4{,}000\delta^2}{|z - \zeta|^3} \tag{17}$$

hold for $z \notin E_j$ and $\zeta \in D_j$, if

$$Q_j(\zeta, z) = g_j(z) + (\zeta - b_j)g_j^2(z). \tag{18}$$

Let Ω be the complement of $E_1 \cup \cdots \cup E_n$. Then Ω is an open set which contains K.

Put $X_1 = X \cap D_1$ and $X_j = (X \cap D_j) - (X_1 \cup \cdots \cup X_{j-1})$, for $2 \leq j \leq n$. Define

$$R(\zeta, z) = Q_j(\zeta, z) \qquad (\zeta \in X_j, z \in \Omega) \tag{19}$$

and

$$F(z) = \frac{1}{\pi} \iint_X (\bar{\partial}\Phi)(\zeta) R(\zeta, z)\, d\xi\, d\eta \qquad (z \in \Omega). \tag{20}$$

Since

$$F(z) = \sum_{j=1}^{n} \frac{1}{\pi} \iint_{X_j} (\bar{\partial}\Phi)(\zeta) Q_j(\zeta, z)\, d\xi\, d\eta, \tag{21}$$

(18) shows that F is a finite linear combination of the functions g_j and g_j^2. Hence $F \in H(\Omega)$.

By (20), (4), and (5) we have

$$|F(z) - \Phi(z)| < \frac{2\omega(\delta)}{\pi\delta} \iint_X \left| R(\zeta, z) - \frac{1}{z - \zeta} \right| d\xi\, d\eta \qquad (z \in \Omega). \tag{22}$$

Observe that the inequalities (16) and (17) are valid with R in place of Q_j if $\zeta \in X$ and $z \in \Omega$. For if $\zeta \in X$ then $\zeta \in X_j$ for some j, and then $R(\zeta, z) = Q_j(\zeta, z)$ for all $z \in \Omega$.

Now fix $z \in \Omega$, put $\zeta = z + \rho e^{i\theta}$, and estimate the integrand in (22) by (16) if $\rho < 4\delta$, by (17) if $4\delta \leq \rho$. The integral in (22) is then seen to be less than the sum of

$$2\pi \int_0^{4\delta} \left(\frac{50}{\delta} + \frac{1}{\rho} \right) \rho\, d\rho = 808\pi\delta \tag{23}$$

and

$$2\pi \int_{4\delta}^{\infty} \frac{4{,}000\delta^2}{\rho^3}\, \rho\, d\rho = 2{,}000\pi\delta. \tag{24}$$

Hence (22) yields

$$|\tilde{F}(z) - \Phi(z)| < 6{,}000\omega(\delta) \qquad (z \in \Omega). \tag{25}$$

Since $F \in H(\Omega)$, $K \subset \Omega$, and $S^2 - K$ is connected, Runge's theorem shows that F can be uniformly approximated on K by polynomials. Hence (3) and (25) show that (2) can be satisfied.

This completes the proof. ////

One unusual feature of this proof should be pointed out. We had to prove that the given function f is in the closed subspace $P(K)$ of $C(K)$. (We use the terminology of Sec. 20.1.) Our first step consisted in approximating f by Φ. But this step took us outside $P(K)$, since Φ was so constructed that in general Φ will not be holomorphic in the whole interior of K. Hence Φ is at some positive distance from $P(K)$. However, (25) shows that this distance is less than a constant multiple of $\omega(\delta)$. [In fact, having proved the theorem, we know that this distance is at most $\omega(\delta)$, by (3), rather than 6,000 $\omega(\delta)$.] The proof of (25) depends on the inequality (4) and on the fact that $\bar{\partial}\Phi = 0$ in G. Since holomorphic functions φ are characterized by $\bar{\partial}\varphi = 0$, (4) may be regarded as saying that Φ is not too far from being holomorphic, and this interpretation is confirmed by (25).

Exercises

1 Extend Mergelyan's theorem to the case in which $S^2 - K$ has finitely many components: Prove that then every $f \in C(K)$ which is holomorphic in the interior of K can be uniformly approximated on K by rational functions.

2 Show that the result of Exercise 1 does not extend to arbitrary compact sets K in the plane, by verifying the details of the following example. For $n = 1, 2, 3, \ldots$, let $D_n = D(\alpha_n; r_n)$ be disjoint open discs in U whose union V is dense in U, such that $\Sigma r_n < \infty$. Put $K = \bar{U} - V$. Let Γ and γ_n be the paths

$$\Gamma(t) = e^{it}, \qquad \gamma_n(t) = \alpha_n + r_n e^{it}, \qquad 0 \le t \le 2\pi,$$

and define

$$L(f) = \int_{\Gamma} f(z)\, dz - \sum_{n=1}^{\infty} \int_{\gamma_n} f(z)\, dz \qquad (f \in C(K)).$$

Prove that L is a bounded linear functional on $C(K)$, prove that $L(R) = 0$ for every rational function R whose poles are outside K, and prove that there exists an $f \in C(K)$ for which $L(f) \neq 0$.

3 Show that the function g constructed in the proof of Lemma 20.2 has the smallest supremum norm among all $f \in H(\Omega)$ such that $zf(z) \to 1$ as $z \to \infty$. (This motivates the proof of the lemma.)

Show also that $b = c_0$ in that proof and that the inequality $|b| < 4r$ can therefore be replaced by $|b| < r$. In fact, b lies in the convex hull of the set E.

APPENDIX
HAUSDORFF'S MAXIMALITY THEOREM

We shall first prove a lemma which, when combined with the axiom of choice, leads to an almost instantaneous proof of Theorem 4.21.

If $\mathscr{F}$ is a collection of sets and $\Phi \subset \mathscr{F}$, we call Φ a *subchain* of $\mathscr{F}$ provided that Φ is totally ordered by set inclusion. Explicitly, this means that if $A \in \Phi$ and $B \in \Phi$, then either $A \subset B$ or $B \subset A$. The union of all members of Φ will simply be referred to as the *union of* Φ.

Lemma *Suppose $\mathscr{F}$ is a nonempty collection of subsets of a set X such that the union of every subchain of $\mathscr{F}$ belongs to $\mathscr{F}$. Suppose g is a function which associates to each $A \in \mathscr{F}$ a set $g(A) \in \mathscr{F}$ such that $A \subset g(A)$ and $g(A) - A$ consists of at most one element. Then there exists an $A \in \mathscr{F}$ for which $g(A) = A$.*

PROOF Fix $A_0 \in \mathscr{F}$. Call a subcollection $\mathscr{F}'$ of $\mathscr{F}$ a *tower* if $\mathscr{F}'$ has the following three properties:

(*a*) $A_0 \in \mathscr{F}'$.
(*b*) The union of every subchain of $\mathscr{F}'$ belongs to $\mathscr{F}'$.
(*c*) If $A \in \mathscr{F}'$, then also $g(A) \in \mathscr{F}'$.

The family of all towers is nonempty. For if $\mathscr{F}_1$ is the collection of all $A \in \mathscr{F}$ such that $A_0 \subset A$, then $\mathscr{F}_1$ is a tower. Let $\mathscr{F}_0$ be the intersection of all towers. Then $\mathscr{F}_0$ is a tower (the verification is trivial), but no proper subcollection of $\mathscr{F}_0$ is a tower. Also, $A_0 \subset A$ if $A \in \mathscr{F}_0$. The idea of the proof is to show that $\mathscr{F}_0$ is a subchain of $\mathscr{F}$.

Let Γ be the collection of all $C \in \mathscr{F}_0$ such that every $A \in \mathscr{F}_0$ satisfies either $A \subset C$ or $C \subset A$.

For each $C \in \Gamma$, let $\Phi(C)$ be the collection of all $A \in \mathscr{F}_0$ such that either $A \subset C$ or $g(C) \subset A$.

Properties (a) and (b) are clearly satisfied by Γ and by each $\Phi(C)$. Fix $C \in \Gamma$, and suppose $A \in \Phi(C)$. We want to prove that $g(A) \in \Phi(C)$. If $A \in \Phi(C)$, there are three possibilities: Either $A \subset C$ and $A \neq C$, or $A = C$, or $g(C) \subset A$. If A is a proper subset of C, then C cannot be a proper subset of $g(A)$, otherwise $g(A) - A$ would contain at least two elements; since $C \in \Gamma$, it follows that $g(A) \subset C$. If $A = C$, then $g(A) = g(C)$. If $g(C) \subset A$, then also $g(C) \subset g(A)$, since $A \subset g(A)$. Thus $g(A) \in \Phi(C)$, and we have proved that $\Phi(C)$ is a tower. The minimality of $\mathscr{F}_0$ implies now that $\Phi(C) = \mathscr{F}_0$, for every $C \in \Gamma$.

In other words, if $A \in \mathscr{F}_0$ and $C \in \Gamma$, then either $A \subset C$ or $g(C) \subset A$. But this says that $g(C) \in \Gamma$. Hence Γ is a tower, and the minimality of $\mathscr{F}_0$ shows that $\Gamma = \mathscr{F}_0$. It follows now from the definition of Γ that $\mathscr{F}_0$ is totally ordered.

Let A be the union of $\mathscr{F}_0$. Since $\mathscr{F}_0$ satisfies (b), $A \in \mathscr{F}_0$. By (c), $g(A) \in \mathscr{F}_0$. Since A is the largest member of $\mathscr{F}_0$ and since $A \subset g(A)$, it follows that $A = g(A)$. ////

Definition A *choice function* for a set X is a function f which associates to each nonempty subset E of X an element of E: $f(E) \in E$.

In more informal terminology, f "chooses" an element out of each nonempty subset of X.

The Axiom of Choice *For every set there is a choice function.*

Hausdorff's Maximality Theorem *Every nonempty partially ordered set P contains a maximal totally ordered subset.*

PROOF Let $\mathscr{F}$ be the collection of all totally ordered subsets of P. Since every subset of P which consists of a single element is totally ordered, $\mathscr{F}$ is not empty. Note that the union of any chain of totally ordered sets is totally ordered.

Let f be a choice function for P. If $A \in \mathscr{F}$, let A^* be the set of all x in the complement of A such that $A \cup \{x\} \in \mathscr{F}$. If $A^* \neq \varnothing$, put

$$g(A) = A \cup \{f(A^*)\}.$$

If $A^* = \varnothing$, put $g(A) = A$.

By the lemma, $A^* = \varnothing$ for at least one $A \in \mathscr{F}$, and any such A is a maximal element of $\mathscr{F}$. ////

NOTES AND COMMENTS

Chapter 1

The first book on the modern theory of integration and differentiation is Lebesgue's "Leçons sur l'intégration," published in 1904.

For an illuminating history of the earlier attempts that were made toward the construction of a satisfactory theory of integration, see [42]†, which contains interesting discussions of the difficulties that even first-rate mathematicians had with simple set-theoretic concepts before these were properly defined and well understood.

The approach to abstract integration presented in the text is inspired by Saks [28]. Greater generality can be attained if σ-algebras are replaced by σ-rings (*Axioms*: $\bigcup A_i \in \mathscr{R}$ and $A_1 - A_2 \in \mathscr{R}$ if $A_i \in \mathscr{R}$ for $i = 1, 2, 3, \ldots$; it is not required that $X \in \mathscr{R}$), but at the expense of a necessarily fussier definition of measurability. See Sec. 18 of [7]. In all classical applications the measurability of X is more or less automatic. This is the reason for our choice of the somewhat simpler theory based on σ-algebras.

Sec. 1.11. This definition of $\mathscr{B}$ is as in [12]. In [7] the Borel sets are defined as the σ-ring generated by the compact sets. In spaces which are not σ-compact, this is a smaller family than ours.

Chapter 2

Sec. 2.12. The usual statement of Urysohn's lemma is: If K_0 and K_1 are disjoint closed sets in a *normal* Hausdorff space X, then there is a continuous function on X which is 0 on K_0 and 1 on K_1. The proof is exactly as in the text.

Sec. 2.14. The original form of this theorem, with $X = [0, 1]$, is due to F. Riesz (1909). See [5], pp. 373, 380–381, and [12], pp. 134–135 for its further history. The theorem is here presented in the same generality as in [12]. The set function μ which is defined for all subsets of X in the proof of Theorem 2.14 is called an *outer measure* because of its countable subadditivity (Step I). For systematic exploitations (originated by Carathéodory) of this notion, see [25] and [28].

Sec. 2.20. For direct constructions of Lebesgue measure, along more classical lines, see [31], [35], [26] and [53].

† Numbers in brackets refer to the Bibliography.

Sec. 2.21. A proof that the cardinality of every countably generated σ-algebra is $\leq c$ may be found on pp. 133–134 of [44]. That this cardinality is either finite or $\geq c$ should be clear after doing Exercise 1 of Chap. 1.

Sec. 2.22. A very instructive paper on the subject of nonmeasurable sets in relation to measures invariant under a group is: J. von Neumann, Zur allgemeinen Theorie des Masses, *Fundamenta Math.*, vol. 13, pp. 73–116, 1929. See also Halmos's article in the special (May, 1958) issue of *Bull. Am. Math. Soc.* devoted to von Neumann's work.

Sec. 2.24. [28], p. 72.

Sec. 2.25. [28], p. 75. There is another approach to the Lebesgue theory of integration, due to Daniell (*Ann. Math.*, vol. 19, pp. 279–294, 1917–1918) based on extensions of positive linear functionals. When applied to $C_c(X)$ it leads to a construction which virtually turns the Vitali-Carathéodory theorem into the definition of measurability. See [17] and, for the full treatment, [18].

Exercise 8. Two interesting extensions of this phenomenon have appeared in *Amer. Math. Monthly*: see vol. 79, pp. 884–886, 1972 (R. B. Kirk) and vol. 91, pp. 564–566, 1984 (F. S. Cater).

Exercise 17. This example appears in A Theory of Radon Measures on Locally Compact Spaces, by R. E. Edwards, *Acta Math.*, vol. 89, p. 160, 1953. Its existence was unfortunately overlooked in [27].

Exercise 18. [7], p. 231; originally due to Dieudonné.

Chapter 3

The best general reference is [9]. See also Chap. 1 of [36].

Exercise 3. Vol. 1 (1920) of *Fundamenta Math.* contains three papers relevant to the parenthetical remark.

Exercise 7. A very complete answer to this question was found by A. Villani, in *Am. Math. Monthly*, vol. 92, pp. 485–487, 1985.

Exercise 16. [28], p. 18. When t ranges over all positive real numbers, a measurability problem arises in the suggested proof. This is the reason for including (ii) as a hypothesis. See W. Walter, *Am. Math. Monthly*, vol. 84, pp. 118–119, 1977.

Exercise 17. The second suggested proof of part (*b*) was published by W. P. Novinger, in *Proc. Am. Math. Soc.*, vol. 34, pp. 627–628, 1972. It was also discovered by David Hall.

Exercise 18. Convergence in measure is a natural concept in probability theory. See [7], Chap. IX.

Chapter 4

There are many books dealing with Hilbert space theory. We cite [6] and [24] as main references. See also [5], [17], and [19].

The standard work on Fourier series is [36]. For simpler introductions, see [10], [31], [43], and [45].

Exercise 2. This is the so-called Gram-Schmidt orthogonalization process.

Exercise 18. The functions that are uniform limits (on R^1) of members of X are called *almost periodic.*

Chapter 5

The classical work here is [2]. More recent treatises are [5], [14], and [24]. See also [17] and [49].

Sec. 5.7. The relations between measure theory on the one hand and Baire's theorem on the other are discussed in great detail in [48].

Sec. 5.11. Although there are continuous functions whose Fourier series diverge on a dense G_δ, the set of divergence must have measure 0. This was proved by L. Carleson (*Acta Math.*, vol. 116, pp. 135–157, 1966) for all f in $L^2(T)$; the proof was extended to $L^p(T)$, $p > 1$, by R. A. Hunt. See also [45], especially Chap. II.

Sec. 5.22. For a deeper discussion of representing measures see Arens and Singer, *Proc. Am. Math. Soc.*, vol. 5, pp. 735–745, 1954. Also [39], [52].

Chapter 6

Sec. 6.3. The constant $1/\pi$ is best possible. See R. P. Kaufman and N. W. Rickert, *Bull. Am. Math. Soc.*, vol. 72, pp. 672–676, 1966, and (for a simpler treatment) W. W. Bledsoe, *Am. Math. Monthly*, vol. 77, pp. 180–182, 1970.

Sec. 6.10. von Neumann's proof is in a section on measure theory in his paper: On Rings of Operators, III, *Ann. Math.*, vol. 41, pp. 94–161, 1940. See pp. 124–130.

Sec. 6.15. The phenomenon $L^\infty \neq (L^1)^*$ is discussed by J. T. Schwartz in *Proc. Am. Math. Soc.*, vol. 2, pp. 270–275, 1951, and by H. W. Ellis and D. O. Snow in *Can. Math. Bull.*, vol. 6, pp. 211–229, 1963. See also [7], p. 131, and [28], p. 36.

Sec. 6.19. The references to Theorem 2.14 apply here as well.

Exercise 6. See [17], p. 43.

Exercise 10(*g*). See [36], vol. I, p. 167.

Chapter 7

Sec. 7.3. This simple covering lemma seems to appear for the first time in a paper by Wiener on the ergodic theorem (*Duke Math. J.*, vol. 5, pp. 1-18, 1939). Covering lemmas play a central role in the theory of differentiation. See [50], [53], and, for a very detailed treatment, [41].

Sec. 7.4. Maximal functions were first introduced by Hardy and Littlewood, in *Acta Math.*, vol. 54, pp. 81–116, 1930. That paper contains proofs of Theorems 8.18, 11.25(*b*), and 17.11.

Sec. 7.21. The same conclusion can be obtained under somewhat weaker hypotheses; see [16], Theorems 260–264. Note that the proof of Theorem 7.21 uses the existence and integrability of only the *right-hand* derivative of f, plus the continuity of f. For a further refinement, see P. L. Walker, *Amer. Math. Monthly*, vol. 84, pp. 287–288, 1977.

Secs. 7.25, 7.26. This treatment of the change of variables formula is quite similar to D. Varberg's in *Amer. Math. Monthly*, vol. 78, pp. 42–45, 1971.

Exercise 5. A very simple proof, due to K. Stromberg, is in *Proc. Amer. Math. Soc.*, vol. 36, p. 308, 1972.

Exercise 12. For an elementary proof that every monotone function (hence every function of bounded variation) is differentiable a.e., see [24], pp. 5–9. In that work, this theorem is made the starting point of the Lebesgue theory. Another, even simpler, proof by D. Austin is in *Proc. Amer. Math. Soc.*, vol. 16, pp. 220–221, 1965.

Exercise 18. These functions φ_n are the so-called Rademacher functions. Chap. V of [36] contains further theorems about them.

Chapter 8

Fubini's theorem is developed here as in [7] and [28]. For a different approach, see [25]. Sec. 8.9(*c*) is in *Fundamenta Math.*, vol. 1, p. 145, 1920.

Sec. 8.18. This proof of the Hardy-Littlewood theorem (see the reference to Sec. 7.4) is essentially that of a very special case of the Marcinkiewicz interpolation theorem. A full discussion of the latter may be found in Chap. XII of [36]. See also [50].

Exercise 2. Corresponding to the idea that an integral is an area under a curve, the theory of the Lebesgue integral can be developed in terms of measures of ordinate sets. This is done in [16].

Exercise 8. Part (*b*), in even more precise form, was proved by Lebesgue in *J. Mathématiques*, ser. 6, vol. 1, p. 201, 1905, and seems to have been forgotten. It is quite remarkable in view of another example of Sierpinski (*Fundamenta Math.*, vol. 1, p. 114, 1920): There is a plane set E which is not Lebesgue measurable and which has at most two points on each straight line. If $f = \chi_E$, then f is not Lebesgue measurable, although all of the sections f_x and f^y are upper semicontinuous; in fact, each has at most two points of discontinuity. (This example depends on the axiom of choice, but not on the continuum hypothesis.)

Chapter 9

For another brief introduction, see [36], chap. XVI. A different proof of Plancherel's theorem is in [33]. Group-theoretic aspects and connections with Banach algebras are discussed in [17], [19], and [27]. For more on invariant subspaces (Sec. 9.16) see [11]; the corresponding problem in L^1 is described in [27], Chap. 7.

Chapter 10

General references: [1], [4], [13], [20], [29], [31] and [37].

Sec. 10.8. Integration can also be defined over arbitrary rectifiable curves. See [13], vol. I, Appendix C.

Sec. 10.10. The topological concept of index is applied in [29] and is fully utilized in [1]. The computational proof of Theorem 10.10 is as in [1], p. 93.

Sec. 10.13. Cauchy published his theorem in 1825, under the additional assumption that f' is continuous. Goursat showed that this assumption is redundant, and stated the theorem in its present form. See [13], p. 163, for further historical remarks.

Sec. 10.16. The standard proofs of the power series representation and of the fact that $f \in H(\Omega)$ implies $f' \in H(\Omega)$ proceed via the Cauchy integral formula, as here. Recently proofs have been constructed which use the winding number but make no appeal to integration. For details see [34].

Sec. 10.25. A very elementary proof of the algebraic completeness of the complex field is in [26], p. 170.

Sec. 10.30. The proof of part (*b*) is as in [47].

Sec. 10.32. The open mapping theorem and the discreteness of $Z(f)$ are topological properties of the class of all nonconstant holomorphic functions which characterize this class up to homeomorphisms. This is Stoilov's theorem. See [34].

Sec. 10.35. This strikingly simple and elementary proof of the global version of Cauchy's theorem was discovered by John D. Dixon, *Proc. Am. Math. Soc.*, vol. 29, pp. 625–626, 1971. In [1] the proof is based on the theory of exact differentials. In the first edition of the present book it was deduced from Runge's theorem. That approach was used earlier in [29], p. 177. There, however, it was applied in simply connected regions only.

Chapter 11

General references: [1], Chap. 5; [20], Chap. 1.

Sec. 11.14. The reflection principle was used by H. A. Schwarz to solve problems concerning conformal mappings of polygonal regions. See Sec. 17.6 of [13]. Further results along these lines were obtained by Carathéodory; see [4], vol. II, pp. 88–92, and *Commentarii Mathematici Helvetici*, vol. 19, pp. 263–278, 1946–1947.

Secs. 11.20, 11.25. This is the principal result of the Hardy-Littlewood paper mentioned in the reference to Sec. 7.4. The proof of the second inequality in Theorem 11.20 is as in [40], p. 23.

Sec. 11.23. The first theorems of this type are in Fatou's thesis, Séries trigonométriques et séries de Taylor, *Acta Math.*, vol. 30, pp. 335–400, 1906. This is the first major work in which Lebesgue's theory of integration is applied to the study of holomorphic functions.

Sec. 11.30. Part (*c*) is due to Herglotz, *Leipziger Berichte*, vol. 63, pp. 501–511, 1911.

Exercise 14. This was suggested by W. Ramey and D. Ullrich.

Chapter 12

Sec. 12.7. For further examples, see [31], pp. 176–186.

Sec. 12.11. This theorem was first proved for trigonometric series by W. H. Young (1912; $q = 2, 4, 6, \ldots$) and F. Hausdorff (1923; $2 \leq q \leq \infty$). F. Riesz (1923) extended it to uniformly bounded

orthonormal sets, M. Riesz (1926) derived this extension from a general interpolation theorem, and G. O. Thorin (1939) discovered the complex-variable proof of M. Riesz's theorem. The proof of the text is the Calderón-Zygmund adaptation (1950) of Thorin's idea. Full references and discussions of other interpolation theorems are in Chap. XII of [36].

Sec. 12.13. In slightly different form, this is in *Duke Math. J.*, vol. 20, pp. 449–458, 1953.

Sec. 12.14. This proof is essentially that of R. Kaufman (*Math. Ann.*, vol. 169, p. 282, 1967). E. L. Stout (*Math. Ann.*, vol. 177, pp. 339–340, 1968) obtained a stronger result.

Chapter 13

Sec. 13.9. Runge's theorem was published in *Acta Math.*, vol. 6, 1885. (Incidentally, this is the same year in which the Weierstrass theorem on uniform approximation by polynomials on an interval was published; see *Mathematische Werke*, vol. 3, pp. 1–37.) See [29], pp. 171–177, for a proof which is close to the original one. The functional analysis proof of the text is known to many analysts and has probably been independently discovered several times in recent years. It was communicated to me by L. A. Rubel. In [13], vol. II, pp. 299–308, attention is paid to the closeness of the approximation if the degree of the polynomial is fixed.

Exercises 5, 6. For yet another method, see D. G. Cantor, *Proc. Am. Math. Soc.*, vol. 15, pp. 335–336, 1964.

Chapter 14

General reference: [20]. Many special mapping functions are described there in great detail.

Sec. 14.3. More details on linear fractional transformations may be found in [1], pp. 22–35; in [13], pp. 46–57; in [4]; and especially in Chap. 1 of L. R. Ford's book "Automorphic Functions," McGraw-Hill Book Company, New York, 1929.

Sec. 14.5. Normal families were introduced by Montel. See Chap. 15 of [13].

Sec. 14.8. The history of Riemann's theorem is discussed in [13], pp. 320–321, and in [29], p. 230. Koebe's proof (Exercise 26) is in *J. für reine und angew. Math.*, vol. 145, pp. 177–223, 1915; doubly connected regions are also considered there.

Sec. 14.14. Much more is true than just $|a_2| \leq 2$: In fact, $|a_n| \leq n$ for all $n \geq 2$. This was conjectured by Bieberbach in 1916, and proved by L. de Branges in 1984 [*Acta Math.*, vol. 154, pp. 137–152, (1985)]. Moreover, if $|a_n| = n$ for just one $n \geq 2$, then f is one of the Koebe functions of Example 14.11.

Sec. 14.19. The boundary behavior of conformal mappings was investigated by Carathéodory in *Math. Ann.*, vol. 73, pp. 323–370, 1913. Theorem 14.19 was proved there for regions bounded by Jordan curves, and the notion of prime ends was introduced. See also [4], vol. II, pp. 88–107.

Exercise 24. This proof is due to Y. N. Moschovakis.

Chapter 15

Sec. 15.9. The relation between canonical products and entire functions of finite order is discussed in Chap. 2 of [3], Chap. VII of [29], and Chap. VIII of [31].

Sec. 15.25. See Szasz, *Math. Ann.*, vol. 77, pp. 482–496, 1916, for further results in this direction. Also Chap. II of [21].

Chapter 16

The classical work on Riemann surfaces is [32]. (The first edition appeared in 1913.) Other references: Chap. VI of [1], Chap. 10 of [13], Chap. VI of [29], and [30].

Sec. 16.5. Ostrowski's theorem is in *J. London Math. Soc.*, vol. 1, pp. 251–263, 1926. See J.-P. Kahane, Lacunary Taylor and Fourier Series, *Bull. Am. Math. Soc.*, vol. 70, pp. 199–213, 1964, for a more recent account of gap series.

Sec. 16.15. The approach to the monodromy theorem was a little simpler in the first edition of this book. It used the fact that every simply connected plane region is homeomorphic to a convex

one, namely U. The present proof is so arranged that it applies without change to holomorphic functions of several complex variables. (Note that when $k > 2$ there exist simply connected open sets in R^k which are not homeomorphic to any convex set; spherical shells furnish examples of this.)

Sec. 16.17. Chap. 13 of [13], Chap. VIII of [29], and part 7 of [4].

Sec. 16.21. Picard's big theorem is proved with the aid of modular functions in part 7 of [4]. "Elementary" proofs may be found in [31], pp. 277–284, and in Chap. VII of [29].

Exercise 10. Various classes of removable sets are discussed by Ahlfors and Beurling in Conformal Invariants and Function-theoretic Null-Sets, *Acta Math.*, vol. 83, pp. 101–129, 1950.

Chapter 17

The classical reference here is [15]. See also [36], Chap. VII. Although [15] deals mainly with the unit disc, most proofs are so constructed that they apply to other situations which are described there. Some of these generalizations are in Chap. 8 of [27]. Other recent books on these topics are [38], [40], and [46].

Sec. 17.1. See [22] for a thorough treatment of subharmonic functions.

Sec. 17.13. For a different proof, see [15], or the paper by Helson and Lowdenslager in *Acta Math.*, vol. 99, pp. 165–202, 1958. An extremely simple proof was found by B. K. Øksendal, in *Proc. Amer. Math. Soc.*, vol. 30, p. 204, 1971.

Sec. 17.14. The terms "inner function" and "outer function" were coined by Beurling in the paper in which Theorem 17.21 was proved: On Two Problems Concerning Linear Transformations in Hilbert Space, *Acta Math.*, vol. 81, pp. 239–255, 1949. For further developments, see [11].

Secs. 17.25, 17.26. This proof of M. Riesz's theorem is due to A. P. Calderón. See *Proc. Am. Math. Soc.*, vol. 1, pp. 533–535, 1950. See also [36], vol. I, pp. 252–262.

Exercise 3. This forms the basis of a definition of H^p-spaces in other regions. See *Trans. Am. Math. Soc.*, vol. 78, pp. 46–66, 1955.

Chapter 18

General references: [17], [19], and [23]; also [14]. The theory was originated by Gelfand in 1941.

Sec. 18.18. This was proved in elementary fashion by P. J. Cohen in *Proc. Am. Math. Soc.*, vol. 12, pp. 159–163, 1961.

Sec. 18.20. This theorem is Wermer's, *Proc. Am. Math. Soc.*, vol. 4, pp. 866–869, 1953. The proof of the text is due to Hoffman and Singer. See [15], pp. 93–94, where an extremely short proof by P. J. Cohen is also given. (See the reference to Sec. 18.18.)

Sec. 18.21. This was one of the major steps in Wiener's original proof of his Tauberian theorem. See [33], p. 91. The painless proof given in the text was the first spectacular success of the Gelfand theory.

Exercise 14. The set Δ can be given a compact Hausdorff topology with respect to which the function $\hat{x}$ is continuous. Thus $x \to \hat{x}$ is a homomorphism of A into $C(\Delta)$. This representation of A as an algebra of continuous functions is a most important tool in the study of commutative Banach algebras.

Chapter 19

Secs. 19.2, 19.3: [21], pp. 1–13. See also [3], where functions of exponential type are the main subject.

Sec. 19.5. For a more detailed introduction to the classes $C\{M_n\}$, see S. Mandelbrojt, "Séries de Fourier et classes quasi-analytiques," Gauthier-Villars, Paris, 1935.

Sec. 19.11. In [21], the proof of this theorem is based on Theorem 19.2 rather than on 19.3.

Exercise 4. The function Φ is called the Borel transform of f. See [3], Chap. 5.

Exercise 12. The suggested proof is due to H. Mirkil, *Proc. Am. Math. Soc.*, vol. 7, pp. 650–652, 1956. The theorem was proved by Borel in 1895.

Chapter 20

See S. N. Mergelyan, Uniform Approximations to Functions of a Complex Variable, *Uspehi Mat. Nauk* (N.S.) 7, no. 2 (48), 31–122, 1952; Amer. Math. Soc. Translation No. 101, 1954. Our Theorem 20.5 is Theorem 1.4 in Mergelyan's paper.

A functional analysis proof, based on measure-theoretic considerations, has recently been published by L. Carleson in *Math. Scandinavica*, vol. 15, pp. 167–175, 1964.

Appendix

The maximality theorem was first stated by Hausdorff on p. 140 of his book "Grundzüge der Mengenlehre," 1914. The proof of the text is patterned after Sec. 16 of Halmos's book [8]. The idea to choose g so that $g(A) - A$ has at most one element appears there, as does the term "tower." The proof is similar to one of Zermelo's proofs of the well-ordering theorem; see *Math. Ann.*, vol. 65, pp. 107–128, 1908.

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LIST OF SPECIAL SYMBOLS AND ABBREVIATIONS†

$\exp(z)$	1	$\hat{x}(\alpha)$	82
τ	8	T	88
$\mathfrak{M}$	8	$L^p(T)$, $C(T)$	88
χ_E	11	Z	89
lim sup	14	$\hat{f}(n)$	91
lim inf	14	c_0	104
f^+, f^-	15	$\|f\|_E$	108
$L^1(\mu)$	24	U	110
a.e.	27	$P_r(\theta - t)$	111
$\bar{E}$	35	Lip α	113
$C_c(X)$	38	$\|\mu\|(E)$	116
$K \prec f \prec V$	39	μ^+, μ^-	119
$\mathfrak{M}_F$	42	$\lambda \ll \mu$	120
m, m_k	51	$\lambda_1 \perp \lambda_2$	120
$\Delta(T)$	51, 150	$B(x, r)$	136
$L^1(R^k)$, $L^1(E)$	53	$Q_r \mu$	136
$\|f\|_p$, $\|f\|_\infty$	65, 66	$D\mu$	136, 241
$L^p(\mu)$, $L^p(R^k)$, ℓ^p	65	$M\mu$	136, 241
$L^\infty(\mu)$, $L^\infty(R^k)$, ℓ^∞	66	Mf	138
$C_0(X)$, $C(X)$	70	AC	145
(x, y)	76	$T'(x)$	150
$\|x\|$	76	J_T	150
$x \perp y$, $M^\perp$	79	BV	157

† The standard set-theoretic symbols are described on pages 6 to 8 and are not listed here.

E_x, E^y	161	$P[d\mu]$	240
f_x, f^y	162	Ω_α	240
$\mu \times \lambda$	164	N_α	241
$f * g$	171	M_{rad}	241
$\mu * \lambda$	175	σ	241
$\hat{f}(t)$	178	H^∞	248
C^∞	194	$f^*(e^{it})$	249
C_c^∞	194	$\varphi_\alpha(z)$	254
$D(a; r)$, $D'(a; r)$, $\bar{D}(a; r)$	196	S^2	266
Ω	197	$\mathscr{S}$	285
$H(\Omega)$	197	$E_p(z)$	301
γ, γ^*	200	$\log^+ t$	311
$\partial\Delta$	202	N	311
$\text{Ind}_\gamma(z)$	203	$(f_0, D_0) \sim (f_1, D_1)$	323
$Z(f)$	208	H^p	338
$\dot{+}$	217	M_f, Q_f	344
$I^2 = I \times I$	222	$\sigma(x)$	357
∂, $\bar{\partial}$	231	$\rho(x)$	360
Δ	232	$C\{M_n\}$	377
$P[f]$	233	$P(K)$	386
Π^+, Π^-	237	$C_c'(R^2)$	388

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部分习题答案

chapter one

3. Prove that if f is a real function on a measurable space X such that $\{x : f(x) \geq r\}$ is measurable for every rational r, then f is measurable.

[**proof**]:For each real number α, there exists an descending sequence $\{r_n\}$ of rational numbers such that $\lim_{n\to\infty} r_n = \alpha$. Moreover,we have

$$(\alpha, +\infty) = \bigcup_{n=1}^{\infty} [r_n, +\infty).$$

Hence,

$$f^{-1}((\alpha, +\infty)) = \bigcup_{n=1}^{\infty} f^{-1}([r_n, +\infty)).$$

Since sets $f^{-1}([r_n, +\infty))$ are measurable for each n, the set $f^{-1}((\alpha, +\infty))$ is also measurable. Then f is measurable.

4. Let $\{a_n\}$ and $\{b_n\}$ be sequences in $[-\infty, +\infty]$, and prove the following assertions:

(a)

$$\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n.$$

(b)

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

provided none of the sums is of the form $\infty - \infty$.

(c) If $a_n \leq b_n$ for all n, then

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n.$$

Show by an example that strict inequality can hold in (b).

[**proof**]: (a) Since

$$\sup_{k\geq n}(-a_k) = -\inf_{k\geq n} a_k, n = 1, 2, \cdots.$$

Therefore, let $n \to \infty$, it obtains

$$\lim_{n\to\infty} \sup_{k\geq n}(-a_k) = -\lim_{n\to\infty} \inf_{k\geq n} a_k.$$

By the definations of the upper and the lower limits, that is

$$\limsup_{n\to\infty}(-a_n) = -\liminf_{n\to\infty} a_n.$$

(b) Since

$$\sup_{k\geq n}(a_k + b_k) \leq \sup_{k\geq n} a_k + \sup_{k\geq n} b_k, n = 1, 2, \cdots.$$

Hence

$$\lim_{n\to\infty}\sup_{k\geq n}(a_k + b_k) \leq \lim_{n\to\infty}[\sup_{k\geq n} a_k + \sup_{k\geq n} b_k] = \lim_{n\to\infty}\sup_{k\geq n} a_k + \lim_{n\to\infty}\sup_{k\geq n} b_k.$$

By the definations of the upper and the lower limits, that is

$$\limsup_{n\to\infty}(a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n.$$

example: we define

$$a_n = (-1)^n, \quad b_n = (-1)^{n+1}, \quad n = 1, 2, \cdots.$$

Then we have

$$a_n + b_n = 0, \quad n = 1, 2, \cdots.$$

But

$$\limsup_{n\to\infty} a_n = \limsup_{n\to\infty} b_n = 1.$$

(c)Because $a_n \leq b_n$ for all n, then we have

$$\inf_{k\geq n}(a_k) \leq \inf_{k\geq n} b_k, n = 1, 2, \cdots.$$

By the definations of the lower limits, it follows

$$\liminf_{n\to\infty} a_n \leq \liminf_{n\to\infty} b_n.$$

5. (a)Supose $f : X \to [-\infty, +\infty]$ and $g : X \to [-\infty, +\infty]$ are measurable. Prove that the sets

$$\{x : f(x) < g(x)\}, \{f(x) = g(x)\}$$

are measurable.

(b)Prove that the set of points at which a sequence of measurable real-valued functions converges (to a finite limit) is measurable.

[**proof**]: (a)Since for each rational r, the set

$$\{x : f(x) < r < g(x)\} = \{x : f(x) < r\} \cap \{x : r < g(x)\}$$

is measurable. And

$$\{x : f(x) < g(x)\} = \bigcup_{r \in \mathbb{Q}} \{x : f(x) < r < g(x)\}$$

Therefore the set $\{x : f(x) < g(x)\}$ is measurable. Also beacause $|f - g|$ is measurable function and the set

$$\{x : f(x) = g(x)\} = \bigcap_{n=1}^{\infty} \{x : |f(x) - g(x)| < \frac{1}{n}\}$$

is also measurable.

(b) Let $\{f_n(x)\}$ be the sequence of measurable real-valued functions, A be the set of points at which the sequence of measurable real-valued functions $\{f_n(x)\}$ converges (to a finite limit). Hence

$$A = \bigcap_{r=1}^{\infty} \bigcup_{k=1}^{\infty} \bigcap_{n,m \geq k}^{\infty} \{x : |f_n(x) - f_m(x)| < \frac{1}{r}\}.$$

Therefore, A is measurable.

7. Suppose that $f_n : X \to [0, +\infty]$ is measurable for $n = 1, 2, 3, \cdots, f_1 \geq f_2 \geq f_3 \geq \cdots \geq 0, f_n(x) \to f(x)$ as $n \to \infty$,for every $x \in X$and $f_1 \in L^1(\mu)$. Prove that then

$$\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu.$$

and show that this conclusion does not follow if the condition "$f_1 \in L^1(\mu)$"is omitted.

[**proof**]: Define $g_n = f_1 - f_n, n = 1, 2, \cdots$, then g_n is increasing measurable on X for $n = 1, 2, 3, \cdots$. Moreover, $g_n(x) \to f_1(x) - f(x)$ as $n \to \infty$, for every $x \in X$. By the Lebesgue's monotone convergence theorem, we have

$$\lim_{n \to \infty} \int_X g_n d\mu = \int_X (f_1 - f) d\mu.$$

Since $f_1 \in L^1(\mu)$, it shows

$$\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu.$$

[**counterexample**]: Let $X = (-\infty, +\infty)$, and we define

$$f_n = \chi_{E_n}, E_n = [n, +\infty), n = 1, 2, \cdots.$$

8. Suppose that E is measurable subset of measure space (X, μ) with $\mu(E) > 0; \mu(X - E) > 0$, then we can prove that the strict inequality in the Fatou's lemma can hold.

10. Suppose that $\mu(X) < \infty, \{f_n\}$ is a sequence of bounded complex measurable functions on X ,and $f_n \to f$ uniformly on X. Proved that

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu,$$

and show that the hypothesis "$\mu(X) < \infty$"cannot be omitted.

[**proof**]: There exist $M_n > 0, n = 1, 2, \cdots$ such that $|f_n(x)| \leq M_n$ for every $x \in X, n = 1, 2, \cdots$. Since $f_n \to f$ uniformly on X, there exists $M > 0$ such that

$$|f_n(x)| \leq M, |f(x)| \leq M \text{ on } X, n = 1, 2, \cdots.$$

By the Lebesgue's dominated convergence theorem, we have

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu.$$

[**counterexample**]: Let $X = (-\infty, +\infty)$, and we define

$$f_n = \frac{1}{n} \text{ on } X, n = 1, 2, \cdots.$$

12. Suppose $f \in L^1(\mu)$. Proved that to each $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| d\mu < \varepsilon$ whenever $\mu(E) < \delta$.

[**proof**]: Since

$$\int_X |f| d\mu = \sup \int_X s d\mu,$$

the supremum being taken over all simple measurable functions s such that $0 \leq s \leq |f|$. Hence, for each $\varepsilon > 0$, there exists a simple measurable function s with $0 \leq s \leq |f|$ such that

$$\int_X |f| d\mu \leq \int_X s d\mu + \frac{\varepsilon}{2}.$$

Suppose that $M > 0$, satisfying $0 \leq s(x) \leq M$ on X, and let $\delta = \frac{\varepsilon}{2M}$, we have

$$\int_E |f| d\mu \leq \int_E s d\mu + \frac{\varepsilon}{2} \leq M m(E) + \frac{\varepsilon}{2} < \varepsilon$$

when $m(E) < \delta$.

部分习题答案

chapter two

1. Let $\{f_n\}$ be a sequence of real nonnegative functions on $\mathbb{R}^1$, and consider the following four statements:

(a)If f_1 and f_2 are upper semicontinuous, then $f_1 + f_2$ is upper semicontinuous.

(b)If f_1 and f_2 are lower semicontinuous, then $f_1 + f_2$ is lower semicontinuous.

(c)If each $\{f_n\}$ is upper semicontinuous, then $\sum_1^\infty f_n$ is upper semicontinuous.

(d)If each $\{f_n\}$ is lower semicontinuous, then $\sum_1^\infty f_n$ is lower semicontinuous.

Show that tree of these are true and that one is false. What happens if the word "nonnegative" is omitted? Is the truth of the statements affected if $\mathbb{R}^1$ is replaced by a general topological space?

[**proof**]: (a) Because for every real number α,

$$\{x : f_1(x) + f_2(x) < \alpha\} = \bigcup_{r\in\mathbb{Q}} [\{x : f_1(x) < r\} \cap \{x : f_2(x) < \alpha - r\}]$$

and f_1 and f_2 are upper semicontinuous, therefore $f_1 + f_2$ is upper semicontinuous.

(b) Because for every real number α,

$$\{x : f_1(x) + f_2(x) > \alpha\} = \bigcup_{r\in\mathbb{Q}} [\{x : f_1(x) > r\} \cap \{x : f_2(x) > \alpha - r\}]$$

and f_1 and f_2 are upper semicontinuous, therefore $f_1 + f_2$ is upper semicontinuous.

(c) This conclusion is false. counterexample: for each $n \in \mathbb{N}$, define

$$f_n = \begin{cases} 1; & x = r_n \\ 0; & \text{others} \end{cases}$$

which $\{r_n\}$ are the all rationals on $\mathbb{R}^1$. Then $\{f_n\}$ is a sequence of real nonnegative upper semicontinuous functions on $\mathbb{R}^1$, moreover we have

$$f(x) = \sum_1^\infty f_n(x) = \chi_{\mathbb{Q}}.$$

But $f(x)$ is not upper semicontinuous.

(d) According to the conclusion of (b), it is easy to obtain.

If the word "nonnegative" is omitted, (a) and (b) are still true but (c) and (d) is not. The truth of the statements is not affected if $\mathbb{R}^1$ is replaced by a general topological space.

2. Let f be an arbitrary complex function on $\mathbb{R}^1$, and define

$$\phi(x,\delta) = \sup\{|f(s) - f(t)| : s,t \in (x-\delta, x+\delta)\}$$

$$\phi(x) = \inf\{\phi(x,\delta) : \delta > 0\}$$

Prove that ϕ is upper semicontinuous, that f is continuous at a point x if and only if $\phi(x) = 0$, and hence that the set of points of continuity of an arbitrary complex function is a G_δ.

[**proof**]: It is obvious that

$$\phi(x) = \inf\{\phi(x, \frac{1}{k}) : k \in \mathbb{N}\} \geq 0$$

and

$$\phi(x, \frac{1}{k}) \geq \phi(x, \frac{1}{k+1}), \quad k \in \mathbb{N}$$

For any $\alpha > 0$, we will prove that the set $\{x : \phi(x) < \alpha\}$ is open. Suppose that $\phi(x) < \alpha$, then there exists a $N > 0$ such that

$$\phi(x, \frac{1}{k}) < \alpha, \quad k > N.$$

Take $y \in U(x, \frac{1}{2k})$, since $U(y, \frac{1}{2k}) \subset U(x, \frac{1}{k})$ and then

$$\phi(y) \leq \phi(y, \frac{1}{2k}) \leq \phi(x, \frac{1}{k}) < \alpha$$

So we have $y \in \{x : \phi(x) < \alpha\}$. Therefore,

$$U(x, \frac{1}{2k}) \subset \{x : \phi(x) < \alpha\}$$

Thus, it show that $\{x : \phi(x) < \alpha\}$ is open. Then it follows that ϕ is upper semicontinuous. In the following, we first suppose that $\phi(x) = 0$. for any $\epsilon > 0$, it exists $\delta > 0$ such that

$$\phi(x, \delta) < \epsilon$$

And so for any $y \in (x - \delta, x + \delta)$ we have

$$|f(x) - f(y)| \leq \phi(x, \delta) < \epsilon$$

That is f is continuous at x. Next, provided that f is continuous at x. Then for any $\epsilon > 0$ it exists $k \in \mathbb{N}$ satifying

$$|f(x) - f(y)| \leq \frac{\epsilon}{2}, \quad \forall y \in (x - \frac{1}{k}, x + \frac{1}{k})$$

and so $\phi(x, \frac{1}{k}) \leq \epsilon$. It shows

$$\phi(x) = \inf\{\phi(x, \frac{1}{k}) : k \in \mathbb{N}\} = 0$$

Finally, since

$$\{x : \phi(x) = 0\} = \bigcap_{n=1}^{\infty} \{x : \phi(x) < \frac{1}{n}\}$$

then it shows that the set of points of continuity of an arbitrary complex function is a G_δ.

3. Let X be a metric space, with metric ρ. For any nonempty $E \subset X$, define

$$\rho_E(x) = \inf\{\rho(x, y) : \ y \in E\}$$

Show that ρ_E is a uniformly continuous function on X. If A and B are disjoint nonempty closed subsets of X, examine the relevance of the function

$$f(x) = \frac{\rho_A(x)}{\rho_A(x) + \rho_B(x)}$$

to Urysohn's lemma.

[**Proof**]: Since for any $x_1, x_2 \in X$ and $y \in E$, we have

$$\rho(x_1, y) \leq \rho(x_1, x_2) + \rho(x_2, y)$$

Hence

$$\rho_E(x_1) \leq \rho(x_1, x_2) + \rho_E(x_2)$$

and so

$$|\rho_E(x_1) - \rho_E(x_2)| \leq \rho(x_1, x_2)$$

Therefore, it shows that ρ_E is a uniformly continuous function on X. In the following, suppose that $K \subset V \subset X$, K is compact subset, and V is open subset. Define

$$f(x) = \frac{\rho_{V^c}(x)}{\rho_{V^c}(x) + \rho_K(x)}$$

then $0 \leq f \leq 1$ and f is continuous on X. Moreover, $K \prec f \prec V$.

4. Examine the proof of the Riesz theorem and prove the following two statements:

(a) If $E_1 \subset V_1$ and $E_2 \subset V_2$, where V_1 and V_2 are disjoint open sets, then

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2),$$

even if E_1 and E_2 are not in $\mathcal{M}$.

(b) If $E \in \mathcal{M}_F$, then $E = N \cup K_1 \cup K_2 \cup \cdots$, where $\{K_i\}$ is a disjoint countable collection of compact sets and $\mu(N) = 0$.

[**proof**]: (a) By the proof of the Riesz theorem, there exists a G_δ set G such that

$$E_1 \cup E_2 \subset G \ \text{ and } \ \mu(E_1 \cup E_2) = \mu(G).$$

Set $G_1 = V_1 \cap G,\ G_2 = V_2 \cap G$, then $E_1 \subset G_1,\ E_2 \subset G_2$ and $G_1 \cap G_2 = \emptyset$. Therefore

$$\mu(E_1)+\mu(E_2) \leq \mu(G_1)+\mu(G_2) = \mu(G_1 \cup G_2) = \mu(G \cap (V_1 \cup V_2)) \leq \mu(G) = \mu(E_1 \cup E_2).$$

And since

$$\mu(E_1 \cup E_2) \leq \mu(E_1) + \mu(E_2),$$

It shows

$$\mu(E_1 \cup E_2) = \mu(E_1) + \mu(E_2).$$

(b) Since $E \in \mathcal{M}_F$, there exist a sequence of disjoint compact subsets $\{K_n\}$ such that

$$K_n \subset E - \bigcup_{i=0}^{n-1} K_i, m(E - \bigcup_{i=0}^{n-1} K_i) < m(K_n) + \frac{1}{n}, n = 1, 2, \cdots,$$

which we order $K_0 = \emptyset$.

Set $K = \bigcup_{n=1}^{\infty} K_n$. Since $m(E) < +\infty$, we obtain that

$$m(E - \bigcup_{i=1}^{n} K_i) < \frac{1}{n}, n = 1, 2, \cdots.$$

Whence we have

$$m(E - K) \leq m(E - \bigcup_{i=1}^{n} K_i) < \frac{1}{n}, n = 1, 2, \cdots.$$

Set $N = E - K$, then it shows that $m(N) = 0$ and $E = K \cup N$.

5. Let E be Cantor's familiar "middle thirds" set. Show that $m(E) = 0$. even though E and $\mathbb{R}^1$ have the same cardinality.

[**proof**]: According to the construction of Cantor's set, it shows that the set E is closed set and

$$m(E) = m([0, 1]) - \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} = 0.$$

11. Let μ be a regular Borel measure on a compact Hausdorff space X; assume $\mu(X) = 1$. prove that there is a compact set $K \subset X$ such that $\mu(K) = 1$ but $\mu(H) < 1$ for every proper compact subset H of K.

[**Proof**]: Define the set

$$\Omega = \{K_\alpha : K_\alpha \subset X, \quad K_\alpha \text{ is compact set with } \mu(k_\alpha) = 1\}$$

then $X \in \Omega$. Order

$$K = \bigcap_{K_\alpha \in \Omega} K_\alpha$$

Thus K is compact subset. Assume that $V \subset X$ is an open set with $K \subset V$. Since X is compact, V^c is also compact. Again since

$$V^c \subset \bigcup_{K_\alpha \in \Omega} K_\alpha^c$$

then there exist finite sets $K_{\alpha_1}, \cdots, K_{\alpha_n}$ such that

$$V^c \subset \bigcup_{k=1}^{n} K_{\alpha_k}^c$$

Because $\mu(X) = 1 = \mu(K_\alpha)$,

$$\mu(K_{\alpha_k}^c) = 0, \quad k = 1, 2, \cdots, n$$

and so $\mu(V^c) = 0$. We therefore have $\mu(V) = 1$. Since μ is regular, hence it shows that $\mu(K) = 1$. By the construction of K, it follows that $\mu(H) < 1$ for any proper compact subset of K. If $U \subset X$ is open set with $\mu(U) = 0$, it follows $\mu(U^c) = 1$. U^c is compact subset since X is compact, and so $K \subset U^c$ by the construction of K. Hence, $U \subset K^c$. It thus shows that K^c is the largest open set in X whose measure is 0.

12. Show that every compact subset of $\mathbb{R}^1$ is the support of a Borel measure.

[**Proof**]: Assume that μ is the Lebesgue measure on $\mathbb{R}^1$ and K is a compact subset of $\mathbb{R}^1$. Define

$$\phi(f) = \int_K f d\mu; \quad f \in C_c(\mathbb{R}^1).$$

Then ϕ is a positive linear functional on $C_c(\mathbb{R}^1)$. By Riesz representation theorem, there exists a σ-algebra $\mathfrak{M}$ in $\mathbb{R}^1$ which contains all Borel sets in $\mathbb{R}^1$, and there exists a unique positive measure σ on $\mathfrak{M}$ such that

$$\phi(f) = \int_{\mathbb{R}^1} f d\sigma, \quad f \in C_c(\mathbb{R}^1).$$

By theorem2.14, it shows

$$\sigma(K) = \inf\{\phi(f) : K \prec f\} = \mu(K)$$

$$\sigma(X) = \sup\{\phi(f) : f \prec X\} \leq \mu(K)$$

and so $\sigma(X) = \sigma(K) = \mu(K) < +\infty$. Therefore, it follows for any measurable set $A \subset X$,

$$\sigma(A) = \sigma(A \cap K).$$

That is, K is the support of σ.

14. Let f be a real-valued Lebesgue measurable function on $\mathbb{R}^k$. Prove that there exist Borel functions g and h such that $g(x) = h(x)$ a.e.$[m]$ and $g(x) \leq f(x) \leq h(x)$ for every $x \in \mathbb{R}^k$.

[**Proof**]: Firstly, assume that $1 > f \geq 0$. By the proof of Theorem 2.24 in text, we have

$$f(x) = \sum_{n=1}^{\infty} t_n(x), \quad \forall x \in \mathbb{R}^k$$

and $2^n t_n$ is the characteristic function of a Lebesgue measurable set $T_n \subset \mathbb{R}^k (n \in \mathbb{N})$. According to the construction of Lebesgue measurable set, there exist Borel measurable sets $F_n, E_n (n \in \mathbb{N})$ satisfying

$$F_n \subset T_n \subset E_n; \quad m(E_n - F_n) = 0; \quad n \in \mathbb{N}.$$

Define

$$g = \sum_{n=1}^{\infty} 2^{-n} \chi_{F_n}; \quad h = \sum_{n=1}^{\infty} 2^{-n} \chi_{E_n}$$

then g, h are Borel measurable functions on $\mathbb{R}^k$ and $g(x) \leq f(x) \leq h(x)$. Moreover, $g(x) = h(x)$ a.e.$[m]$.

Secondly, it is easy that the conclusion is correct for $0 \leq f < M$, and hence it is also for bounded real-valued Lebesgue measurable function on $\mathbb{R}^k$.

Finally, if f be a real-valued Lebesgue measurable function on $\mathbb{R}^k$ and if

$$B_n = \{x : |f(x)| \leq n\}, \quad n = 1, 2, \cdots$$

then we have

$$\chi_{B_n}(x) f(x) \to f(x), \quad \text{as } n \to \infty.$$

According to the conclusion above, there exist Borel functions g_n and h_n such that $g_n(x) = h_n(x)$ a.e.$[m]$ and $g_n(x) \leq \chi_{B_n}(x) f(x) \leq h_n(x)$ for every $x \in \mathbb{R}^k$. Take

$$g = \limsup g_n; \quad h = \liminf h_n$$

then $g(x) = h(x)$ a.e.$[m]$ and $g(x) \leq f(x) \leq h(x)$ for every $x \in \mathbb{R}^k$.

15. It is easy to guess the limits of

$$\int_0^n (1 - \frac{x}{n})^n e^{\frac{x}{2}} dx \text{ and } \int_0^n (1 + \frac{x}{n})^n e^{-2x} dx$$

as $n \to \infty$. Prove that your guesses are correct.

[**proof**]: Define

$$f_n(x) = \chi_{[0,n]}(1 - \frac{x}{n})^n e^{\frac{x}{2}}; \quad g_n(x) = \chi_{[0,n]}(1 + \frac{x}{n})^n e^{-2x}; \quad n \in \mathbb{N}$$

Then it is easy that

$$|f_n(x)| \le e^{-\frac{x}{2}}, \text{ and } f_n(x) \to e^{-\frac{x}{2}};$$

$$|g_n(x)| \le e^{-x}, \text{ and } f_n(x) \to e^{-x}.$$

By the Lebesgue's Dominated Convergence Theorem, it shows

$$\lim_{n\to\infty} \int_0^n (1-\frac{x}{n})^n e^{\frac{x}{2}} dx = \lim_{n\to\infty} \int_0^\infty f_n(x)dx = \int_0^\infty e^{-\frac{x}{2}} dx = 2$$

$$\lim_{n\to\infty} \int_0^n (1+\frac{x}{n})^n e^{-2x} dx = \lim_{n\to\infty} \int_0^\infty g_n(x)dx = \int_0^\infty e^{-x} dx = 1$$

17. Define the distance between points (x_1, y_1) and (x_2, y_2) in the plane to be

$$|y_1 - y_2| \quad \text{if } x_1 = x_2; \qquad 1 + |y_1 - y_2| \quad \text{if} \quad x_1 \ne x_2.$$

show that this is indeed a metric, and that the resulting metric space X is locally compact.

If $f \in C_c(X)$, let $x_1, x_2, \cdots, x_n$ be those values of x for which $f(x, y) \ne 0$ for at least one y (there are only finitely many such x!), and define

$$\Lambda f = \sum_{j=1}^{n} \int_{-\infty}^{+\infty} f(x_j, y)dy$$

Let μ be the measure associated with this Λ by Theorem2.14. If E is the x-axis, show that $\mu(E) = \infty$ although $\mu(K) = 0$ for every compact $K \subset E$.

[**proof**]:Order

$$P_i = (x_i, y_i), \quad i = 0, 1, 2, 3, \cdots.$$

$$d(P_1, P_2) = \begin{cases} |y_1 - y_2| & \text{if} \quad x_1 = x_2; \\ 1 + |y_1 - y_2| & \text{if} \quad x_1 \ne x_2 \end{cases}$$

Since

$$d(P_1, P_2) \le \begin{cases} |y_1 - y_3| + |y_2 - y_3| & \text{if} \quad x_1 = x_2; \\ 1 + |y_1 - y_3| + |y_2 - y_3| & \text{if} \quad x_1 \ne x_2 \end{cases}$$

Therefore,

$$d(P_1, P_2) \le d(P_1, P_3) + d(P_2, P_3).$$

That is d is metric. Take $X = (\mathbb{R}^2, d)$. Then

$$P_n \to P \Leftrightarrow \exists\ N > 0, \text{such that if } n > N, x_n = x; \text{and} |y_n - y| \to 0.$$

and for any $0 < \delta < 1$,

$$U(P_0, \delta) = \{P \in X : d(P_0, P) < \delta\} = \{P \in X : x = x_0, \text{and } |y - y_0| < \delta\}$$

$$\overline{U(P_0, \delta)} = \{P \in X : d(P_0, P) \leq \delta\} = \{P \in X : x = x_0, \text{and } |y - y_0| \leq \delta\}$$

Moreover, $\overline{U(P_0, \delta)}$ is compact subset of X. Thus X is locally compact Hausdorff space.

In the following, we order the set for $f \in C_c(X)$

$$\Omega_f = \{x \in \mathbb{R} : \exists\ y \in \mathbb{R} \text{ such that } f(x, y) \neq 0\}$$

Let K be the support of f, K is compact. Then there exists finite $P_1, P_2, \cdots, P_n \in K$ for any $0 < \delta < 1$ such that

$$K \subset \bigcup_{j=1}^{n} \overline{U(P_j, \delta)}$$

and so $\Omega_f = \{x_1, x_2, \cdots, x_n\}$. Since

$$\Lambda f = \sum_{j=1}^{n} \int_{-\infty}^{+\infty} f(x_j, y) dy$$

it is easy that Λ is a positive linear functional on $C_c(X)$. There thus exist a measure μ associated with this Λ by Theorem2.14. Let E be the x-axis and $0 < \delta < 1$. Define

$$V = \bigcup_{P_i \in E} U(P_i, \delta); \quad \text{where } P_i = (x_i, 0);$$

it shows V is open and $E \subset V$. For any finite elements $P_1, P_2, \cdots, P_n \in E$, the set

$$F = \bigcup_{j=1}^{n} \overline{U(P_j, \frac{\delta}{2})}$$

is compact with $F \subset V$. By the Urysohn's lemma, there exists $f \in C_c(X)$ satisfying $F \prec f \prec V$. So we have

$$\mu(V) \geq \Lambda f \geq n\delta.$$

Hence, it shows $\mu(V) = \infty$ as $n \to \infty$. Therefore, $\mu(E) = \infty$. Assume that $K \subset E$ is compact, then it is obvious that K is finite set. Take

$$K = \{P_1, P_2, \cdots, P_m\}$$

For any $0 < \epsilon < 1$, define

$$G = \bigcup_{j=1}^{m} U(P_j, \epsilon)$$

then g is open and $K \subset G$. By the Urysohn's lemma, there exists $g \in C_c(X)$ satisfying $K \prec g \prec G$. Since

$$\mu(K) \leq \Lambda g \leq 2\epsilon$$

whence $\mu(K) = 0$.

21. If X is compact and $f : X \to (-\infty, +\infty)$ is upper semicontinuous, prove that f attains its maximum at some point of X.

[**proof**]: for every $t \in f(X)$, define

$$E_t = \{x \in X : f(x) < t\}$$

Assume that f does not attain its maximum at some point of X. Then it shows $X = \bigcup_{t \in f(X)} E_t$. Since f is continuous, E_t is open sets. Again since X is compact, there exists finite $t_i (i = 1, 2, \cdots, n) \in f(X)$ such that $X = \bigcup_{i=1}^{n} E_{t_i}$. Take

$$t_0 = \max\{t_1, t_2, \cdots, t_n\}$$

then we have

$$f(x) < t_0, \quad \forall x \in X.$$

But it is in contradiction with $t_0 \in f(X)$. It therefore is proved that f attains its maximum at some point of X.

部分习题答案

chapter three

1. Prove that the supremum of any collection of convex functions on (a, b) is convex on (a, b) and that pointwise limits of sequences of convex functions are convex. What can you say about upper and lower limits of sequences of convex functions?

[**Proof**]: Suppose that $\alpha \geq 0, \beta \geq 0$ and $\alpha + \beta = 1$. Let $\{f_i\}_{i \in I}$ be any collection of convex functions on (a, b). for any $x, y \in (a, b)$, we have

$$f_i(\alpha x + \beta y) \leq \alpha f_i(x) + \beta f_i(y), \quad \forall i \in I.$$

It prove that

$$\sup_{i \in I} f_i(\alpha x + \beta y) \leq \alpha \sup_{i \in I} f_i(x) + \beta \sup_{i \in I} f_i(y).$$

So $\sup\limits_{i \in I} f_i$ is convex.
Assume $I = \mathbb{N}$ and f is pointwise limits of sequences of convex functions $\{f_i\}$. according to the properties of limit, we have

$$\lim_{i \to \infty} f_i(\alpha x + \beta y) \leq \alpha \lim_{i \to \infty} f_i(x) + \beta \lim_{i \to \infty} f_i(y)$$

That is $f(\alpha x + \beta y) \leq \alpha f(x) + \beta f(y)$ and so f is convex.
By the definition of upper limits and the conclusion above, it is easy that the upper limits of sequences of convex functions is convex. But the lower limits of sequences of convex functions is false.

2. If ϕ is convex on (a, b) and if ψ is convex and nondecreasing on the range of ϕ, prove that $\psi \circ \phi$ is convex on (a, b). For $\phi > 0$, show that the convexity of $\ln \phi$ implies the convexity of ϕ , but not vice versa.

[**Proof**]: Assume that

$$x, y \in (a, b); \alpha, \beta \geq 0 \text{and } \alpha + \beta = 1.$$

Then

$$\psi \circ \phi(\alpha x + \beta y) \leq \psi(\alpha \phi(x) + \beta \phi(y)) \leq \alpha \psi \circ \phi(x) + \beta \psi \circ \phi(y).$$

and so $\psi \circ \phi$ is convex on (a, b). Since e^t is nondecreasing convex on $\mathbb{R}$ and $\phi = e^{\ln \phi}$, it shows from the conclusion above that the convexity of $\ln \phi$ implies the convexity of ϕ for $\phi > 0$.

[**counterexample**]: $\phi(x) = x^2$ is convex on $(0, \infty)$ but $\ln \phi(x) = 2 \ln x$ is not convex. Moreover, for $\phi > 0$, it can show that the convexity of $\log_c \phi (c > 1)$ implies the convexity of ϕ.

3. Assume that ϕ is a continuous real function on (a, b) such that

$$\phi(\frac{x+y}{2}) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(y)$$

for all x and $y \in (a, b)$. Prove that ϕ is convex.

[**Proof**]: According to the definition of convex function,we only need to prove the case $0 < \lambda < \frac{1}{2}$. Take

$$E = \{\frac{k}{2^n} \mid k = 1, 2, \cdots, 2^n - 1;\ n \in \mathbb{N}\}.$$

For $n = 1$, it is obvious that

$$\phi(\frac{x+y}{2}) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(y)$$

for all x and $y \in (a, b)$.

For $n = 2$, suppose that $\lambda_k = \frac{k}{4}, k = 1$.

$$\begin{aligned}
\phi(\lambda_1 x + (1 - \lambda_1)y) &= \phi(\tfrac{1}{2}(\tfrac{x+y}{2} + y)) \\
&\leq \tfrac{1}{2}\phi(\tfrac{x+y}{2}) + \tfrac{1}{2}\phi(y) \\
&\leq \tfrac{1}{2}[\tfrac{1}{2}\phi(x) + \tfrac{1}{2}\phi(y)] + \tfrac{1}{2}\phi(y) \\
&= \tfrac{1}{4}\phi(x) + \tfrac{3}{4}\phi(y) \\
&= \lambda_1\phi(x) + (1 - \lambda_1)\phi(y)
\end{aligned}$$

Assume that the case

$$\lambda = \frac{k}{2^{n-1}}, k = 1, 2, \cdots, 2^{n-1} - 1$$

is correct. For any $\lambda \in E$ with $\lambda = \frac{k}{2^n}; 1 \leq k \leq 2^{n-1}, n \geq 2$, and for all x and $y \in (a, b)$, it shows that

(i)

$$\phi(\lambda x + (1 - \lambda)y) = \phi(\frac{x+y}{2}) \leq \frac{1}{2}\phi(x) + \frac{1}{2}\phi(y) = \lambda\phi(x) + (1 - \lambda)\phi(y)$$

when $k = 2^{n-1}$.

(ii)for $1 \le k < 2^{n-1}$,

$$\lambda x + (1-\lambda y) = \frac{k}{2^n}x + (1-\frac{k}{2^n})y = \frac{1}{2}(\frac{k}{2^{n-1}}x + (1-\frac{k}{2^{n-1}})y + y)$$

Since $\frac{k}{2^{n-1}}x + (1-\frac{k}{2^{n-1}})y \in (a,b)$ and so

$$\begin{aligned}\phi(\lambda x + (1-\lambda)y) &\le \tfrac{1}{2}\phi(\tfrac{k}{2^{n-1}}x + (1-\tfrac{k}{2^{n-1}})y) + \tfrac{1}{2}\phi(y)\\ &\le \tfrac{k}{2^n}\phi(x) + \tfrac{1}{2}(1-\tfrac{k}{2^{n-1}})\phi(y) + \tfrac{1}{2}\phi(y)\\ &= \lambda\phi(x) + (1-\lambda)\phi(y)\end{aligned}$$

For any $0 < \lambda < 1$, there exists a sequence $\{\lambda_n\} \subset E$ with $\lim_{n\to\infty} \lambda_n = \lambda$. Thus

$$\begin{aligned}\phi(\lambda x + (1-\lambda)y) &= \lim_{n\to\infty} \phi(\lambda_n x + (1-\lambda_n)y)\\ &\le \lim_{n\to\infty} (\lambda_n\phi(x) + (1-\lambda_n)\phi(y))\\ &= \lambda\phi(x) + (1-\lambda)\phi(y)\end{aligned}$$

Therefore, it shows that ϕ is convex on (a,b).

4. Suppose f is a complex measurable function on X, μ is a positive measure on X, and

$$\phi(p) = \int_X |f|^p d\mu = \|f\|_p^p \quad (0 < p < \infty).$$

Let $E = \{p : \phi(p) < \infty\}$, Assume $\|f\|_\infty > 0$.

(a) If $r < p < s$, $r \in E$, and $s \in E$, Prove that $p \in E$.

(b) Prove that $\ln \phi$ is convex in the interior of E and that ϕ is continuous on E.

(c) By (a), E is connected. Is E necessarily open? losed? Can E consist of a single point? Can E be any connected subset of $(0,\infty)$?

(d) If $r < p < s$, prove that $\|f\|_p \le \max(\|f\|_r, \|f\|_s)$. Show that this implies the inclusion $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e) Assume that $\|f\|_r < \infty$ for some $r < \infty$ and prove that

$$\|f\|_p \to \|f\|_\infty \text{ as } p \to \infty.$$

[**Proof**]: (a) If $r < p < s$, there exist $0 < \alpha < 1$ and $0 < \beta < 1$ such that $\alpha + \beta = 1$ and $p = \alpha r + \beta s$. By Hölder's inequality, it follows that

$$\|f\|_p^p = \phi(p) = \int_X |f|^p d\mu = \int_X |f|^{\alpha r+\beta s} d\mu \le \{\int_X |f|^r d\mu\}^\alpha \cdot \{\int_X |f|^s d\mu\}^\beta = \|f\|_r^{\alpha r} \cdot \|f\|_s^{\beta s}.$$

As $r \in E$ and $s \in E$, it prove that $p \in E$.

(b) By (a), it reduces that E is convex. Moreover, E is connected. Since $E \subset (0, \infty)$, the interior of E is an interval and denoted by (a, b). For any s, t in the interior of E and $0 < \alpha < 1$ and $0 < \beta < 1$, it follows from (a) that

$$\phi(\alpha s + \beta t) \leq \phi(s)^{\alpha} \cdot \phi(t)^{\beta}.$$

So it proves

$$\ln \phi(\alpha s + \beta t) \leq \ln[\phi(s)^{\alpha} \cdot \phi(t)^{\beta}] = \alpha \ln \phi(s) + \beta \ln \phi(t).$$

That is, $\ln \phi$ is convex in the interior of E. Moreover, $\ln \phi$ is continuous on (a, b). Thus, ϕ is also continuous on (a, b). Provided that $a \in E$. For any decreasing sequence $\{s_n\}$ of (a, b) with $\lim\limits_{n\to\infty} s_n = a$, it is easy to show that

$$\lim_{n\to\infty} |f(x)|^{s_n} = |f(x)|^{a}, \quad \forall x \in X.$$

Since $|f(x)|^{s_n} \leq \max |f(x)|^{a}, |f(x)|^{s_1}, \quad \forall x \in X$ and $|f(x)|^{a}, |f(x)|^{s_1} \in L^1(\mu)$, by the Lebesgue control convergence theorem, it follows that

$$\lim_{n\to\infty} \phi(s_n) = \phi(a).$$

Similarly, the other cases can be proved. Hence ϕ is continuous on E.

(c) By (a), it shows that E is connected and convex. E can be any interval on $(0, \infty)$. For examples: take $X = (0, \infty)$ and μ is the Lebesgue measure on X.

(i) For $E = (a, b), (0 < a < b)$, take

$$f(x) = \begin{cases} x^{-\frac{1}{b}}, & x \in (0, 1); \\ x^{-\frac{1}{a}}, & x \in [1, \infty) \end{cases}$$

(ii)For $E = [a, b], (0 < a < b)$, take

$$f(x) = \begin{cases} x, & x \in (0, 1); \\ x^{-\frac{1}{a}-1}, & x \in [1, \infty) \end{cases}$$

(iii)For $E = [a, b), (0 < a < b)$, take

$$f(x) = \begin{cases} x^{-\frac{1}{b}}, & x \in (0, 1); \\ x^{-\frac{1}{a}-1}, & x \in [1, \infty) \end{cases}$$

(iv)For $E = (a, b], (0 < a < b)$, take

$$f(x) = \begin{cases} x^{-\frac{1}{b+1}}, & x \in (0, 1); \\ x^{-\frac{1}{a}}, & x \in [1, \infty) \end{cases}$$

(v)For $E = \emptyset$, take $f \equiv 1$.

(d) If $r < p < s$,by (a), there exist $0 < \alpha < 1$ and $0 < \beta < 1$ such that $\alpha + \beta = 1$ and $p = \alpha r + \beta s$. And it is easy to know

$$\phi(p) = \phi(\alpha r + \beta s) \leq \phi(r)^{\alpha} \cdot \phi(s)^{\beta} \leq \max \phi(r), \phi(s).$$

Thus, $\|f\|_p \leq \max \|f\|_r, \|f\|_s$. Moreover, it shows that $L^r(\mu) \cap L^s(\mu) \subset L^p(\mu)$.

(e)Firstly since $\|f\|_\infty > 0$, it shows that $\|f\|_p > 0$ for any $0 < p < \infty$. secondly, assume that $\|f\|_\infty = +\infty$. Hence,$\mu(\{x \in x : |f(x)| > 2\}) > 0$. For any $p > r$,

$$0 < 2^p \mu(\{x \in x : |f(x)| > 2\}) \leq \int_X |f|^p d\mu = \|f\|_p^p.$$

Therefore, $\lim\limits_{n \to \infty} \|f\|_p = +\infty$.
Thirdly, assume that $\|f\|_\infty = 1$ and $\|f\|_r > 0$. Without generality, assume that $|f(x)| \leq 1, \forall x \in X$. Thus For any $p > r$, it shows that

$$\|f\|_p^p \leq \|f\|_r^r.$$

That is $\|f\|_p \leq \|f\|_r^{\frac{r}{p}}$. And so

$$\lim_{p \to \infty} \|f\|_p = 1 = \|f\|_\infty.$$

The general case can reduce to the case $\|f\|_\infty = 1$ and $\|f\|_r > 0$. This is completed the proof.

5. Assume, in addition to the hypotheses of Exercise 4, that

$$\mu(X) = 1.$$

(a) Prove that $\|f\|_r \leq \|f\|_s$, if $0 < r < s \leq \infty$.
(b) Under what conditions does it happen that $0 < r < s \leq \infty$ and $\|f\|_r = \|f\|_s < \infty$?
(c) Prove that $L^r(\mu) \supset L^s(\mu)$ if $0 < r < s$. Under what conditions do these

two spaces contain the same functions?

(d) Assume that $\|f\|_r < \infty$ for some $r > 0$, and prove that

$$\lim_{p\to 0} \|f\|_p = \exp\{\int_X \log|f| d\mu\}$$

if $\exp\{-\infty\}$ is defined to be 0.

[**Proof**]: (a) When $s = \infty$, it is easy that

$$\int_X |f|^r d\mu \le \|f\|_\infty^r \mu(X)$$

and so $\|f\|_r \le \|f\|_\infty$. If $s < \infty$, $\phi(x) = x^{\frac{s}{r}}$ is convex on $(0, \infty)$ and by the Jensen's Inequality, it shows

$$\int_X |f|^s d\mu = \int_X (|f|^r)^{\frac{s}{r}} d\mu \ge (\int_X (|f|^r) d\mu)^{\frac{s}{r}}$$

and whence $\|f\|_s \ge \|f\|_r$.

(b) Assume that $s = \infty$, and $\|f\|_r = \|f\|_\infty < \infty (0 < r < s)$. Then for any $1 < p < \infty$, it shows by (a) that

$$\|f\|_1 = \|f\|_p = \|f\|_q < \infty$$

and so f is constant almost everywhere by the Holder's inequality. Secondly, assume $0 < r < s < \infty$. Take $\alpha = \frac{s}{r} > 1$, then by the Holder's inequality

$$\int_X |f|^r \cdot 1 d\mu \le (\int_X |f|^s d\mu)^{\frac{r}{s}} = \|f\|_s^r = \|f\|_r^r$$

Therefore, $|f|^r =$ constant *a.e.* on X. So it happen that $0 < r < s \le \infty$ and $\|f\|_r = \|f\|_s < \infty$ if and only if $f =$ constant *a.e.* on X.

(c)If $0 < r < s$, then by (a) it is easy to prove $L^s(\mu) \subset L^r(\mu)$. Moreover, we will prove the following:

$L^r(\mu) = L^s(\mu) \Leftrightarrow$ there exist finite pairwise disjoint measurable sets at most in

$$\Omega = \{E \subset X : \mu(E) > 0\}.$$

Firstly, assume that there are infinte pairwise disjoint members in Ω. Then it exists a sequence $\{E_n\} \subset X$ of pairwise disjoint measurable sets such that

$$0 < \mu(E_n) < 1, \ n = 1, 2, \cdots.$$

Take $\frac{1}{s} < t < \frac{1}{r}$, and define

$$f = \sum_{n=1}^{\infty} n^{-t} \chi_{E_n}$$

Then

$$\begin{aligned} \int_X |f|^s d\mu &= \int_X \sum_{n=1}^{\infty} n^{-ts} \chi_{E_n} d\mu \\ &= \sum_{n=1}^{\infty} n^{-ts} \mu E_n \\ &\leq \sum_{n=1}^{\infty} n^{-ts} \end{aligned}$$

Since $st > 1$, it has $\sum_{n=1}^{\infty} n^{-t} < +\infty$ and so $f \in L^s(\mu)$. But

$$\begin{aligned} \int_X |f|^r d\mu &= \int_X \sum_{n=1}^{\infty} n^{-tr} \chi_{E_n} d\mu \\ &= \sum_{n=1}^{\infty} n^{-tr} \mu E_n \\ &\geq \sum_{n=1}^{\infty} n^{-tr} \cdot n^{-\alpha} \\ &= \sum_{n=1}^{\infty} n^{-(tr+\alpha)} \end{aligned}$$

which $\alpha = \frac{1-tr}{2}$. Since $0 < \alpha + tr < 1$, it shows that $\sum_{n=1}^{\infty} n^{-(tr+\alpha)} = +\infty$. Hence, $f \notin L^r(\mu)$.

Next, we assume that there exist finite pairwise disjoint measurable sets at most in Ω. Provided that $f \in L^r(\mu)$, then there exists an increasing seqnence $\{h_n\}$ of simple measurable functions such that

$$0 \leq h_n(x) \leq h_{n+1}(x) \leq \cdots \leq |f(x)|, \ forallx \in X$$

and

$$h_n(x) \to |f(x)|, n \to \infty, \forall x \in X.$$

Put

$$t_1(x) = h_1(x); t_n(x) = h_n(x) - h_{n-1}(x), \ n = 2, 3, \cdots$$

then $2^n t_n$ is the characteristic function of a measurable set $T_n \subset X$, and $\{T_n\}$ are pairwise disjoint. Moreover,

$$f(x) = \sum_{n=1}^{\infty} t_n(x), \quad \forall x \in X.$$

According to the assumption, there are finite members of $\{T_n\}$ with nonzero measure at most. Thus f equal to a simple function almost everywhere and so $f \in L^s(\mu)$. Therefore, $L^r(\mu) = L^s(\mu)$.

(d) Assume that $\|f\|_r < \infty$ for some $r > 0$, then

$$\|f\|_p \le \|f\|_r < +\infty;\ \|f\|_p \ge e^{\int_X \ln|f|d\mu};\ \ 0 < p < r.$$

and $\{\|f\|_p\}$ is increasing relative to p. Therefore, the limit $\lim\limits_{p\to 0}\|f\|_p = c$ exists. If c=0, it is easy that $e^{\int_X \ln|f|d\mu} = 0$. In the following, we assume $c > 0$. Since $\|f\|_r < +\infty$, we may suppose that f is finite everywhere. Let

$$E = [|f| > 1];\ \ F = [|f| < 1];\ \ G = [|f| = 1]$$

So we have

$$\begin{aligned}\|f\|_p &= (\textstyle\int_X |f|^p d\mu)^{\frac{1}{p}} \\ &= (\textstyle\int_E(|f|^p - 1 + 1)d\mu + \int_F(|f|^p - 1 + 1)d\mu + \int_G(|f|^p - 1 + 1)d\mu)^{\frac{1}{p}} \\ &= (1 + \textstyle\int_E(|f|^p - 1)d\mu + \int_F(|f|^p - 1)d\mu)^{\frac{1}{p}}\end{aligned}$$

Take $\alpha > 1$; $\quad 0 < \beta < 1$, because

$$\lim_{x\to 1}\frac{x-1}{\ln x} = 1; \quad |f(x)|^p \to 1,\ p \to 0$$

there exists $\delta > 0$ such that

$$|f(x)|^p - 1 < \alpha \ln|f(x)|^p,\ \forall x \in E;\ \ |f(x)|^p - 1 < \beta \ln|f(x)|^p,\ \forall x \in F$$

when $p < \delta$. Thus

$$\begin{aligned}\|f\|_p &\le (1 + \textstyle\int_E \alpha \ln|f|^p d\mu + \int_F \beta \ln|f|^p d\mu)^{\frac{1}{p}} \\ &= (1 + p(\alpha \textstyle\int_E \ln|f|d\mu + \beta \int_F \ln|f|d\mu)^{\frac{1}{p}}\end{aligned}$$

Then

$$0 < c = \lim_{p\to 0}\|f\|_p \le e^{\alpha\int_E \ln|f|d\mu + \beta\int_F \ln|f|d\mu}$$

Let

$$\alpha \to 1^+;\ \ \beta \to 1^-$$

then

$$c = \lim_{p\to 0}\|f\|_p \le e^{\int_E \ln|f|d\mu + \int_F \ln|f|d\mu} = e^{\int_X \ln|f|d\mu}.$$

Hence

$$\lim_{p\to 0}\|f\|_p = e^{\int_X \ln|f|d\mu}.$$

11. Suppose $\mu(\Omega) = 1$, and suppose f and g are positive measurable functions on Ω such that $fg \geq 1$. Prove that

$$\int_\Omega f d\mu \cdot \int_\Omega g d\mu \geq 1.$$

[**Proof**]: Since f and g are positive measurable functions on Ω such that $fg \geq 1$, $f^{\frac{1}{2}}$ and $g^{\frac{1}{2}}$ are positive measurable functions on Ω such that $(fg)^{\frac{1}{2}} \geq 1$. By Hölder's inequality, it follows that

$$1 \leq (\int_\omega (fg)^{\frac{1}{2}} d\mu)^2 \leq \int_\Omega f d\mu \cdot \int_\Omega g d\mu.$$

12. Suppose $\mu(\Omega) = 1$ and $h : \Omega \to [0, \infty]$ is measurable. If

$$A = \int_\Omega h d\mu,$$

prove that

$$\sqrt{1 + A^2} \leq \int_\Omega \sqrt{1 + h^2} d\mu \leq 1 + A.$$

[**Proof**]: It is obvious correct when $A = \infty$ or 0 and so we assume $0 < A < +\infty$ and $h : \Omega \to (0, \infty)$. Since $\phi(x) = \sqrt{1 + x^2}$ is convex on $(0, \infty)$, it shows by the Jensen's Inequality that

$$\sqrt{1 + A^2} \leq \int_\Omega \sqrt{1 + h^2} d\mu.$$

Because $\sqrt{1 + h^2} \leq 1 + h$,

$$\int_\Omega \sqrt{1 + h^2} d\mu \leq 1 + A.$$

This complete the proof.

14. Suppose $1 < p < \infty$, $f \in L^p = L^p((0, \infty))$, relative to Lebesgue measure and

$$F(x) = \frac{1}{x} \int_0^x f(t) dt \quad (0 < x < \infty).$$

(a) Prove Hardy's inequality

$$\|F\|_p \leq \frac{p}{p - 1} \|f\|_p$$

which shows that the mapping $f \to F$ carries L^p into L^p.

(b) Prove that equality holds only if $f = 0 a.e.$

(c) Prove that the constant $p/(p-1)$ cannot be replaced by a smaller one

.

(d) If $f > 0$ and $f \in L^1$, prove that $F \notin L^1$.

[**Proof**]: (a) Firstly, assume that $f \geq 0$ and $f \in C_c((0,\infty))$. Then

$$F(x) = \frac{1}{x}\int_0^x f(t)dt \quad (0 < x < \infty)$$

is differential on $(0,\infty)$ and so $xF'(x) = f(x) - F(x), \forall x \in (0,\infty)$. Therefore

$$\begin{aligned}\int_0^\infty F^p(x)dx &= F^p(x)x\mid_0^\infty - p\int_0^\infty F^{p-1}(x)F'(x)dx \\ &= -p\int_0^\infty F^{p-1}(x)xF'(x)dx \\ &= -p\int_0^\infty F^{p-1}(x)(f(x)-F(x))dx \\ &= -p\int_0^\infty F^{p-1}(x)f(x)dx + p\int_0^\infty F^p(x)dx\end{aligned}$$

Thus

$$(p-1)\int_0^\infty F^p(x)dx = p\int_0^\infty F^{p-1}(x)f(x)dx \leq p(\int_0^\infty F^{(p-1)q}(x)dx)^{\frac{1}{q}}(\int_0^\infty f^p(x)dx)^{\frac{1}{p}}$$

and so

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p.$$

Moreover, it is easy that

$$\|F\|_p \leq \frac{p}{p-1}\|f\|_p, \quad \forall f \in C_c((0,\infty)).$$

Given $f \in L^p((0,\infty))$, since $C_c((0,\infty))$ is dense in $L^p((0,\infty))$, there exist a sequence $\{f_n\} \subset C_c((0,\infty))$ such that

$$\|f_n - f\|_p \to 0, \text{ as } n \to \infty.$$

That is, for any $\epsilon > 0$, there exists $N > 0$ such that $\|f_n - f\|_p < \epsilon$ if $n > N$. It shows by the Jensen's Inequality that

$$\begin{aligned}|F_n(x) - F(x)| &\leq \tfrac{1}{x}\int_0^x |f_n(t) - f(t)|dt) \\ &\leq \tfrac{1}{x}(\int_0^x |f_n(t) - f(t)|^p dt)^{\frac{1}{p}} \cdot x^{\frac{1}{q}} \\ &\leq x^{-\frac{1}{p}}\|f_n - f\|_p\end{aligned}$$

and thus

$$F_n(x) \to F(x), \ n \to \infty, \ \forall x \in (0,\infty).$$

Since

$$\|F_n\|_p \leq \frac{p}{p-1}\|f_n\|_p,\ n = 1, 2, \cdots$$

it shows that $F_n \in L^p$ and $\{F_n\} \subset L^p$ is Cauchy sequence. By the completion of L^p, it exists $G(x) \in L^p$ satisfying

$$\|F_n - G\|_p \to 0, n \to \infty$$

and so $\|F_n\|_p \to \|G\|_p < +\infty$. By the Fatou's Lemma,

$$\|F\|_p \leq \lim_{n\to\infty} \|F_n\|_p = \|G\|_p < +\infty$$

whence $F \in L^p$. Moreover,

$$\|F\|_p \leq \|G\|_p = \lim_{n\to\infty} \|F_n\|_p \leq \lim_{n\to\infty} \frac{p}{p-1}\|f_n\|_p = \frac{p}{p-1}\|f\|_p.$$

Therefore, the map $f \to F$ is continuous from $L^p(0,\infty)$ to $L^p(0,\infty)$.

(b) Firstly, assume that $f \geq 0$ and $f \in C_c(0,\infty)$. Then, by (a), we have

$$\begin{aligned}\int_0^\infty F^p(x)dx &= \tfrac{p}{p-1}\int_0^\infty F^{p-1}(x)f(x)dx \\ &\leq \tfrac{p}{p-1}(\int_0^\infty F^{(p-1)q}(x)dx)^{\frac{1}{q}}(\int_0^\infty f^p(x)dx)^{\frac{1}{p}} \\ &\leq \tfrac{p}{p-1}(\int_0^\infty F^p(x)dx)^{\frac{1}{q}}(\int_0^\infty f^p(x)dx)^{\frac{1}{p}}\end{aligned}$$

and so

$$\begin{aligned}\|F\|_p &= \tfrac{p}{p-1}\|f\|_p \\ &\Leftrightarrow \exists\ \alpha, \beta \in \mathbb{R} \text{ and } \alpha\beta \neq 0 \text{ satisfying } \alpha F^p = \beta f^p\ a.e. \\ &\Leftrightarrow \alpha F = \beta f\ a.e. \\ &\Leftrightarrow f = 0\ a.e.\end{aligned}$$

.

Therefore, it shows the conclusion $f = 0$ *a.e.* if $f \in C_c(0,\infty)$ and $\|F\|_p = \frac{p}{p-1}\|f\|_p$.

(c)

(d)Suppose that $f > 0$ and $f \in L^1$, then $G = \int_0^\infty f d\mu < +\infty$ and so it exists $N > 0$ such that

$$\int_0^x f(t)dt > \frac{G}{2}$$

when $x > N$. Thus

$$\int_0^\infty F(x)dx \geq \int_N^\infty \frac{1}{x}\int_0^x f(t)dt \geq \frac{G}{2}\int_N^\infty \frac{1}{x}dx = +\infty$$

and therefore $F \notin L^1$.

18. Let μ be a positive measure on X. A sequence $\{f_n\}$ of complex measurable functions on X is said to converge in measure to the measurable function f if to every $\varepsilon > 0$ there corresponds an N such that

$$\mu(\{x : |f_n(x) - f(x)| > \varepsilon\}) < \varepsilon$$

for all $n > N$. (This notion is of importance in probability theory.) Assume $\mu(X) < \infty$ and prove the following statements:

(a) If $f_n(x) \to f(x) a.e.$, then $f_n \to f$ in measure.

(b) If $f_n \in L^p(\mu)$ and $\|f_n - f\|_p \to 0$, then $f_n \to f$ in measure; here $1 \leq p \leq \infty$.

(c) If $f_n \to f$ in measure, then $\{f_n\}$ has a subsequence which converges to $f a.e.$

Investigate the converses of (a) and (b). What happen to (a), (b),and (c) if $\mu(X) = \infty$, for instance, if μ is Lebesgue measure on $\mathbb{R}^1$?

[**Proof**]: (a) Set $E = \{x \in X : \lim_{n\to\infty} f_n(x) \neq f(x)\}$, then $\mu(E) = 0$. For every $\varepsilon > 0$, we define

$$E_k = \{x \in X : |f_n(x) - f(x)| > \varepsilon\},\ k = 1, 2, \cdots;$$
$$F_n = \bigcup_{k\geq n}^{\infty} E_k,\ n = 1, 2, \cdots; F = \bigcap_{n=1}^{\infty} F_n.$$

It is easy that $F \subset E$ and so $\mu(F) = 0$. Since $\{F_n\}$ is a sequence of decreasing measurable sets and $\mu(X) < +\infty$, it shows that

$$\lim_{n\to\infty} \mu(F_n) = \mu(F) = 0.$$

As $E_n \subset F_n,\ n = 1, 2, \cdots$, it is trival that

$$\lim_{n\to\infty} \mu(E_n) = 0.$$

It is then proved that $f_n \to f$ in measure.

(b) For every $\varepsilon > 0$, suppose that

$$E_k = \{x \in X : |f_n(x) - f(x)| > \varepsilon\},\ k = 1, 2, \cdots.$$

Then it follows that for $1 \leq p < \infty$,

$$\varepsilon^p \mu(E_n) \leq \int_X |f_n - f|^p d\mu = \|f_n - f\|_p^p.$$

Since $\|f_n - f\|_p \to 0$, it shows that

$$\lim_{n\to\infty} \mu(E_n) = 0.$$

It is then proved that $f_n \to f$ in measure. If $p = +\infty$ and $\|f_n - f\|_p \to 0$, there exists a national $N > 0$ satisfying

$$\|f_n - f\|_p < \varepsilon, \text{ whenever } n > N.$$

Hence it shows

$$\mu(E_n) = 0, \text{ whenever } n > N.$$

It is then proved that $f_n \to f$ in measure.

(c)Suppose that $f_n \to f$ in measure. For every national k,there exists national n_k such that

$$\mu(E_k) < \frac{1}{2^k}, \ k = 1, 2, \cdots.$$

where $E_k = \{x \in X : |f_{n_k}(x) - f(x)| > \frac{1}{2^k}\}$.
Define

$$F = \bigcap_{n=1}^{\infty} \bigcup_{k\geq n}^{\infty} E_k$$

then we have

$$\mu(F) \leq \mu(\bigcup_{k\geq n}^{\infty} E_k) \leq \sum_{k=n}^{\infty} \mu(E_k) = \frac{1}{2^{n-1}}, \ n = 1, 2, \cdots.$$

It is obvious that $\mu(F) = 0$. For any $x \in X - F$, it follows that there exists national N such that

$$|f_{n_k}(x) - f(x)| \leq \frac{1}{2^k}, \text{ for each national } k > N.$$

Therefore, we obtain

$$\lim_{k\to\infty} f_{n_k}(x) = f(x).$$

It then shows that $\{f_n\}$ has a subsequence which converges to f $a.e.$

20. Suppose ϕ is a real function on $\mathbb{R}$ such that

$$\phi(\int_0^1 f(x)dx) \leq \int_0^1 \phi(f)dx$$

for every real bounded measurable f. Prove that ϕ is then convex.

[**Proof**]: Assume that

$$x, y \in \mathbb{R}; \quad 0 < \lambda < 1.$$

Then

$$\lambda x + (1-\lambda)y = \int_0^1 (x\chi_{[0,\lambda]}(t) + y\chi_{[\lambda,1]})(t)dt.$$

Thus

$$\begin{aligned} \phi(\lambda x + (1-\lambda)y) &= \phi(\textstyle\int_0^1 (x\chi_{[0,\lambda]}(t) + y\chi_{[\lambda,1]})(t)dt) \\ &\leq \textstyle\int_0^1 \phi \circ (x\chi_{[0,\lambda]}(t) + y\chi_{[\lambda,1]})(t)dt \\ &= \lambda\phi(x) + (1-\lambda)\phi(y). \end{aligned}$$

Therefore, ϕ is convex.

部分习题答案

chapter four

In this set of exercises, H always denotes a Hilbert space.

1. If M is a closed subspace of H, prove that $M = (M^{\perp})^{\perp}$. Is there a similar true statement for subspaces M which are not necessarily closed?

[**Proof**]: It is easy $M \subset (M^{\perp})^{\perp}$. on the other hand, for any $z \in (M^{\perp})^{\perp}$, it exist $x \in M$ and $y \in M^{\perp}$ with $z = x + y$ when M is a closed subspace of H. Thus,

$$0 = (z, y) = (x, y) + (y, y) = (y, y)$$

and so $y = 0$. Therefore, $z = x \in M$. It thus follows that $M \supset (M^{\perp})^{\perp}$ and whence $M = (M^{\perp})^{\perp}$.

Next, assume that M is not closed subspace. Since $M \subset (M^{\perp})^{\perp}$ and $(M^{\perp})^{\perp}$ is closed, it is obvious that $\overline{M} \subset (M^{\perp})^{\perp}$. since $M \subset \overline{M}$, it shows $\overline{M}^{\perp} \subset M^{\perp}$. Moreover, $(M^{\perp})^{\perp} \subset (\overline{M}^{\perp})^{\perp}$. Again since $\overline{M} = (\overline{M}^{\perp})^{\perp}$, it is proved that $\overline{M} = (M^{\perp})^{\perp}$.

2. Let $\{x_n : n = 1, 2, 3, \cdots\}$ be a linearly independent set of vectors in H. Show that the following construction yields an orthonormal set $\{u_n\}$ such that $\{x_1, x_2, \cdots, x_N\}$ and $\{u_1, u_2, \cdots, u_N\}$ have the same span for all N.

[**Proof**]: Let $\{x_n : n = 1, 2, 3, \cdots\}$ be a linearly independent set of vectors in H. Put

$$u_1 = \frac{x_1}{\|x_1\|};\ v_n = x_n - \sum_{k=1}^{n-1}(x_n, u_k)u_k;\ u_n = \frac{v_n}{\|v_n\|};\ n = 2, 3, \cdots$$

Then it is easy that

$$(v_2, u_1) = 0; (u_2, u_1) = 0$$

and so $(v_n, u_k) = 0, k = 1, 2, \cdots, n-1$. Therefore, $\{u_n\}$ is an orthonormal set. Moreover,

$$span\{x_n\} = span\{u_n\}.$$

3. Show that $L^p(T)$ is separable if $1 \leq p < \infty$, but that $L^{\infty}(T)$ is not separable.

[**Proof**]: Firstly, assume that $1 \leq p < \infty$, then $C(T)$ is dense in $L^p(T)$. Let $\mathcal{P}$ be the set of all trigonometric polynomials and $\tilde{\mathcal{P}}$ be the set of all trigonometric polynomials with rational coefficients, it is obvious that $\mathcal{P}$ is dense in $C(T)$and $\tilde{\mathcal{P}}$ is also. Since

$$\|f - g\|_p \leq \|f - g\|_\infty, \quad \forall f, g \in C(T)$$

and so it shows that $\tilde{\mathcal{P}}$ is countable dense subset in $L^p(T)$. This proves that $L^p(T)$ is separable.

Next, we prove that $L^\infty(T)$ is not separable. Assume that $L^\infty(T)$ is separable and let $\{u_n\}$ is the countable dense subset of $L^\infty(T)$. Put

$$f_t(s) = \chi_{[0,t]}(s); 0 < t < 2\pi$$

and extend it to the real axis $\mathbb{R}$ with period 2π. Thus,

$$f_t \in L^\infty(T) \text{ and } \|f_t\|_\infty = 1.$$

Take $0 < \epsilon < \frac{1}{2}$, according to the assumption, it exists some u_k satisfying

$$U(u_k, \epsilon) \cap \{u_n\} \text{ is infinite set.}$$

Put $f_t, f_{t'} \in U(u_k, \epsilon); 0 < t < t' \leq 2\pi$, then

$$1 = \|f_t - f_{t'}\|_\infty \leq \|f_t - u_k\|_\infty + \|f_{t'} - u_k\|_\infty < 2\epsilon < 1$$

It is contradiction. Therefore, $L^\infty(T)$ is not separable.

4. Show that H is separable if and only if H contains a maximal orthonormal system which is at most countable.

[**Proof**]: First, there exists a countable dense subset M of H if H is separable. Let N is the maximal linearly independent subset of M. By the conclusion of Exercise 2, it shows the existence of a maximal orthonormal system of H which is at most separable.

Second, suppose that $\{u_n\}$ is a countabe maximal orthonormal system of H. Put $M = \overline{span\{u_n\}}$. If $M \neq H$, ther exists an element $v \in H$ and $v \notin M$. Furthermore,

$$\overline{v} = \sum_{n=1}^{\infty} (v, u_n) u_n \in M; \quad v - \overline{v} \neq 0.$$

Let $u_0 = \frac{v-\overline{v}}{\|v-\overline{v}\|}$, then

$$\|u_0\| = 1;\ (u_0, u_n) = 0,\ n = 1, 2, \cdots.$$

Therefore, $\{u_0, u_1, u_2, \cdots\} \subset H$ is a orthonormal set but this contradict the maximality of $\{u_1, u_2, \cdots\} \subset H$. It thus follows that $M = H$. So H is separable.

5. If $M = \{x : Lx = 0\}$, where L is a continuous linear functional on H, prove that $M^\perp$ is a vector space of dimension 1(unless $M = H$).

[**Proof**]: If $L = 0$, then $M = H$ and $M^\perp = \{0\}$. in the following, assume that $L \neq 0$. Then it exists unique element $y \neq 0$ of H such that

$$Lx = (x, y);\quad \forall x \in H.$$

Thus, $y \perp M$ and so $M = span\{y\}^\perp$. That is, $M^\perp = span\{y\}$ since M is a closed subspace by the continuity of L. This complete the proof.

7. Suppose that $\{a_n\}$ is a sequence of positive numbers such that $\sum a_n b_n < \infty$ whenever $b_n \geq 0$ and $\sum b_n^2 < \infty$. Prove that $\sum a_n^2 < \infty$.

[**Proof**]: Assume that $\sum\limits_{n=1}^{\infty} a_n^2 = +\infty$. For any $k \in \mathbb{N}$, it then exists $n_k \in \mathbb{N}$ such that

$$\sum_{n=1}^{n_k} a_n^2 > k;\quad 1 < n_1 < n_2 < n_3 < \cdots$$

Hence

$$c_k^2 = \sum_{n_k+1}^{n_{k+1}} a_n^2 > 1;\quad k = 1, 2, \cdots.$$

Put

$$b_n = \begin{cases} \frac{a_n}{kc_k}; & n_k \leq n < n_{k+1},\quad k = 1, 2, \cdots \\ 0; & 1 \leq n < n_1 \end{cases}$$

Then

$$\sum_{n=1}^{\infty} b_n^2 = \sum_{k=1}^{\infty} \sum_{n=n_k+1}^{n_{k+1}} \frac{a_n^2}{k^2 c_k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2} < +\infty$$

but

$$\sum_{n=1}^{\infty} a_n b_n = \sum_{k=1}^{\infty} \sum_{n=n_k+1}^{n_{k+1}} \frac{a_n^2}{kc_k} = \sum_{k=1}^{\infty} \frac{c_k}{k} = +\infty$$

Therefore, the assumption is error and $\sum a_n^2 < \infty$.

9. If $A \subset [0, 2\pi]$ and A is measurable, prove that

$$\lim_{n\to\infty} \int_A \cos nx dx = \lim_{n\to\infty} \int_A \sin nx dx = 0.$$

[**Proof**]: If $A \subset [0, 2\pi]$ and A is measurable, then $\chi_A \in L^2([0, 2\pi])$. Since $L^2([0, 2\pi])$ is Hilbert space and $\{e^{int} : n \in \mathbb{Z}\}$ is the maximal orthonormal set of $L^2([0, 2\pi])$, and so we have

$$\sum_{n\in\mathbb{Z}} |(\chi_A, e^{int})|^2 = \|\chi_A\|_2 < +\infty$$

Therefore, it shows that

$$|(\chi_A, e^{int})| \to 0, \quad |n| \to \infty$$

Furthermore, it follows that

$$\lim_{n\to\infty} \int_A \cos nx dx = \lim_{n\to\infty} \int_A \sin nx dx = 0.$$

16. If $x_0 \in H$ and M is a closed linear subspace of H, prove that

$$\min\{\|x - x_0\| : x \in M\} = \max\{|(x_0, y)| : y \in M^\perp, \|y\| = 1\}.$$

[**Proof**]: Suppose that P and Q are the projections on M, $M^\perp$ respectively. For $x_0 \in H$, it shows that $x_0 = Px_0 + Qx_0$ and

$$\|Qx_0\| = \min\{\|x - x_0\| : x \in M\}.$$

On the other hand, for any $y \in M^\perp$ with $\|y\| = 1$, then

$$(x_0, y) = (Px_0 + Qx_0, y) = (Qx_0, y)$$

and whence

$$|(x_0, y)| = |(Qx_0, y)| \le \|Qx_0\| \cdot \|y\| = \|Qx_0\|.$$

Since $Qx_0 \in M^\perp$, we may put $y_0 = \frac{Qx_0}{\|Qx_0\|}$ when $Qx_0 \neq 0$. Then $(x_0, y_0) = \|Qx_0\|$. Thus

$$\max\{|(x_0, y)| : y \in M^\perp, \|y\| = 1\} = \|Qx_0\|.$$

This complet the proof.

部分习题答案

chapter five

2. Prove that the unit ball(open or closed) is convex in every normed linear space.

[**Proof**]: Let X be a normed linear space and S be the open nuit ball of X. Assume that $0 \leq \alpha, \beta \leq 1$ and $\alpha + \beta = 1$. For any $x, y \in S$,

$$\|\alpha x + \beta y\| \leq \|\alpha x\| + \|\beta y\| = \alpha\|x\| + \beta\|y\| < \alpha + \beta = 1$$

and so $\alpha x + \beta y \in S$. It shows that S is convex.

4. Let C be the space of all continuous functions on $[0, 1]$, with the supremum norm. Let M consist of all $f \in C$ for which

$$\int_0^{\frac{1}{2}} f(t)dt - \int_{\frac{1}{2}}^1 f(t)dt = 1.$$

Prove that M is closed convex subset of C which contains no element of minimal norm.

[**Proof**]: By use of the Lebesgue dominated convergence theorem, it is easy that M is closed convex set of C. Moreover, for any $f \in M$,

$$1 = \int_0^{\frac{1}{2}} f(t)dt - \int_{\frac{1}{2}}^1 f(t)dt \leq \int_0^1 |f(t)|dt = \|f\|_1 \leq \|f\|_\infty.$$

If $f \in C$ and $\|f\|_\infty = 1$, it holds that

$$\int_0^1 |f(t)|dt = 1; \quad \int_0^1 (1 - |f(t)|)dt = 0; \quad 1 - |f(t)| \geq 0, \ \forall t \in [0, 1]$$

and so $|f(t)| = 1, \ \forall t \in [0, 1]$. But in this case, $f \notin M$. Therefore, M is closed convex subset of C which contains no element of minimal norm.

5. Let M be the set of all $f \in L^1([0, 1])$, relative to Lebesgue measure, such that

$$\int_0^1 f(t)dt = 1.$$

Show that M is a closed convex subset of $L^1([0, 1])$ which contains infinitely many elements of minimal norm.

[**Proof**]: Assume that $0 \leq \alpha, \beta \leq 1$ and $\alpha + \beta = 1$. For any $f, g \in M \subset L^1([0,1])$,

$$\int_0^1 [\alpha f + \beta g](t)dt = \alpha \int_0^1 f(t)dt + \beta \int_0^1 g(t)dt = \alpha + \beta = 1$$

whence $\alpha f + \beta g \in M$. So M is convex. Next, given

$$\{f_n\} \subset M; \text{ and } \|f_n - f\|_1 \to 0, \ n \to \infty.$$

Then

$$\int_0^1 f_n(t)dt - \int_0^1 f(t)dt = \int_0^1 \leq \int_0^1 |f_n(t) - f(t)|dt = \|f_n - f\|_1 \to 0$$

and so

$$\lim_{n\to\infty} \int_0^1 f_n(t)dt = \int_0^1 f(t).$$

It follows that $f \in M$ since $\{f_n\} \subset M$ and so M is closed convex subset. Moreover,

$$\|f\|_1 \geq 1, \quad \forall f \in M.$$

Since

$$nt^{n-1} \in M, \ \forall n \in \mathbb{N}; \text{ and } \|nt^{n-1}\|_1 = 1$$

it shows that M contains infinitely many elements of minimal norm.

6. Let f be a bounded linear functional on a subspace M of a Hilbert space H. Prove that f has a unique norm-preserving extension to a bounded linear functional on H, and that this extension vanishes on $M^\perp$.

[**Proof**]: If $f = 0$, we only need to take the bounded linear functional $F = 0$ on H. In the following, we assume $f \neq 0$. Since f is a bounded linear functional on a subspace M, it is easy to extend f to a bounded linear functional $\tilde{f}$ on $\overline{M}$ and $\|f\| = \|\tilde{f}\|$. In fact, for any $x \in \overline{M}$, there exists a sequence $\{x_n\} \subset M$ such that $\|x_n - x\| \to 0$ and then define $\tilde{f}(x) = \lim_{n\to\infty} f(x_n)$. So we can define a linear functional of H such that

$$F(x) = \begin{cases} \tilde{f}(x); & x \in \overline{M} \\ 0; & x \in M^\perp \end{cases}$$

and then it is proved that F is the norm-preserving linear extension on H of f and this extension vanishes on $M^\perp$. Moreover, for any norm-preserving linear

extension g on H of f, $g\,|_{\overline{M}} = \tilde{f}$. Since $M^\perp = \overline{M}^\perp$ and $H = \overline{M} \oplus M^\perp$, it shows that it is unique that the norm-preserving extension of f to a bounded linear functional on H which vanishes on $M^\perp$.

8. Let X be a normed linear space, and let X^* be its dual space, as defined in Sec. 5.21, with the norm

$$\|f\| = \sup\{|f(x)| : \|x\| \le 1\}.$$

(a)Prove that X^* is a Banach space.
(b) Prove that the mapping $f \to f(x)$ is, for each $x \in X$, a bounded linear functional on X^*, of norm $\|x\|$.
(c) Prove that $\{\|x_n\|\}$ is bounded if $\{x_n\}$ is a sequence in X such that $\{f(x_n)\}$ is bounded for every $f \in X^*$.

[**Proof**]: (a) It is obvious that X^* is a normed linear space. Given $\{f_n\} \subset X^*$ is a cauchy sequence, then for any $\epsilon > 0$, there is a $N > 0$ such that

$$\|f_n - f_m\| < \epsilon; \quad \forall n, m > N$$

and so

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\| \cdot \|x\| < \|x\| \cdot \epsilon; \quad \forall n, m > N, \ \forall x \in X.$$

Thus $\{f_n(x)\} \subset \mathbb{C}$ is also a cauchy sequence and so we can define

$$f(x) = \lim_{n\to\infty} f_n(x); \quad \forall x \in X.$$

It is easy that f is a linear functional on X. Since $\{\|f_n\|\}$ is a cauchy sequence, ther is $M > 0$ satisfying

$$\|f_n\| \le M, \ n = 1, 2, \cdots.$$

Hence, for ϵ, N as above, it shows for $n > N$ the following

$$|f(x)| \le |f_n(x)| + \|x\|\epsilon \le \|x\|(M + \epsilon); \quad \forall x \in X.$$

It forces

$$|f(x)| \le M\|x\|; \quad \forall x \in X$$

and so $\|f\| \le M$. That is, $f \in X^*$. Therefore, X^* is a Banach space.

(b) For each $x \in X$, define

$$\lambda(f) = f(x); \quad \forall f \in X^*$$

Then for any $f, g \in X^*$ and $\alpha, \beta \in \mathbb{C}$,

$$\Lambda(\alpha f + \beta g) = (\alpha f + \beta g)(x) = \alpha f(x) + \beta g(x) = \alpha\Lambda(f) + \beta\Lambda(g)$$

and we see that Λ is a linear functional on X^*. Since

$$|\Lambda(f)| = |f(x)| \leq \|f\| \cdot \|x\|; \quad \forall f \in X^*$$

whence $\|\Lambda\| \leq \|x\|$. From the Hahn-Banach Theorem, there is $g \in X^*$ such that $g(x) = \|x\|$ and $\|g\| = 1$. It implies that $\|\Lambda\| = \|x\|$.

(c) Suppose that $\{x_n\} \subset X$ such that $\{f(x_n)\}$ is bounded for every $f \in X^*$. By (a),(b) and the Banach-Steinhaus Theorem, it is easy to prove that $\{\|x_n\|\}$ is bounded.

9. Let c_0, l^1, l^∞ be the Banach spaces consisting of all complex sequences $x = \{\xi_i\},\ i = 1, 2, \cdots$, define as follows:

$$x \in l^1 \text{if and only if} \|x\|_1 = \sum |\xi_i| < \infty.$$

$$x \in L^\infty \text{if and only if} \|x\|_\infty = \sup |\xi_i| < \infty.$$

c_0 is the subspace of l^∞ consisting of all $x \in l^\infty$ for which $\xi_i \to 0$ as $i \to \infty$. Prove the following four statements.

(a) If $y = \{\eta_i\} \in l^1$ and $\Lambda x = \sum \xi_i \eta_i$ for every $x \in c_0$, then Λ is a bounded linear functional on c_0, and $\|\Lambda\| = \|y\|_1$. Moreover, every $\Lambda \in (c_0)^*$ is obtained in this way. In brief, $(c_0)^* = l^1$.

(b) In the same sense, $(l^1)^* = l^\infty$.

(c) Every $y \in l^1$ induces a bounded linear functional on l^∞, as in (a). However, this does not give all of $(l^\infty)^*$, since $(l^\infty)^*$ contains nontrivial functionals that vanish on all of c_0.

(d) c_0 and l^1 are separable but l^∞ is not.

[**Proof**]: (a) For each $y = \{\eta_i\} \in l^1$, we define

$$\Lambda x = \sum_{i=1}^{\infty} \xi_i \eta_i; \quad \forall x = \{\xi_i\} \in c_0.$$

Then the above series converges absolutely and so Λ is linear mapping. Since

$$|\Lambda x| \leq \|x\|_\infty \cdot \|y\|_1$$

it follows that $\|\Lambda\| \leq \|y\|_1$. Thus, $\Lambda \in (c_0)^*$. On the other hand, put

$$e_i = \{\overbrace{0, 0, \cdots, 1}^{i}, 0, 0, \cdots, \}; \quad i = 1, 2, \cdots.$$

It is obvious that $e_i \in c_0$ and $\|e_i\|_\infty = 1,\ i = 1, 2, \cdots$. For any $\Lambda \in (c_0)^*$,take

$$\eta_i = \Lambda e_i, \quad i = 1, 2, \cdots.$$

Then

$$|\eta_i| = |\Lambda e_i| \leq \|\Lambda\| \cdot \|e_i\|_\infty = \|\Lambda\|.$$

Let $y = \{\eta_i\}$ and then $y \in l^\infty$ with $\|y\|_\infty \leq \|\Lambda\|$. Assume $x_n = \{\xi_i^n\}$, where

$$\xi_i^n = \begin{cases} \frac{\overline{\eta_i}}{|\eta_i|}; & \eta_i \neq 0,\ i = 1, 2, \cdots, n; \\ 0; & \eta_i = 0, \text{or } i > n. \end{cases}$$

then $x_n \in c_0$ with $\|x_n\|_\infty = 1, n = 1, 2, \cdots$. Moreover,

$$\Lambda x_n = \sum_{i=1}^{n} |\eta_i| \leq \|\Lambda\| \cdot \|x_n\|_\infty = \|\Lambda\|$$

whence

$$\sum_{i=1}^{\infty} |\eta_i| \leq \|\Lambda\|$$

Thus $y \in l^1$ and $\|y\|_1 \leq \|\Lambda\|$. Since for any $x = \{\xi_i\} \in c_0$, $x = \sum\limits_{i=1}^{\infty} \xi_i e_i$ and so $\Lambda x = \sum\limits_{i=1}^{\infty} \xi_i \eta_i$. From the proof above, it implies that $\|\Lambda\| \leq \|y\|_1$ and so $\|\Lambda\| = \|y\|_1$. Therefore, $(c_0)^* = l^1$.

(b) For each $y = \{\eta_i\} \in l^\infty$, define

$$\Lambda x = \sum_{i=1}^{\infty} \xi_i \eta_i; \quad \forall x = \{\xi_i\} \in l^1.$$

Then the above series converges absolutely and so Λ is linear mapping. Since

$$|\Lambda x| \leq \|y\|_\infty \cdot \|x\|_1$$

it shows that $\|\Lambda\| \le \|y\|_\infty$. Thus, $\Lambda \in (l^1)^*$. On the other hand, suppose that $\Lambda \in (l^1)^*$. Put

$$\eta_i = \Lambda e_i, \quad i = 1, 2, \cdots.$$

Then

$$|\eta_i| = |\Lambda e_i| \le \|\Lambda\| \cdot \|e_i\|_1 = \|\Lambda\|.$$

Let $y = \{\eta_i\}$ and then $y \in l^\infty$ with $\|y\|_\infty \le \|\Lambda\|$. Since for any $x = \{\xi_i\} \in l^1$, $x = \sum_{i=1}^{\infty} \xi_i e_i$ and so $\Lambda x = \sum_{i=1}^{\infty} \xi_i \eta_i$. It implies that $\|\Lambda\| \le \|y\|_\infty$ and so $\|\Lambda\| = \|y\|_\infty$. Therefore, $(l^1)^* = l^\infty$.

(c) For each $y = \{\eta_i\} \in l^\infty$, define

$$\Lambda x = \sum_{i=1}^{\infty} \xi_i \eta_i; \quad \forall x = \{\xi_i\} \in l^\infty.$$

Then the above series converges absolutely and so Λ is linear mapping. Since

$$|\Lambda x| \le \|x\|_\infty \cdot \|y\|_1$$

it shows that $\|\Lambda\| \le \|y\|_1$. Thus, $\Lambda \in (l^\infty)^*$. Since $c_0 \subset l^\infty$ is closed subspace, by the Hahn-Banach theorem, there is $f \in (l^\infty)^*$ such that $f\,|_{c_0} = 0$. By (a), $(c_0)^* = l^1$ and so $f \notin l^1$. Therefore, $l^1 \ne (l^\infty)^*$.

(d) Since

$$\overline{span\{e_i\}}^{c_0} = c_0; \quad \overline{span\{e_i\}}^{l^1} = l^1$$

it is easy to prove that c_0, l^1 are separable.

10. If $\sum \alpha_i \xi_i$ converges for every sequence (ξ_i) such that $\xi_i \to 0$ as $i \to \infty$, prove that $\sum |\alpha_i| < \infty$.

[**Proof**]: Put

$$x = \{\alpha_i\}; \quad x_n = \{\alpha_1, \alpha_2, \cdots, \alpha_n, 0, 0, \cdots\}; \quad \xi = \{\xi_n\} \in c_0.$$

and define

$$\rho(\xi) = \sum_{i=1}^{\infty} \alpha_i \xi_i; \quad \rho_n(\xi) = \sum_{i=1}^{n} \alpha_i \xi_i.$$

Then ρ is linear and $\rho_n (n \in \mathbb{N})$ is bounded linear functionals on c_0. Moreover,

$$\|\rho_n\| = \sum_{i=1}^{n} |\alpha_i|, \quad n \in \mathbb{N}.$$

Since

$$\lim_{n\to\infty} \rho_n(\xi) = \rho(\xi); \quad \forall \xi \in c_0$$

exists, by the Banach-Steinhaus Theorem, there is $M > 0$ satisfying

$$\|\rho_n\| \leq M; \quad n \in \mathbb{N}$$

That is,

$$\sum_{i=1}^{n} |\alpha_i| \leq M, \quad n \in \mathbb{N}.$$

It thus forces that $\sum_{i=1}^{\infty} |\alpha_i| < +\infty$.

11. For $0 < \alpha \leq 1$, let Lipα denote the space of all complex functions f on $[a, b]$ for which

$$M_f = \sup_{s \neq t} \frac{|f(s) - f(t)|}{|s - t|^\alpha} < \infty.$$

Prove that Lipα is a Banach space, if $\|f\| = |f(a)| + M_f$; also, if

$$\|f\| = M_f + \sup_{x \in [a,b]} |f(x)|.$$

[**Proof**]: It is easy to prove that Lipα is a normed linear space with the corresponding norm. Let $\{f_n\}$ be a cauchy sequence of Lipα.

(a) Assume

$$\|f\| = |f(a)| + M_f.$$

There thus exists $G > 0$ such that $M_{f_n} \leq G$. For any $\epsilon > 0$, there is $N > 0$ such that

$$|f_n(a) - f_m(a)| < \epsilon; M_{f_n - f_m} < \epsilon$$

when $n, m > N$. For every $s \in (a, b]$,

$$\begin{aligned} \frac{|f_n(s)-f_m(s)|}{|s-a|^\alpha} &\leq \frac{|f_n(a)-f_m(a)|}{|s-a|^\alpha} + \frac{|f_n(s)-f_m(s)-f_n(a)+f_m(a)|}{|s-a|^\alpha} \\ &\leq M_{f_n-f_m} + \frac{|f_n(a)-f_m(a)|}{|s-a|^\alpha} \end{aligned}$$

It implies that $\{f_n(s)\}$ is cauchy sequence for each $s \in [a, b]$. Therefore, we can define

$$\lim_{n\to\infty} f_n(s) = f(s); \quad \forall s \in [a, b].$$

Since for any $s, t \in [a, b]$ and $s \neq t$, for each $n \in \mathbb{N}$,

$$\begin{aligned} \frac{|f(s)-f(t)|}{|s-t|^\alpha} &\leq \frac{|f_n(s)-f_n(t)|}{|s-t|^\alpha} + \frac{|f_n(s)-f(s)|}{|s-t|^\alpha} + \frac{|f_n(t)-f(t)|}{|s-t|^\alpha} \\ &\leq \frac{|f_n(s)-f(s)|}{|s-t|^\alpha} + \frac{|f_n(t)-f(t)|}{|s-t|^\alpha} + G \end{aligned}$$

Let $n \to \infty$, then

$$\sup_{s \neq t} \frac{|f(s) - f(t)|}{|s - t|^\alpha} \leq G < +\infty.$$

That is, $f \in \text{Lip}\alpha$ and so $\text{Lip}\alpha$ is a Banach space.

(b) Assume

$$\|f\| = M_f + \sup_{x \in [a,b]} |f(x)|.$$

There thus exists $G > 0$ such that $M_{f_n} \leq G$. Put $\|f\|_\infty = \sup_{x \in [a,b]} |f(x)|$ and so $\{\|f_n\|_\infty\}$ is cauchy sequence. Then it exists a complex function f on $[a, b]$ such that

$$\sup_{x \in [a,b]} |f_n(x) - f(x)| \to 0; \quad n \to 0.$$

Moreover, for any $s, t \in [a, b]$ and $s \neq t$, for each $n \in \mathbb{N}$,

$$\begin{aligned} \tfrac{|f(s)-f(t)|}{|s-t|^\alpha} &\leq \tfrac{|f_n(s)-f_n(t)|}{|s-t|^\alpha} + \tfrac{|f_n(s)-f(s)|}{|s-t|^\alpha} + \tfrac{|f_n(t)-f(t)|}{|s-t|^\alpha} \\ &\leq \tfrac{|f_n(s)-f(s)|}{|s-t|^\alpha} + \tfrac{|f_n(t)-f(t)|}{|s-t|^\alpha} + G \end{aligned}$$

Let $n \to \infty$, then

$$\sup_{s \neq t} \frac{|f(s) - f(t)|}{|s - t|^\alpha} \leq G < +\infty.$$

That is, $f \in \text{Lip}\alpha$ and so $\text{Lip}\alpha$ is a Banach space.

16. Suppose X and Y are Banach spaces, and suppose Λ is a linear mapping of X into Y, with the following property: For every sequence $\{x_n\}$ in X for which $x = \lim x_n$ and $y = \lim \Lambda x_n$ exist, it is true that $y = \Lambda x$. Prove that Λ is continuous.

[**Proof**]: Provided that X, Y are Banach spaces. Define

$$X \oplus Y = \{(x, y) \mid x \in X,\ y \in Y\}$$

The addition and scalar multiplication on $X \oplus Y$ are difined in the obvious manner as follows:

$$(x_1, y_1)+(x_2, y_2) = (x_1+x_2, y_1+y_2);\ \ \alpha(x, y) = (\alpha x, \alpha y);\ \ \forall x, x_i \in X,\ y, y_i \in Y, i = 1, 2.$$

If put

$$\|(x, y)\| = \|x\| + \|y\|; \quad \forall x \in X, \quad y \in Y$$

it then implies that $X \oplus Y$ is normed linear space. Given $\{(x_n, y_n)\} \subset X \oplus Y$ is cauchy sequence, and so $\{x_n\} \subset X$ and $\{y_n\} \subset Y$ are also cauchy sequences. Therefore, it exist $x \in X$ and $y \in Y$ such that

$$\|x_n - x\| \to 0; \quad \|y_n - y\| \to 0; \text{ if } \ n \to \infty.$$

Hence,

$$\|(x_n, y_n) - (x, y)\| = \|(x_n - x, y_n - y)\| = \|x_n - x\| + \|y_n - y\| \to 0; \ n \to \infty.$$

This implies that $X \oplus Y$ is Banach space. Put

$$G = \{(x, \Lambda x) \mid x \in X\}$$

it is easy that G is a subspace of $X \oplus Y$. Assume that $\{(x_n, \Lambda x_n)\} \subset G$ is cauchy sequence, and then $\{x_n\} \subset x$ and $\{\Lambda x_n\} \subset Y$ are aslo cauchy sequences. So there exist $x \in X$ and $y \in Y$ such that

$$\|x_n - x\| \to 0; \quad \|\Lambda x_n - y\| \to 0; \quad n \to \infty.$$

According to the property of Λ, it holds that $\Lambda x = y$ and $(x, \Lambda x) \in G$. It thus shows that G is closed and so G is a Banach space. Take $T(x, \Lambda x) = x$ for each $x \in x$, then T is linear mapping and

$$\|T(x, \Lambda x)\| = \|x\| \leq \|x\| + \|\Lambda x\| = \|(x, \Lambda x)\|; \quad \forall x \in X$$

whence $\|T\| \leq 1$. Moreover, T is injective and surjective bounded linear mapping. By the open mapping theorem, it show that T^{-1} is continuous and so there is $M > 0$ such that $\|T^{-1}\| \leq M$. Thus

$$\|x\| + \|\Lambda x\| = \|(x, \Lambda x)\| = \|T^{-1}x\| \leq M\|x\|; \quad \forall x \in X$$

and so

$$\|\Lambda x\| \leq M\|x\|; \quad \forall x \in X.$$

It follows that Λ is continuous.

17. If μ is a positive measure, each $f \in L^\infty(\mu)$ defines a pultiplication operator M_f on $L^2(\mu)$ into $L^2(\mu)$, such that $M_f g = fg$. Prove that $\|M_f\| \leq \|f\|_\infty$. For which measure μ is it true that $\|M_f\| = \|f\|_\infty$ for all $f \in L^\infty(\mu)$? For which $f \in L^\infty(\mu)$ does M_f map $L^2(\mu)$ onto $L^2(\mu)$?

[**Proof**]: For each $f \in L^\infty(\mu)$, define

$$M_f g = fg; \quad \forall g \in L^2(\mu)$$

Then it is easy that M_f is a linear operator from $L^2(\mu)$ into $L^2(\mu)$. Furthermore,

$$\|M_f g\|_2 = \|fg\|_2 \leq \|f\|_\infty \cdot \|g\|_2; \quad \forall g \in L^2(\mu)$$

Thus it shows that $\|M_f\| \leq \|f\|_\infty$.

If $f \in L^\infty(\mu)$ with $\frac{1}{f} \in L^\infty(\mu)$, then M_f maps $L^2(\mu)$ onto $L^2(\mu)$.

18. Suppose $\{\Lambda_n\}$ is a sequence of bounded linear transformations from a normed linear space X to a Banach space Y, suppose $\|\Lambda_n\| \leq M < \infty$ for all n, and suppose there is a dense set $E \subset X$ such that $\{\Lambda_n x\}$ converges for each $x \in E$. Prove that $\{\Lambda_n x\}$ converges for each $x \in X$.

[**Proof**]: For every $x \in X$, there is a sequence $\{x_k\} \subset E$ such that $\|x_k - x\| \to 0 (k \to \infty)$ since $E \subset X$ is a dense subset. Moreover, $\{\Lambda_n x_k\} \subset Y$ converges for each $k \in \mathbb{N}$. It thus exists a sequence $\{y_k\} \subset Y$ such that

$$\Lambda_n x_k \to y_k, \quad n \to \infty; \quad k = 1, 2, \cdots.$$

For any $\epsilon > 0$, there exists $N > 0$ such that

$$\|x_k - x_{k'}\| < \frac{\epsilon}{M}; \quad \forall k, k' > N.$$

and so

$$\|y_k - y_{k'}\| = \lim_{n\to\infty} \|\Lambda_n(x_k - x_{k'})\| \leq \lim_{n\to\infty} \|\Lambda_n\| \cdot \|x_k - x_{k'}\| \leq M \cdot \frac{\epsilon}{M} = \epsilon$$

It hence implies that $\{y_k\} \subset Y$ is a cauchy sequence. Since Y is a Banach space, there is $y \in Y$ with $\|y_k - y\| \to 0$. Therefore,

$$\begin{aligned} \|\Lambda_n x - y\| &= \|\Lambda_n x - \Lambda_n x_k + \Lambda_n x_k - y_k + y_k - y\| \\ &\leq \|\Lambda_n\| \cdot \|x_k - x\| + \|\Lambda_n x_k - y_k\| + \|y_k - y\| \\ &\leq M \cdot \|x_k - x\| + \|\Lambda_n x_k - y_k\| + \|y_k - y\| \end{aligned}$$

This implies that $\lim_{n\to\infty} \Lambda_n x = y$.

部分习题答案

chapter six

2. Prove that the example given at the end of Sec. 6.10 has stated properties.

[**Proof**]:

3. Prove that the vector space $M(X)$ of all complex regular Borel measures on a locally compact hausdorff space X is a Banach space if $\|\mu\| = |\mu|(X)$.

[**Proof**]: Firstly, assume $\mu, \lambda \in M(X)$ and $\alpha \in \mathbb{C}$. Then $\alpha\mu \in M(X)$ and $|\mu|, |\lambda|$ are regular Borel measures. For any measurable set E in X and $\epsilon > 0$, it exist compact sets $K_1, K_2 \subset E$ and open sets $G_1, G_2 \supset E$ satisfying:

$$|\mu|(E) < |\mu|(K_1) + \frac{\epsilon}{2}; \quad |\lambda(E)| < |\lambda|(K_2) + \frac{\epsilon}{2}$$

$$|\mu|(G_1) < |\mu|(E) + \frac{\epsilon}{2}; \quad |\lambda|(G_2) < |\lambda|(E) + \frac{\epsilon}{2}$$

Take

$$K = K_1 \cup K_2; \quad g = G_1 \cap G_2$$

it implies that $K$is compact and G is open. Moreover,

$$(|\mu| + |\lambda|)(E) < (|\mu| + |\lambda|)(K) + \epsilon; \quad (|\mu| + |\lambda|)(G) < (|\mu| + |\lambda|)(E) + \epsilon$$

and so $|\mu|+|\lambda|$ is regular Norel measure. Since $|\mu+\lambda|$ and $|\mu|+|\lambda|$ are bounded positive measures and

$$|\mu + \lambda| \leq |\mu| + |\lambda|$$

it shows that $|\mu + \lambda|$ is regular Borel measure. In fact, we have the following conclusion:

Suppose that μ is bounded positive measure, then

μ is regular $\Leftrightarrow$ for any measurable set E and $\epsilon > 0$, there are compact subset $K \subset E$ and open set $G \supset E$ such that

$$\mu(E - K) < \epsilon; \quad \mu(G - E) < \epsilon.$$

Therefore, it shows that $M(X)$ is a linear space. Define

$$\|\mu\| = |\mu|(X), \quad \forall \mu \in M(X)$$

it is easy to show that $\|\mu\|$ is norm and $M(X)$ is normed linear space. By Theorem 6.19, it is obtained that $M(X) \cong (C_0(X))^*$. Since $(C_0(X))^*$ is a Banach space and so $M(X)$ is also.

4. Suppose $1 \leq p \leq \infty$, and q is the exponent conjugate to p. Suppose μ is a positive σ-finite measure and g is a measurable function such that $fg \in L^1(\mu)$ for every $f \in L^p(\mu)$. Prove that then $g \in L^q(\mu)$.

[**Proof**]: Since μ is a positive σ-finite measure, there is a sequence of disjoint measurable sets $\{X_n\}$ of X such that

$$X = \bigcup_{n=1}^{\infty} X_n; \quad \mu(X_n) < +\infty, \quad n = 1, 2, \cdots.$$

It thus shows that

$$\chi_{X_n} \in L^p(\mu), \quad n = 1, 2, \cdots.$$

Therefore,

$$g \in L^1(X_n, \mu), \quad n = 1, 2, \cdots.$$

It implies that g is finite almost everywhere on X and so we may assume that g is finite on X. Let

$$E_n = \{x \in X \mid |g(x)| \leq n\}, \quad n = 1, 2, \cdots.$$

Then $\{E_n\}$ is a sequence of increasing measurable sets and $X = \bigcup_{n=1}^{\infty} E_n$. Put $g_n = \chi_{E_n} \cdot g$ and then $\{g_n\}$ are measurable on X. Moreover,

(a) $g_n(x) \to g(x)$; $\forall x \in X$.

(b) $|g_n(x)| \leq n$; $\forall x \in X$, $n = 1, 2, \cdots$.

(c) $g_n \in L^q(\mu)$, $n = 1, 2, \cdots$.

Define

$$T_n(f) = \int_X f g_n d\mu; \quad \forall f \in L^p(\mu)$$

it is easy to prove that $\{T_n\}$ is bounded linear functionals on $L^p(\mu)$ with

$$\|T_n\| = \|g_n\|_q; \quad n = 1, 2, \cdots.$$

Since

$$|T_n(f)| \leq \int_X |f g_n| d\mu = \int_{E_n} |fg| d\mu \leq \int_X |fg| d\mu \leq \|fg\|_1; \quad n = 1, 2, \cdots$$

By use of the Banach-Steinhaus Theorem, there is $M > 0$ such that

$$\|T_n\| \leq M; \quad n = 1, 2, \cdots.$$

It hence implies that

$$\|g_n\|_q \leq M; \quad n = 1, 2, \cdots.$$

If $q = \infty$, it shows from $X = \bigcup_{n=1}^{\infty} E_n$ that there is $n > 0$ for any $x \in X$ with $x \in E_n$ and so

$$|g(x)| = |g_n(x)| \leq \|g_n\|_\infty \leq M.$$

it shows that $\|g\|_\infty \leq M$ and $g \in L^\infty(\mu)$.
In the case $1 \leq q < \infty$, because

$$|g_n(x)| \leq |g_{n+1}(x)|, \quad n = 1, 2, \cdots; \quad |g_n(x)| \to |g(x)|, \quad \forall x \in X$$

By the Lebesgue's monotone convergence Theorem, it shows

$$\int_x |g_n|^q d\mu \to \int_X |g|^q d\mu, \quad n \to \infty$$

and so

$$(\int_X |g|^q d\mu)^{\frac{1}{q}} \leq M.$$

Therefore, $\|g\|_q \leq M$ and $g \in L^q(\mu)$.

6. Sppose $1 < p < \infty$ and prove that $L^q(\mu)$ is the dual space of $L^p(\mu)$ even if μ is not σ-finite.

[**Proof**]: Sppose $1 < p < \infty$ and μ is not σ-finite.
(a) Since for any $f \in L^p(\mu)$ and $g \in L^q(\mu)$, it is easy that $f \cdot g \in L^1(\mu)$. Then we can define

$$T_g(f) = \int_x f \cdot g d\mu, \quad \forall f \in L^q(\mu)$$

and $T_g \in (L^p(\mu))^*$ with $\|T_g\| \leq \|g\|_q$. Assume

$$E_n = \{x \in X : |g| \geq \frac{1}{n}\}, \quad n = 1, 2, \cdots.$$

It is obvious that $\{E_n\}$ is an increasing sequence of measurable sets and

$$\bigcup_{n=1}^{\infty} E_n = E = \{x \in X : |g| > 0\}.$$

Put

$$f_n = \chi_{E_n}|g|^{q-1}\alpha$$

where $\alpha g = |g|;\ |\alpha| = 1$. Then $f_n \in L^p(\mu)$ and

$$\|f_n\|_p = (\int_{E_n} |g|^q d\mu)^{\frac{1}{p}};\quad n = 1, 2, \cdots.$$

Moreover, $\{|f_n|\}$ converges increasingly to $|g|^{q-1}$. Therefore,

$$T_g(f_n) = \int_X f_n \cdot g d\mu = \int_{E_n} |g|^q d\mu;\quad n = 1, 2, \cdots.$$

and so

$$\int_{E_n} |g|^q d\mu \le \|T_g\| \cdot \|f_n\|_p = \|T_g\| \cdot (\int_{E_n} |g|^q d\mu)^{\frac{1}{p}}$$

It shows that

$$\int_{E_n} |g|^q d\mu \le \|T_g\|^q;\quad n = 1, 2, \cdots.$$

By the Lebesgue's monotone convergence Theorem, it shows that

$$\int_{E_n} |g|^q d\mu = \int_X |f_n| \cdot |g| d\mu \to \int_X |g|^q d\mu;\quad n \to \infty.$$

Hence, $\int_X |g|^q d\mu \le \|T_g\|^q$ and so $\|g\|_q = \|T_g\|$.

(b) Assume $\Phi \in (L^p(\mu))^*$ and $\|\Phi\| \neq 0$. At first, for each $f \in L^p(\mu)$, define

$$D_f = \{x \in X : f(x) \neq 0\};\quad E_n = \{x \in X : |f(x)| \ge \frac{1}{n}\}.$$

Since $D_f = \bigcup_{n=1}^{\infty} E_n$, and

$$\mu(E_n)(\frac{1}{n})^p \le \int_{E_n} |f|^p d\mu \le \int_X |f|^p d\mu < +\infty$$

it thus follows that $\mu(E_n) < +\infty,\ n = 1, 2, \cdots;$ and so D_f is σ-finite measurable set.

Secondly, given $E \subset X$ is σ-finite measurable set, and put

$$M = \{f \in L^p(\mu) : D_f \subset E\}$$

then $M \subset L^p(\mu)$ is a Banach subspace. In fact, provided that $\{f_n\} \subset M$ is a cauchy sequence, there is $f \in L^p(\mu)$ such that

$$\|f_n - f\|_p \to 0,\quad n \to \infty.$$

So it exists a subsequence of $\{f_n\} \subset M$ which converges to f almost everywhere. It implies that $f \in M$. Therefore, $\Phi \mid_M$ is a bounded linear functional on M. Since it also that $M \subset L^p(E,\mu)$, by the Hahn-Banach Theorem, it can extend $\Phi \mid_M$ to a bounded linear functional $\Phi \mid_E$ of $L^p(E,\mu)$ and so there is unique $h_E \in L^q(E,\mu)$ such that

$$\Phi(f) = \int_E f \cdot h_E d\mu, \quad \forall f \in M; \quad \|h_E\|_q = \|\Phi \mid_E \| = \|\Phi \mid_M \| \leq \|\Phi\|.$$

Take $g_E = h_E$ on E and $g_E = 0$ when $x \in X - E$. It is easy that

$$g_E \in L^q(\mu); \quad D_{g_E} \subset E; \ \|g_E\|_q = \|h_E\|_q \leq \|\Phi\|$$

and

$$\Phi(f) = \int_E f \cdot h_E d\mu = \int_X f \cdot g_E d\mu; \quad \forall f \in M.$$

Order

$$\alpha = \sup\{\|g_E\|_q : E \subset X \text{ is } \sigma - \text{finite measurable subset}\}$$

it is obvious that $0 \leq \alpha \leq \|\Phi\|$. Because there is a sequence $\{f_n\} \subset L^p(\mu)$ satisfying:

$$\|f_n\|_p = 1, \ n = 1, 2, \cdots; \quad |\Phi(f_n)| \to \|\Phi\|, \ n \to \infty.$$

Take $F = \bigcup_{n=1}^{\infty} D_{f_n}$, then F is a σ-finite measurable set. Moreover,

$$|\Phi(f_n)| \leq \|\Phi \mid_F \| = \|g_F\|_q \leq \alpha \leq \|\Phi\|, \quad n = 1, 2, \cdots.$$

Therefore, it holds that $\|\Phi\| = \|g_F\|_q = \alpha$. Write $g = g_F$, then $g \in L^q(\mu)$ and $\|\Phi\| = \|g\|_q$.

Thirdly, assume that $A \subset X$ is σ-finite measurable set and write $B = A \cup F$, B is also σ-finite measurable set. Then

$$\|\Phi\| = \alpha = \|g_F\|_q \leq \|g_B\|_q \leq \alpha \leq \|\Phi\|.$$

That is,

$$\|\Phi\| = \|g_F\|_q = \|g_B\|_q.$$

Since $L^p(F,\mu) \subset L^p(B,\mu)$, it show sthat $h_B = h_F$ on F almost everywhere

and so $g_B = g_F$ a.e. on F. Again since

$$\begin{aligned}\|\Phi\|^q &= \int_X |g_B|^q d\mu \\ &= \int_B |g_B|^q d\mu \\ &= \int_{A-F} |g_B|^q d\mu + \int_F |g_B|^q d\mu \\ &= \int_{A-F} |g_B|^q d\mu + \int_F |g_F|^q d\mu \\ &\geq \int_F |g_F|^q d\mu \\ &= \|\Phi\|^q\end{aligned}$$

Hence, it holds that $\int_{A-F} |g_B|^q d\mu = 0$ and so $g_B = 0$ a.e. on $A - F$. It implies that $g = g_F = g_B$ a.e. on X.

finally, from the conclusion above, it shows that

$$\Phi(f) = \int_X g_{D_f \cup F} f d\mu = \int_X g f d\mu.$$

Therefore, $(L^p(\mu))^* = L^q(\mu)$.

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引言　指数函数

在数学中这是个最重要的函数．它是用公式来定义的，对每个复数 z，规定

$$\exp(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!} \tag{1}$$

级数(1)对每个 z 绝对收敛，对复平面的每个有界子集一致收敛．因此，exp 是连续函数．(1)的绝对收敛指出了算式

$$\sum_{k=0}^{\infty}\frac{a^k}{k!}\sum_{m=0}^{\infty}\frac{b^m}{m!}=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\frac{n!}{k!\,(n-k)!}a^k b^{n-k}$$

$$=\sum_{n=0}^{\infty}\frac{(a+b)^n}{n!}$$

是正确的．它给出了重要的加法公式

$$\exp(a)\exp(b)=\exp(a+b) \tag{2}$$

此公式对所有复数 a 和 b 是正确的．

我们规定数 e 是 $\exp(1)$，习惯上常用较短的表达式 e^z 代替 $\exp(z)$．注意，由(1)可得

$$e^0=\exp(0)=1$$

定理

(a) 对每一个复数 z，$e^z\neq 0$．

(b) exp 的导数是它自己：$\exp'(z)=\exp(z)$．

(c) exp 限制在实轴上是单调增加的正函数，且当 $x\to\infty$ 时，$e^x\to\infty$；当 $x\to-\infty$ 时，$e^x\to 0$．

(d) 存在一个正数 π 使得 $e^{\pi i/2}=i$，并使得 $e^z=1$ 当且仅当

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$z/(2\pi i)$是整数时才成立.

(e) exp 是周期函数,其周期是 $2\pi i$.

(f) 映射 $t\to e^{it}$ 将实轴映到单位圆周上.

(g) 若 w 是复数且 $w\neq 0$,则存在某些 z,使 $w=e^z$.

证明 由(2), $e^z\cdot e^{-z}=e^{z-z}=e^0=1$. 由此得到(a). 其次

$$\begin{aligned}\exp'(z)&=\lim_{h\to 0}\frac{\exp(z+h)-\exp(z)}{h}\\&=\exp(z)\lim_{h\to 0}\frac{\exp(h)-1}{h}\\&=\exp(z)\end{aligned}$$

在上述等式中,第一个是定义,第二个从(2)得到,而第三个从(1)得到,因此证明了(b).

由于(1),显然 exp 在正实轴上是单调增加的,而且当 $x\to\infty$ 时, $e^x\to\infty$. (c)的另一个断言是 $e^x\cdot e^{-x}=1$ 的结果.

对于任何实数 t,(1)表示 e^{-it} 是 e^{it} 的共轭复数. 因此

$$|e^{it}|^2=e^{it}\cdot\overline{e^{it}}=e^{it}\cdot e^{-it}=e^{it-it}=e^0=1$$

或

(3) $$|e^{it}|=1 \quad (t \text{ 是实数})$$

换句话说,若 t 为实数,则 e^{it} 位于单位圆周上. 我们定义 $\cos t$, $\sin t$ 为 e^{it} 的实部和虚部:

(4) $$\cos t=\mathrm{Re}[e^{it}],\quad \sin t=\mathrm{Im}[e^{it}] \quad (t \text{ 是实数})$$

若对等价于(4)的 Euler 恒等式

(5) $$e^{it}=\cos t+i\sin t$$

两边微分,并且应用(b),则得

$$\cos' t+i\sin' t=ie^{it}=-\sin t+i\cos t$$

于是

(6) $$\cos'=-\sin,\quad \sin'=\cos$$

幂级数(1)给出表示式

$$\cos t=1-\frac{t^2}{2!}+\frac{t^4}{4!}-\frac{t^6}{6!}+\cdots \tag{7}$$

取$t=2$, 则级数(7)的各项按绝对值减少(除首项外), 而且它们的符号是交错的, 因此 cos2 小于级数(7)的前三项之和; 于是$\cos 2<-\frac{1}{3}$. 由于$\cos 0=1$且 cos 是实轴上的实连续函数, 故可断定存在一个最小的正数t_0使得

$\cos t_0=0$. 我们定义

$$\pi=2t_0 \tag{8}$$

从(3)及(5)得到 $\sin t_0=\pm 1$. 由于在开区间$(0, t_0)$上

$$\sin' t=\cos t>0$$

和 $\sin 0=0$, 故有 $\sin t_0>0$, 因此 $\sin t_0=1$, 而且

$$e^{\pi i/2}=i \tag{9}$$

由此可见, $e^{\pi i}=i^2=-1$, $e^{2\pi i}=(-1)^2=1$, 且对每个正整数n, $e^{2\pi in}=1$. 同样立即得到(e):

$$e^{z+2\pi i}=e^z\cdot e^{2\pi i}=e^z \tag{10}$$

若$z=x+iy$, x和y为实数, 则$e^z=e^x e^{iy}$; 因此$|e^z|=e^x$. 若$e^z=1$, 则必须有$e^x=1$, 从而$x=0$; 根据(10), 为了证明$y/2\pi$一定是整数, 只要证明当$0<y<2\pi$时, $e^{iy}\neq 1$就足够了.

设$0<y<2\pi$, 且

$$e^{iy/4}=u+iv(u\text{ 和 }v\text{ 是实数}) \tag{11}$$

由于$0<y/4<\pi/2$, 故有$u>0, v>0$. 同样

$$e^{iy}=(u+iv)^4=u^4-6u^2v^2+v^4+4iuv(u^2-v^2) \tag{12}$$

仅当$u^2=v^2$时, (12)的右边才是实数; 由于$u^2+v^2=1$仅当$u^2=v^2=\frac{1}{2}$时才成立, 因此(12)表示出

$$e^{iy}=-1\neq 1$$

这就证明了(d).

我们已经知道，$t\to e^{it}$ 将实轴映入单位圆周内. 为了证明(f)，现固定 w 使得 $|w|=1$；我们将要证明，对于某些实数 t 有 $w=e^{it}$. 记 $w=u+iv$，u 和 v 为实数，而且首先假定 $u\geqslant 0$ 及 $v\geqslant 0$. 由于 $u\leqslant 1$，则 π 的定义表明存在一个 t，$0\leqslant t\leqslant \pi/2$，使得 $\cos t=u$；因而 $\sin^2 t=1-u^2=v^2$，又由于当 $0\leqslant t\leqslant \pi/2$ 时有 $\sin t\geqslant 0$，故 $\sin t=v$. 因此 $w=e^{it}$.

若 $u<0$ 及 $v\geqslant 0$，则 $-iw$ 满足上述的条件. 因此，对于某些实数 t 有 $-iw=e^{it}$，而且 $w=e^{i(t+\pi/2)}$. 最后，若 $v<0$，则上述两种情况证明了，对于某些实数 t 有 $-w=e^{it}$，因此 $w=e^{i(t+\pi)}$. 这就证明了(f).

若 $w\neq 0$，令 $\alpha=w/|w|$，因而 $w=|w|\alpha$. 根据(c)，有一个实数 x 使得 $|w|=e^x$. 由于 $|\alpha|=1$，则(f)证明了，对于某些实数 y 有 $\alpha=e^{iy}$. 因此 $w=e^{x+iy}$. 这就证明了(g)，且完成了本定理的证明.

我们将遇到 $(1+x^2)^{-1}$ 在实直线上的积分. 为了求它的值，在 $(-\pi/2,\pi/2)$ 内，令 $\varphi(t)=\sin t/\cos t$. 根据(6)有 $\varphi'=1+\varphi^2$. 因此，φ 是一个 $(-\pi/2,\ \pi/2)$ 到 $(-\infty,\ \infty)$ 上的单调增加的映射，而且得到

$$\int_{-\infty}^{\infty}\frac{dx}{1+x^2}=\int_{-\pi/2}^{\pi/2}\frac{\varphi'(t)dt}{1+\varphi^2(t)}=\int_{-\pi/2}^{\pi/2}dt=\pi$$

第一章 抽象积分

大约十九世纪末，许多数学家才搞清楚，Riemann 积分(微积分学教程中学习的内容之一)应当由另外一种类型更广泛、更灵活、更适合于处理极限过程的积分来代替. 在这方面所作努力之中，最著名的有 Jordan, Borel, W. H. Young 及 Lebesgue, 而Lebesgue 的结构终于是最成功的.

简要说来，其主要的思想是：函数 f 在闭区间 $[a,b]$ 上的Riemann 积分能够用和式

$$\sum_{i=1}^{n} f(t_i)m(E_i)$$

逼近，其中 $E_1,\cdots,E_n$ 为不相交的闭区间，它们的并为 $[a,b]$，$m(E_i)$ 表示 E_i 的长度，而 $t_i\in E_i(i=1,2,\cdots,n)$. Lebesgue 发现，当上面和式中的集 E_i 属于直线上较大的一类子集，即所谓"可测集"，而且把所考虑的函数类扩大到所谓"可测函数"时，可以得出一种完全令人满意的积分理论. 涉及到决定性的集论的性质如下：任何可数个可测集族的并和交是可测的；每一个可测集的余集也是可测的；而且，最重要的是"长度"的概念(现称为"测度")可按这样的方法推广：对于两两不相交的可测集的每一个可数集族 $\{E_i\}$，都有

$$m(E_1\cup E_2\cup E_3\cup\cdots)=m(E_1)+m(E_2)+m(E_3)+\cdots$$

m 的这个性质称为可数可加性.

从 Riemann 积分理论过渡到 Lebesgue积分理论是一个完备化的过程(其意义以后将更加精确地叙述). 它在分析上如同从有

理数系构造实数系那样有同样根本的重要性.

上面提到的测度m当然与实直线的几何性质是密切相关的. 在本章中,将介绍关于在任意集上任何可数可加测度的 Lebesgue 积分的一种抽象的(公理化的)叙述(精确定义见后). 这个抽象理论无论从那方面看都不比实直线上的特殊情形更困难;它说明,积分理论中的大部分与基础空间的任何几何(或拓扑)是无关的;而且,它当然也向我们提供了一个广泛有用的工具. 我们将在第二章中证实包括 Lebesgue 测度在内的一大类测度的存在.

集论的记号和术语

1.1 某些集能用列出它们的元素来描述,例如$\{x_1, \cdots, x_n\}$是元素为$x_1, \cdots, x_n$的集;而$\{x\}$是仅有一个元素x的集. 更为常见的是用性质来描述集合. 对于具有性质P的所有元素x的集记为

$$\{x:P\}$$

符号$\varnothing$表示空集. 族及类这些词在使用时与集是同义的.

若x为集A的元素,记$x\in A$;否则$x\notin A$. 若B是A的子集,即若$x\in B$蕴涵着$x\in A$,就记$B\subset A$. 若$B\subset A$且$A\subset B$,则$A=B$. 若$B\subset A$且$A\neq B$,则B是A的真子集. 注意,对于每一个集A有$\varnothing\subset A$.

$A\cup B$及$A\cap B$分别为A与B的并及交. 若$\{A_\alpha\}$是集族,其中α遍取某个指标集I,则$\{A_\alpha\}$的并及交记作

$$\bigcup_{\alpha\in I} A_\alpha \quad 及 \quad \bigcap_{\alpha\in I} A_\alpha$$

即:

$$\bigcup_{\alpha\in I} A_\alpha = \{x: x\in A_\alpha \text{ 对至少一个}\alpha\in I\}$$

$$\bigcap_{\alpha\in I} A_\alpha = \{x: x\in A_\alpha \text{ 对每一个 } \alpha\in I\}$$

若 I 是所有正整数的集，习惯的记号是

$$\bigcup_{n=1}^{\infty} A_n \text{ 及 } \bigcap_{n=1}^{\infty} A_n$$

若$\{A_\alpha\}$的任何两个集没有公共元素，则$\{A_\alpha\}$是一个不相交的集族.

记 $A-B=\{x: x\in A, x\notin B\}$. 当从上下文能清楚地知道是对哪一个比较大的集取余集时，A 的余集就用A^c来表示.

集 $A_1, \cdots, A_n$ 的笛卡儿乘积 $A_1\times A_2\times\cdots\times A_n$是全部 n 元序组 $(a_1, \cdots, a_n)$的集，其中 $a_i\in A_i$, $i=1, \cdots, n$.

实直线(或实数系)记作 R^1，且

$$R^k=R^1\times\cdots\times R^1 \quad (k\text{个因子})$$

广义实数系是R^1加上两个符号∞及$-\infty$，并且具有明显的顺序. 若$-\infty\leqslant a\leqslant b\leqslant\infty$，则闭区间$[a, b]$及开区间$(a, b)$定义为

$$[a, b]=\{x: a\leqslant x\leqslant b\},\ (a, b)=\{x: a<x<b\}$$

同样，记

$$[a, b)=\{x: a\leqslant x<b\},\ (a, b]=\{x: a<x\leqslant b\}$$

若$E\subset[-\infty, \infty]$且 $E\neq\varnothing$，则在$[-\infty, \infty]$内，E 的最小上界(上确界)及最大下界(下确界)存在，并记为$\sup E$ 及 $\inf E$.

有时(仅当$\sup E\in E$ 时)把$\sup E$ 记作 $\max E$.

记号

$$f:\ X\to Y$$

意味着 f 是集X到集Y内的一个函数(或映射，或变换)；即对每个 $x\in X$，f 确定一个元素$f(x)\in Y$. 若$A\subset X$及 $B\subset Y$，则A的象及B的逆象(或原象)是

$$f(A)=\{y: y=f(x)\text{对某个}x\in A\}$$

$$f^{-1}(B)=\{x: f(x)\in B\}$$

要注意，虽然 $B\neq\varnothing$，但 $f^{-1}(B)$可能是空的.

f 的定义域是 X，f 的值域是 $f(X)$.

若 $f(X)=Y$，称 f 把 X 映射到 Y 上.

对每一个 $y\in Y$，用 $f^{-1}(y)$ 代替 $f^{-1}(\{y\})$. 对每个 $y\in Y$，若 $f^{-1}(y)$ 至多由一点组成，则 f 称为一、一的. 若 f 是一、一的，则 f^{-1} 是一个函数，其定义域是 $f(X)$，而值域是 X.

若 $f:X\to[-\infty,\infty]$ 且 $E\subset X$，则习惯上记作 $\sup_{x\in E}f(x)$ 而不是 $\sup f(E)$.

若 $f:X\to Y$ 及 $g:Y\to Z$，则复合函数 $g\circ f:X\to Z$ 用公式

$$(g\circ f)(x)=g(f(x))\quad(x\in X)$$

定义.

可测性概念

可测函数类在积分理论中起着基础的作用. 它与另一最重要的函数类，即连续函数类有些共同的基本性质. 记住这些相似的地方是有益的. 因此，我们的表述是按这样一种方式组织的，即特别强调拓扑空间、开集和连续函数这些概念与可测空间、可测集和可测函数等另一些概念之间的类似性. 看来，当这种处理方式十分抽象时，这些概念之间的关系就显得最清楚不过了. 正是这一点(而不只是为了追求一般性)，促使我们来探讨这个课题.

1.2 定义

(a) 集 X 的子集族 τ 称为 X 上的一个拓扑，若 τ 具有如下三个性质:

(i) $\varnothing\in\tau$ 及 $X\in\tau$.

(ii) 若 $V_i\in\tau$，$i=1,\cdots,n$，则 $V_1\cap V_2\cap\cdots\cap V_n\in\tau$.

(iii) 若 $\{V_\alpha\}$ 是 τ 的元素的任何集族(有限，可数或不可数)，则 $\bigcup_\alpha V_\alpha\in\tau$.

(*b*) 若 τ 是 X 上的拓扑，则 X 称为拓扑空间，且 τ 的元素称为 X 的开集.

(*c*) 若 X 和 Y 为拓扑空间，且 f 是 X 到 Y 内的映射，若对 Y 的每一个开集 V，$f^{-1}(V)$ 是 X 的开集，则称 f 为连续的.

1.3 定义

(*a*) 集 X 的子集族 $\mathfrak{M}$ 称为 X 的一个 σ-代数，若 $\mathfrak{M}$ 具有如下性质:

(i) $X\in\mathfrak{M}$.

(ii) 若 $A\in\mathfrak{M}$，则 $A^c\in\mathfrak{M}$，其中 A^c 是 A 关于 X 的余集.

(iii) 若 $A=\bigcup_{n=1}^{\infty}A_n$ 且 $A_n\in\mathfrak{M}$，$n=1,2,3,\cdots$，则 $A\in\mathfrak{M}$.

(*b*) 若 $\mathfrak{M}$ 是 X 的 σ-代数，则 X 称为可测空间，且 $\mathfrak{M}$ 的元素称为 X 的可测集.

(*c*) 若 X 是可测空间，Y 是拓扑空间，而 f 是 X 到 Y 内的映射，若对 Y 的每一个开集 V，$f^{-1}(V)$ 是 X 的可测集，则 f 称为可测的.

把术语"可测空间"应用到序对 $(X,\ \mathfrak{M})$ 而不是 X 上也许更合适些. 毕竟 X 是一个集，并且 X 也不会由于我们心目中同时有一个由其子集构成的 σ-代数而有任何形式的改变. 类似地，一个拓扑空间是一个序对 $(X,\ \tau)$. 但如果这类事情在全部数学内有系统地去做的话，则术语就会变得非常不方便. 我们将于 1.21 节中在一些较大的范围内再讨论这个问题.

1.4 关于定义1.2的注释 最熟悉的拓扑空间是度量空间. 我们假定对度量空间已有所了解，但为了完整起见，还是给出基本的定义.

度量空间是一个集 X，在此集中定义了一个具有如下性质的距离函数(或度量) ρ:

(*a*) 对所有 $x,y\in X$，$0\leqslant\rho(x,y)<\infty$.

(*b*) 当且仅当 $x=y$ 时，$\rho(x,y)=0$.

(*c*) 对所有 $x,y\in X$，$\rho(x,y)=\rho(y,x)$.

(*d*) 对所有 $x,y,z\in X$，$\rho(x,y)\leqslant\rho(x,z)+\rho(z,y)$.

性质(*d*)称为三角不等式.

若 $x\in X$ 且 $r\geqslant 0$，则以 x 为中心，r 为半径的开球是指集 $\{y\in X:\rho(x,y)<r\}$.

若 X 是度量空间，且 τ 是所有集 $E\subset X$ 的集族，E 是开球的任意并，则 τ 是 X 内的拓扑. 不难证明，交的性质依赖于这样的事实：若 $x\in B_1\cap B_2$ 其中 B_1 及 B_2 为开球，则 x 是一个开球 $B\subset B_1\cap B_2$ 的中心. 我们把它留作练习.

例如，在实直线 R^1 上，当且仅当一个集是一些开区间 (a,b) 的并时，该集是开集. 在平面 R^2 上，开集是开圆盘的并.

我们将常常碰到的另一拓扑空间是扩充了的实直线 $[-\infty,\infty]$；它的拓扑是通过规定如下的集为开集来定义的，即规定 (a,b)，$[-\infty,a)$，$(a,\infty]$ 以及这种类型开区间的任何并为开集.

在 1.2 节定义(*c*)中给出的连续性定义是对整体而言的，通常需要定义局部的连续性，即若对 $f(x_0)$ 的每一个邻域 V，对应有 x_0 的一个邻域 W，使得 $f(W)\subset V$，则称 X 到 Y 内的映射 f 在点 $x_0\in X$ 连续.

（根据定义，点 x 的邻域是一个包含点 x 的开集.）

对于度量空间，此局部性定义当然与习惯上的 ε-δ 定义是相同的.

下面的简易命题将这两个连续性定义按预期的方式联系起来了.

1.5 命题 设 X 和 Y 是拓扑空间，f 是 X 到 Y 内的映射. 当且仅当 f 在 X 的每一点连续时，映射 f 是连续的.

证明 若 f 连续且 $x_0\in X$，则对 $f(x_0)$ 的每一个邻域 V，$f^{-1}(V)$

是 x_0 的一个邻域. 因为 $f(f^{-1}(V))\subset V$, 故 f 在 x_0 连续.

若 f 在 X 的每一点连续, 且 V 是 Y 的开集, 则每一点 $x\in f^{-1}(V)$ 有一个邻域 W_x, 使得 $f(W_x)\subset V$. 因此 $W_x\subset f^{-1}(V)$. 由此可见 $f^{-1}(V)$ 是开集 W_x 的并, 所以 $f^{-1}(V)$ 本身是开集, 因而 f 是连续的.

1.6 关于定义 1.3 的注释 设 $\mathfrak{M}$ 是集 X 内的 σ-代数, 由定义 1.3 的性质(i)至(iii)立即得出如下事实.

(*a*) 由于 $\varnothing=X^c$, 故(i)和(ii)蕴涵着 $\varnothing\in\mathfrak{M}$.

(*b*) 在(iii)中取 $A_{n+1}=A_{n+2}=\cdots=\varnothing$, 我们看出, 若 $A_i\in\mathfrak{M}$, $i=1,\cdots,n$, 则 $A_1\cup A_2\cup\cdots\cup A_n\in\mathfrak{M}$.

(*c*) 由于

$$\bigcap_{n=1}^{\infty}A_n=\left(\bigcup_{n=1}^{\infty}A_n^c\right)^c$$

故 $\mathfrak{M}$ 对可数交(同时对有限交)是封闭的.

(*d*) 由于 $A-B=B^c\cap A$, 若 $A\in\mathfrak{M}$ 及 $B\in\mathfrak{M}$, 就有 $A-B\in\mathfrak{M}$.

前缀 σ 是考虑到这样的事实: (iii)要求对于 $\mathfrak{M}$ 的元的所有可数并成立. 若(iii)只要求对有限并成立, 则 $\mathfrak{M}$ 称为集的代数.

1.7 定理 设 Y 和 Z 为拓扑空间, 且 $g:Y\to Z$ 是连续的.

(*a*) 若 X 是拓扑空间, $f:X\to Y$ 是连续的, 且 $h=g\circ f$, 则 $h:X\to Z$ 是连续的.

(*b*) 若 X 是可测空间, $f:X\to Y$ 是可测的, 且 $h=g\circ f$, 则 $h:X\to Z$ 是可测的.

简言之, 连续函数的连续函数是连续的; 可测函数的连续函数是可测的.

证明 若 V 是 Z 内的开集, 则 $g^{-1}(V)$ 是 Y 内的开集, 且

$$h^{-1}(V)=f^{-1}(g^{-1}(V))$$

若 f 是连续的, 则 $h^{-1}(V)$ 是开集, 因而证明了(*a*).

若 f 是可测的，则 $h^{-1}(V)$ 是可测的，因而证明了 (b).

1.8 定理 设 u 和 v 是可测空间 X 上的实可测函数，设 Φ 是平面到拓扑空间 Y 内的连续映射，且对 $x\in X$ 定义

$$h(x)=\Phi(u(x),v(x))$$

则 $h: X\to Y$ 是可测的.

证明 令 $f(x)=(u(x),v(x))$，则 f 将 X 映射到平面内. 由于 $h=\Phi\cdot f$，定理 1.7 指出，只要证明 f 的可测性就足够了.

若 R 是平面上任一个其边平行于坐标轴的开长方形，则 R 是两个开区间 I_1 及 I_2 的笛卡儿乘积，并根据关于 u 和 v 的假设，

$$f^{-1}(R)=u^{-1}(I_1)\cap v^{-1}(I_2)$$

是可测的. 在平面上的每一个开集 V 是这样的长方形 R_i 的可数并，且由于

$$f^{-1}(V)=f^{-1}\left(\bigcup_{i=1}^{\infty}R_i\right)=\bigcup_{i=1}^{\infty}f^{-1}(R_i)$$

因此，$f^{-1}(V)$ 是可测的.

1.9 设 X 是可测空间. 下面的命题是定理 1.7 和 1.8 的推论：

(a) 若 u 和 v 是 X 上的实可测函数，$f=u+iv$，则 f 为 X 上的复可测函数.

这个结论由定理 1.8，令 $\Phi(z)=z$ 得出.

(b) 若 $f=u+iv$ 是 X 上的复可测函数，则 u, v，及 $|f|$ 都是 X 上的实可测函数.

这个结论由定理 1.7，令 $g(z)=\mathrm{Re}(z)$，$\mathrm{In}(z)$，及 $|z|$ 得出.

(c) 若 f 及 g 是 X 上的复可测函数，则 $f+g$，及 fg 亦然.

对实的 f, g，由定理 1.8，令

$$\Phi(s,t)=s+t$$

和 $\Phi(s,t)=st$ 得出. 对复的情形即由 (a) 及 (b) 得出.

(d) 若 E 是 X 内的可测集，且

$$\chi_E(x)=\begin{cases}1 & \text{当 } x\in E\\ 0 & \text{当 } x\notin E\end{cases}$$

则 χ_E 是可测函数.

这是明显的. 我们称 χ_E 为集 E 的特征函数. 整个书中，字母 χ 将专用于表示特征函数.

(e) 若 f 为 X 上的复可测函数，则存在 X 上的复可测函数 α，使得 $|\alpha|=1$，且 $f=\alpha|f|$.

证明 令 $E=\{x:f(x)=0\}$，Y 为除去原点的复平面. 对 $z\in Y$，定义 $\varphi(z)=z/|z|$，且令

$$\alpha(x)=\varphi(f(x)+\chi_E(x)) \qquad (x\in X)$$

若 $x\in E$，$\alpha(x)=1$；若 $x\notin E$，$\alpha(x)=f(x)/|f(x)|$. 由 φ 在 Y 上连续且 E 可测(为什么?)，α 的可测性由(c)，(d)及定理1.7得出.

现在指出存在大量的 σ-代数.

1.10 定理 若 $\mathscr{F}$ 为 X 的任意子集族，则在 X 内存在一最小的 σ-代数 $\mathfrak{M}^*$，使得 $\mathscr{F}\subset\mathfrak{M}^*$.

$\mathfrak{M}^*$ 有时称为由 $\mathscr{F}$ 生成的 σ-代数.

证明 令 Ω 为 X 内所有包含 $\mathscr{F}$ 的 σ-代数 $\mathfrak{M}$ 的族. 因为 X 的全体子集的集族是具有这种性质的 σ-代数，故 Ω 非空. 设 $\mathfrak{M}^*$ 为所有 $\mathfrak{M}\in\Omega$ 的交. 显然，$\mathscr{F}\subset\mathfrak{M}^*$，且 $\mathfrak{M}^*$ 含于每一个包含 $\mathscr{F}$ 的 σ-代数中. 为完成此证明，必须证明 $\mathfrak{M}^*$ 本身是 σ-代数.

如果对 $n=1,2,3,\cdots$，$A_n\in\mathfrak{M}^*$，且如果 $\mathfrak{M}\in\Omega$，则 $A_n\in\mathfrak{M}$，因为 $\mathfrak{M}$ 是 σ-代数，所以 $\bigcup A_n\in\mathfrak{M}$. 因为对每一个 $\mathfrak{M}\in\Omega$，有 $\bigcup A_n\in\mathfrak{M}$，故得出 $\bigcup A_n\in\mathfrak{M}^*$. σ-代数定义的其余两个性质可用同样方法验证.

1.11 Borel 集 设 X 为拓扑空间. 由定理1.10，在 X 内存在一个最小的 σ-代数 $\mathscr{B}$，使得 X 内每一个开集都属于 $\mathscr{B}$. 称 $\mathscr{B}$ 的元素为 X 的Borel 集.

特别，闭集是 Borel 集(由定义，它是开集的余集)，且闭集的一切可数并及开集的一切可数交都是 Borel 集. 后两者分别称为 F_σ 集及 G_δ 集，并且起着重要的作用. 这符号来源于Hausdorff. 字母 F 及 G 分别用于闭集及开集，而 σ 用于并(Summe). δ 用于交(Durchschnitt). 例如每个半开区间 $[a, b)$ 是 R^1 内的一个 G_δ 集和一个 F_σ 集.

因为 $\mathscr{B}$ 是 σ-代数，我们现在可以把 X 看作一个可测空间，而 Borel 集则起着可测集的作用. 更简洁地，考虑可测空间 $(X, \mathscr{B})$. 若 $f: X \to Y$ 是 X 的连续映射，其中 Y 为任意拓扑空间. 由定义，显然对 Y 内每个开集 V，$f^{-1}(V) \in \mathscr{B}$. 换句话说，$X$ 的每个连续映射是 Borel 可测的.

Borel 可测映射通常称为 Borel 映射，或 Borel 函数.

1.12 定理 假设 $\mathfrak{M}$ 是 X 内的 σ-代数，Y 为拓扑空间，f 映 X 到 Y 内：

(*a*) 如果 Ω 为所有集 $E \subset Y$ 使得 $f^{-1}(E) \in \mathfrak{M}$ 的集族，则 Ω 为 Y 内的 σ-代数.

(*b*) 如果 f 可测且 E 为 Y 的 Borel 集，则 $f^{-1}(E) \in \mathfrak{M}$.

(*c*) 如果 $Y = [-\infty, \infty]$，且对每一个实数 α，$f^{-1}((\alpha, \infty]) \in \mathfrak{M}$，则 f 可测.

(*d*) 如果 f 可测，Z 为拓扑空间，$g: Y \to Z$ 为 Borel 映射，且如果 $h = g \circ f$，则 $h: X \to Z$ 可测.

(*e*) 常用来作为实值函数可测性的判别准则(见习题 3). 注意，(*d*) 推广了定理 1.7(*b*).

证明 从关系式

$$f^{-1}(Y) = X$$

$$f^{-1}(Y - A) = X - f^{-1}(A)$$

和

$$f^{-1}(A_1 \cup A_2 \cup \cdots) = f^{-1}(A_1) \cup f^{-1}(A_2) \cup \cdots$$

可得出(a).

现在证明(b)，设Ω如(a)所规定；f的可测性蕴涵着Ω包含Y内所有开集，且因Ω为σ-代数，Ω包含Y内所有Borel集.

现在证明(c)，令Ω为所有$E\subset[-\infty,+\infty]$，使得$f^{-1}(E)\in\mathfrak{M}$的集族. 选取一实数$\alpha$，并选取$\alpha_n<\alpha$使得当$n\to\infty$时，$\alpha_n\to\alpha$. 因为对于每个$n$，$(\alpha_n,\infty]\in\Omega$，由于

$$[-\infty,\alpha)=\bigcup_{n=1}^{\infty}[-\infty,\ \alpha_n]=\bigcup_{n=1}^{\infty}(\alpha_n,\infty]^c$$

而(a)已证明了Ω为σ-代数，看出$[-\infty,\alpha)\in\Omega$. 而

$$(\alpha,\beta)=[-\infty,\beta)\cap(\alpha,\infty]$$

也有相同的结论. 因为$[-\infty,\ \infty]$内的每个开集都是可数个上述类型开区间的并；故Ω包含每一个开集，于是f可测.

现在证明(d)，令$V\subset Z$为开集. 则$g^{-1}(V)$为Y的Borel集，且因

$$h^{-1}(V)=f^{-1}(g^{-1}(V))$$

(b)指出$h^{-1}(V)\in\mathfrak{M}$.

1.13 定义 设$\{a_n\}$为$[-\infty,\infty]$内的一个序列，令

(1) $$b_k=\sup\{a_k,a_{k+1},a_{k+2},\cdots\}$$
$$(k=1,2,3,\cdots)$$

且

(2) $$\beta=\inf\{b_1,b_2,b_3,\cdots\}$$

我们称β为$\{a_n\}$的上极限，记作

(3) $$\beta=\limsup_{n\to\infty}a_n$$

容易验证下列性质：首先，$b_1\geqslant b_2\geqslant b_3\geqslant\cdots$，所以当$k\to\infty$时，$b_k\to\beta$；其次，存在$\{a_n\}$的子序列$\{a_{n_i}\}$使得当$i\to\infty$时，$a_{n_i}\to\beta$，且$\beta$为具有此性质的最大数.

类似地定义下极限：简单交换(1)及(2)中的sup及inf. 注意

(4) $\liminf_{n\to\infty} a_n = -\limsup_{n\to\infty}(-a_n)$

如果$\{a_n\}$收敛，则显然有

(5) $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = \lim_{n\to\infty} a_n$

设$\{f_n\}$为集X上广义实数的函数的序列，则用

(6) $(\sup_n f_n)(x) = \sup_n (f_n(x))$

(7) $(\limsup_{n\to\infty} f_n)(x) = \limsup_{n\to\infty}(f_n(x))$

来定义在X上的函数$\sup_n f_n$及$\limsup_{n\to\infty} f_n$.

如果

(8) $$f(x) = \lim_{n\to\infty} f_n(x)$$

在每一点$x\in X$此极限都假设存在，则称f为叙列$\{f_n\}$的点态极限.

1.14 定理 如果对$n=1,2,3,\cdots$，$f_n: X\to[-\infty,\infty]$是可测的，且

$$g=\sup_{n\geqslant 1} f_n,\quad h=\limsup_{n\to\infty} f_n$$

则g及h是可测的.

证明 $g^{-1}((\alpha,\infty]) = \bigcup_{n=1}^{\infty} f_n^{-1}((\alpha,\infty])$. 因此由定理1.12(c)推出$g$可测. 当然，以 inf 代替 sup，同样的结论仍然成立.

因为

$$h=\inf_{k\geqslant 1}\{\sup_{i\geqslant k} f_i\}$$

由此得出h可测.

系

(a) 每个点态收敛的复可测函数序列的极限是可测的.

(b) 如果f及g可测(值域在$[-\infty,+\infty]$内)，则$\max\{f,g\}$

及$\min\{f,g\}$也可测. 特别,函数

$f^+=\max\{f,0\}$及 $f^-=-\min\{f,0\}$是可测的.

1.15 上述函数f^+及f^-称为f的正部和负部. 我们有$|f|=f^++f^-$和$f=f^+-f^-$,而后者是一种将f表为具有某种极小性质的,两个非负函数之差的典型形式:

命题 若$f=g-h$, $g\geqslant 0$及$h\geqslant 0$,则$f^+\leqslant g$且$f^-\leqslant h$.

证明 由$f\leqslant g$及$0\leqslant g$,显然有$\max\{f,0\}\leqslant g$.

简单函数

1.16 定义 在可测空间X上,值域仅由$[0,\infty)$内有限个点组成的函数s称为简单函数.

(有时称具有有限值域的任意函数为简单函数是方便的,然而我们主要感兴趣的还是上述情况. 注意,我们从简单函数的值中明显地排除了∞.)

如果$\alpha_1,\cdots,\alpha_n$为简单函数s不同的值,且令 $A_i=\{x:s(x)=\alpha_i\}$,显然

$$s=\sum_{i=1}^{n}\alpha_i\chi_{A_i}$$

在这里χ_{A_i}按照1.9(*d*)所定义,是A_i的特征函数.

同样显然的是,当且仅当每个A_i可测时,s是可测的.

1.17 定理 设$f:X\to[0,\infty]$可测. 则存在X上的简单可测函数s_n使得

(*a*) $0\leqslant s_1\leqslant s_2\leqslant\cdots\leqslant f$.

(*b*) 对每个$x\in X$,当$n\to\infty$,$s_n(x)\to f(x)$.

证明 对$n=1,2,3,\cdots$,及$1\leqslant i\leqslant n2^n$,定义

(1) $$E_{n,i}=f^{-1}\left(\left[\frac{i-1}{2^n},\frac{i}{2^n}\right)\right)\text{及 }F_n=f^{-1}([n,\infty])$$

并且令

$$s_n=\sum_{i=1}^{n2^n}\frac{i-1}{2^n}\chi_{E_{n,i}}+n\chi_{F_n} \tag{2}$$

定理 1.12(*b*)指出 $E_{n,i}$ 及 F_n 均为可测集. 容易看出函数(2)满足(*a*). 若 x 使得 $f(x)<\infty$, 则当 n 足够大时, $s_n\geqslant f(x)-2^{-n}$; 若 $f(x)=\infty$, 则 $s_n(x)=n$; 这样就证明了(*b*).

应该看到, 若 f 有界, 上述构造将给出一个一致收敛的序列 $\{s_n\}$.

测度的初等性质

1.18 定义

(*a*) 正测度为一定义在σ-代数$\mathfrak{M}$上的函数μ,其值域在$[0,\infty]$内,并且是可数可加的. 即若$\{A_i\}$为 $\mathfrak{M}$ 中互不相交的可数集族, 则

$$\mu\left(\bigcup_{i=1}^{\infty}A_i\right)=\sum_{i=1}^{\infty}\mu(A_i) \tag{1}$$

为避免麻烦,我们假设至少对一个 $A\in\mathfrak{M}$, $\mu(A)<\infty$.

(*b*) 测度空间是一个可测空间具有定义在其可测集的σ-代数上的正测度.

(*c*) 复测度是定义在一个 σ-代数上的复值可数可加函数.

注: 我们称作正测度的正好是通常的测度, 为了强调起见我们才加上个"正"字. 若对每个 $E\in\mathfrak{M}$, $\mu(E)=0$, 根据我们的定义, μ 还是正测度. 对正测度取∞值是允许的, 但当我们讲到复测度 μ 时, 对每个$E\in\mathfrak{M}$, $\mu(E)$要理解为复数. 当然, 实测度构成了复测度的一个子类.

1.19 定理 设μ 为 σ-代数 $\mathfrak{M}$ 上的正测度. 则

(a) $\mu(\varnothing)=0$.

(b) 如果 $A_1,\cdots,A_n$ 均为 $\mathfrak{M}$ 的两两不相交的元素，则

$$\mu(A_1\cup\cdots\cup A_n)=\mu(A_1)+\cdots+\mu(A_n)$$

(c) 如果 $A\in\mathfrak{M}, B\in\mathfrak{M}$，则 $A\subset B$ 蕴涵着 $\mu(A)\leqslant\mu(B)$.

(d) 如果 $A=\bigcup_{n=1}^{\infty}A_n, A_n\in\mathfrak{M}$，且

$$A_1\subset A_2\subset A_3\subset\cdots$$

则当 $n\to\infty$ 时，$\mu(A_n)\to\mu(A)$.

(e) 如果 $A=\bigcap_{n=1}^{\infty}A_n, A_n\in\mathfrak{M}$，

$$A_1\supset A_2\supset A_3\supset\cdots$$

且 $\mu(A_1)$ 有限，则当 $n\to\infty$ 时，$\mu(A_n)\to\mu(A)$.

如证明将指出的，这些性质除(c)外，对复测度也成立；(b)称为有限可加性；(c)称为单调性.

证明

(a) 取 $A\in\mathfrak{M}$，使得 $\mu(A)<\infty$，在 1.18(1)中，取 $A_1=A$，及 $A_2=A_3=\cdots=\varnothing$.

(b) 在 1.18(1)中取 $A_{n+1}=A_{n+2}=\cdots=\varnothing$.

(c) 由于 $B=A\cup(B-A)$ 及 $A\cap(B-A)=\varnothing$，可看出(b)蕴涵着 $\mu(B)=\mu(A)+\mu(B-A)\geqslant\mu(A)$.

(d) 令 $B_1=A_1$，且对 $n=2,3,4,\cdots$，令 $B_n=A_n-A_{n-1}$. 则 $B_n\in\mathfrak{M}$，$i\neq j$ 时，$B_i\cap B_j=\varnothing$，$A_n=B_1\cup\cdots\cup B_n$，且 $A=\bigcup_{i=1}^{\infty}B_i$.

因此

$$\mu(A_n)=\sum_{i=1}^{n}\mu(B_i)\quad\text{而}\quad\mu(A)=\sum_{i=1}^{\infty}\mu(B_i)$$

根据无穷级数和的定义得出(*d*).

(*e*) 令 $C_n=A_1-A_n$. 则 $C_1\subset C_2\subset C_3\subset\cdots$,

$$\mu(C_n)=\mu(A_1)-\mu(A_n)$$

$A_1-A=\bigcup C_n$, 于是由(*d*)得出

$$\mu(A_1)-\mu(A)=\mu(A_1-A)=\lim_{n\to\infty}\mu(C_n)=\mu(A_1)-\lim_{n\to\infty}\mu(A_n)$$

这就推出(*e*).

1.20 例 如我们将看到的，有趣的测度空间的构造要求做些艰苦的工作. 然而，可以立即给出少数浅显的例子.

(*a*) X为任意集，对任意 $E\subset X$，若 E 为无穷集，定义 $\mu(E)=\infty$，当 E 为有限集时，令 $\mu(E)$ 为 E 的点数. 这个 μ 称为 X 上的计数测度.

(*b*) 固定 $x_0\in X$. 对任意 $E\subset X$，若 $x_0\in E$，定义 $\mu(E)=1$；若 $x_0\notin E$，定义 $\mu(E)=0$. 这个 μ 称为集中在 x_0 的单位质量.

(*c*) 令 μ 是集$\{1, 2, 3, \cdots\}$上的计数测度，令 $A_n=\{n, n+1, n+2, \cdots\}$. 则 $\bigcap A_n=\varnothing$，但对 $n=1, 2, 3, \cdots$，$\mu(A_n)=\infty$. 这表明定理1.19(*e*)中的假设

$$\mu(A_1)<\infty$$

并非多余的.

1.21 关于术语的注释 人们经常看到把测度空间看成是"三元序组"$(X, \mathfrak{M}, \mu)$，其中 X 为集，$\mathfrak{M}$ 为 X 内的 σ-代数，而 μ 为定义在 $\mathfrak{M}$ 上的测度. 类似地，可测空间是"序对"$(X, \mathfrak{M})$. 这虽然有点多余，但在逻辑上完全正确，通常也是便利的. 例如在 $(X, \mathfrak{M})$ 中，集 X 仅是 $\mathfrak{M}$ 的最大元素. 因此我们若知道 $\mathfrak{M}$，也就知道了 X. 类似地，根据定义，每个测度都有一个 σ-代数作为它的定义域，于是我们若知道测度 μ，也就知道作为 μ 的定义域的 σ-代数 $\mathfrak{M}$，那末作为 σ-代数 $\mathfrak{M}$ 的最大元素集 X 也就知道了.

因此，使用象"设 μ 为一测度"，或者当我们在问题中希望强调

σ-代数或这集的时候说"设 μ 为$\mathfrak{M}$上的测度"或"设 μ 为X上的测度"这类表达语，那是完全合理的.

虽在逻辑上不大有意义但在习惯上(我们通常都是顺从数学上的习惯而不是逻辑)总是说"设X为测度空间"；这里着重的并不在于集，而在于测度. 当然，在使用这种说法时，我们总是默认，有一个定义在X的某个 σ-代数上的测度，而它正好是我们实际上要讨论的.

类似地，一个拓扑空间乃是一个序对(X, τ)，此处 τ 为集X上的拓扑. 并且有意义的东西是在 τ 中而不是在X中，但述及时仍说成"拓扑空间X".

全部数学中均使用这种默契. 大多数的数学系统都是由一些集，连同某些特定的子集类或者某些二元运算或者某些关系(要求它们具有某些性质). 根据需要，人们可以列举出它们，然后把这个系统作为序对，三元序组等等来描述. 例如实直线可描述为四元序组$(R^1, +, \cdot, <)$，此处$+$，$\cdot$，及$<$满足完备阿基米德(Archimedes)有序域的公理. 然而，我敢打赌，只有极少数数学家才想把实数域看作四元序组.

$[0, \infty]$中的算术运算

1.22 在整个积分论中，不可避免地要遇到∞. 一个理由是希望在无限测度的集上能够积分；毕竟，实直线有无限长. 另一个理由是，即使最初只对实值函数有兴趣，但是一个正实函数序列的上极限或一个正实函数序列的和在某些点却很可能是∞. 而当这种情况发生时，人们不得不作出一些特殊规定，这将在很大程度上丧失如定理1.26及1.27那样的优美性.

我们定义，如果 $0 \leqslant a \leqslant \infty$，则 $a + \infty = \infty + a = \infty$，且

$$a\cdot\infty=\infty\cdot a=\begin{cases}\infty, & \text{当 } 0<a\leqslant\infty\\ 0, & \text{当 } a=0\end{cases}$$

实数的和及积当然用通常方法定义.

定义$0\cdot\infty=0$似乎很奇怪，然而，用这定义不难验证，在$[0,\infty]$内交换律，结合律及分配律都是无条件成立的.

对待消去律则要小心处置：仅当$a<\infty$时，$a+b=a+c$才蕴涵着$b=c$；仅当$0<a<\infty$时，$ab=ac$才蕴涵$b=c$.

注意，下列有用的命题成立：

如果$0\leqslant a_1\leqslant a_2\leqslant\cdots$，$0\leqslant b_1\leqslant b_2\leqslant\cdots$，$a_n\to a$且$b_n\to b$则$a_nb_n\to ab$.

把这个命题和定理1.17及1.14结合起来，可以看出到$[0,\infty]$内的可测函数的和及积是可测函数.

正函数的积分

本节中，$\mathfrak{M}$为集X的σ-代数，μ为$\mathfrak{M}$上的正测度.

1.23 定义 如果s为X上的可测简单函数，形如

$$s=\sum_{i=1}^{n}\alpha_i\chi_{A_i}\tag{1}$$

此处$\alpha_1,\cdots,\alpha_n$为s的不同的值(与定义1.16比较)，且如果$E\in\mathfrak{M}$，定义

$$\int_E s\,d\mu=\sum_{i=1}^{n}\alpha_i\mu(A_i\cap E)\tag{2}$$

此处用到约定$0\cdot\infty=0$；因可能对某个i，$\alpha_i=0$且$\mu(A_i\cap E)=\infty$.

如果$f:X\to[0,\infty]$为可测，且$E\in\mathfrak{M}$，定义：

$$\int_E f\,d\mu=\sup\int_E s\,d\mu\tag{3}$$

这上确界取遍所有使得$0\leqslant s\leqslant f$的简单可测函数s.

(3) 的左边称为 f 在 E 上关于测度 μ 的 Lebesgue积分. 它是$[0,\infty]$内的一个数.

注意，当 f 是简单函数的时候，表面上得到了 $\int_E f d\mu$ 的两个定义，就是(2)和(3). 然而，它们得到的积分值是相同的. 因为在这种情况下，出现在(3)右边的函数 s 中，f 是最大的.

1.24 下列命题是定义的直接推论. 其中出现的函数和集，都假定是可测的:

(a) 如果 $0\leqslant f\leqslant g$，则 $\int_E f d\mu\leqslant\int_E g d\mu$.

(b) 如果 $A\subset B$ 而 $f\geqslant 0$，则 $\int_A f d\mu\leqslant\int_B f d\mu$.

(c) 如果 $f\geqslant 0$ 而 c 为常数，$0\leqslant c\leqslant\infty$，则

$$\int_E cf d\mu=c\int_E f d\mu$$

(d) 如果对所有 $x\in E$，$f(x)=0$，即使 $\mu(E)=\infty$，也有 $\int_E f d\mu=0$.

(e) 如果 $\mu(E)=0$，即使对每个 $x\in E$，$f(x)=\infty$，也有 $\int_E f d\mu=0$.

(f) 如果 $f\geqslant 0$，则 $\int_E f d\mu=\int_X \chi_E f d\mu$.

最后的结果表明，我们能把我们的积分定义限制为在全部 X 上的积分而不失去一般性. 若要在子集上积分，我们可以用(f)作为定义. 提出那一方式作为定义，纯粹是一种爱好.

在此要注明，测度空间 X 的每个可测子集 E 可以用完全自然的方法重新构成一个测度空间: 新的可测集简单地说，就是含在 E 内的 X 的可测子集，除了它的定义域受限制之外，测度并不改变.

这再次表明，一旦我们在每个测度空间上定义积分之后，就自动地定义了可测空间的每个可测子集上的积分.

1.25 命题 设 s 和 t 为 X 上的可测简单函数. 对 $E\in\mathfrak{M}$，定义

$$\varphi(E)=\int_E s\,d\mu \tag{1}$$

则 φ 为 $\mathfrak{M}$ 上的测度. 同时

$$\int_X (s+t)\,d\mu=\int_X s\,d\mu+\int_X t\,d\mu \tag{2}$$

(这个命题包括定理 1.27 及 1.29 的特殊形式.)

证明 如果 s 象在定义 1.23 内一样，且 $E_1, E_2, \cdots$ 为 $\mathfrak{M}$ 的互不相交元素，其并为 E. μ 的可数可加性指出

$$\varphi(E)=\sum_{i=1}^{n}\alpha_i\mu(A_i\cap E)=\sum_{i=1}^{n}\alpha_i\sum_{r=1}^{\infty}\mu(A_i\cap E_r)$$

$$=\sum_{r=1}^{\infty}\sum_{i=1}^{n}\alpha_i\mu(A_i\cap E_r)=\sum_{r=1}^{\infty}\varphi(E_r)$$

同样，$\varphi(\varnothing)=0$，因此 φ 不恒等于 ∞.

其次设 s 同前面一样，$\beta_1, \cdots, \beta_m$，是 t 的不同的值，且令 $B_j=\{x: t(x)=\beta_j\}$. 若 $E_{ij}=A_i\cap B_j$，则

$$\int_{E_{ij}}(s+t)\,d\mu=(\alpha_i+\beta_j)\mu(E_{ij})$$

且

$$\int_{E_{ij}} s\,d\mu+\int_{E_{ij}} t\,d\mu=\alpha_i\mu(E_{ij})+\beta_j\mu(E_{ij})$$

于是用 E_{ij} 代替 X 时，(2)成立. 因为 X 是不相交的集 $E_{ij}(1\leqslant i\leqslant n, 1\leqslant j\leqslant m)$ 的并，我们命题的前半部蕴涵着(2)成立.

现在达到理论的有趣的部分，它的最显著的特点之一是利用它可以有把握处理极限运算.

1.26 Lebesgue单调收敛定理 设 $\{f_n\}$ 是一个 X 上的可测函数序列，且假定

(a) 对每一个 $x\in X$，$0\leqslant f_1(x)\leqslant f_2(x)\leqslant\cdots\leqslant\infty$.

(b) 对每一个 $x\in X$，当 $n\to\infty$ 时，$f_n(x)\to f(x)$.

则 f 是可测的，且当 $n\to\infty$ 时，

$$\int_X f_n d\mu\to\int_X f d\mu$$

证明 因为 $\int f_n\leqslant\int f_{n+1}$，存在一个 $\alpha\in[0,\infty)$ 使得当 $n\to\infty$ 时

$$\int_X f_n d\mu\to\alpha \tag{1}$$

根据定理 1.14，f 是可测的. 因为 $f_n\leqslant f$，故对每个 n，有 $\int f_n\leqslant\int f$，因此(1)蕴涵

$$\alpha\leqslant\int_X f d\mu \tag{2}$$

设 s 是使得 $0\leqslant s\leqslant f$ 的任一简单可测函数，设 c 是一个常数，$0<c<1$，且定义

$$E_n=\{x: f_n(x)\geqslant cs(x)\}\qquad(n=1,2,3,\cdots) \tag{3}$$

每一个 E_n 是可测的，$E_1\subset E_2\subset E_3\subset\cdots$，且 $X=\bigcup E_n$. 为了看出这个等式，考虑 $x\in X$；若 $f(x)=0$，则 $x\in E_1$；若 $f(x)>0$，因为 $c<1$，则 $cs(x)<f(x)$；因此对某个 n，$x\in E_n$. 同样有

$$\int_X f_n d\mu\geqslant\int_{E_n} f_n d\mu\geqslant c\int_{E_n} s d\mu\qquad(n=1,2,3,\cdots) \tag{4}$$

令 $n\to\infty$，应用命题 1.25 和定理 1.19(d)于(4)的最后一个积分，结果是

$$\alpha\geqslant c\int_X s d\mu \tag{5}$$

因为(5)对每一个 $c<1$ 成立，故

(6) $$\alpha \geqslant \int_X s d\mu$$

对满足 $0 \leqslant s \leqslant f$ 的每一个简单可测函数 s 成立. 因此,

(7) $$\alpha \geqslant \int_X f d\mu$$

此定理从(1),(2),(7)得出.

1.27 定理 如果对 $n=1,2,3,\cdots, f_n: X \to [0,\infty]$ 是可测的, 且

(1) $$f(x)=\sum_{n=1}^{\infty} f_n(x) \qquad (x \in X)$$

则

(2) $$\int_X f d\mu = \sum_{n=1}^{\infty} \int_X f_n d\mu$$

证明 首先, 象在定理 1.17 一样, 存在简单可测函数序列 $\{s_i'\}$, $\{s_i''\}$, 使得 $s_i' \to f_1$, $s_i'' \to f_2$. 若 $s_i = s_i' + s_i''$, 则 $s_i \to f_1 + f_2$, 单调收敛定理连同命题 1.25 一起指出

(3) $$\int_X (f_1+f_2) d\mu = \int_X f_1 d\mu + \int_X f_2 d\mu$$

其次, 令 $g_N = f_1 + \cdots + f_N$, 序列 $\{g_N\}$ 单调地收敛于 f, 若我们应用归纳法于(3), 看出

(4) $$\int_X g_N d\mu = \sum_{n=1}^{N} \int_X f_n d\mu$$

再一次应用单调收敛定理, 得到(2). 这就完成了证明.

若设 μ 是可数集上的计数测度, 定理 1.27 是一个关于非负实数的二重级数的结论(它当然能用初等方法证明):

系 如果 $a_{ij} \geqslant 0$ 对 i 和 $j=1,2,3,\cdots$ 成立, 则

$$\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}a_{ij}=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}$$

1.28 Fatou 引理 如果对每一个正整数 n, $f_n: X\to[0,\infty]$ 是可测的. 则

(1) $$\int_X(\liminf_{n\to\infty}f_n)d\mu\leqslant\liminf_{n\to\infty}\int_X f_n d\mu$$

(1)中严格不等式是能出现的，见习题 8.

证明 令

(2) $$g_k(x)=\inf_{i\geqslant k}f_i(x)\qquad(k=1,2,3,\cdots;x\in X)$$

则 $g_k\leqslant f_k$, 所以

(3) $$\int_X g_k d\mu\leqslant\int_X f_k d\mu\qquad(k=1,2,3,\cdots)$$

同时, $0\leqslant g_1\leqslant g_2\leqslant\cdots$, 根据定理 1.14, 每一个 g_k 是可测的，且由定义 1.13, 当 $k\to\infty$ 时, $g_k(x)\to\liminf f_n(x)$. 所以单调收敛定理指出，当 $k\to\infty$ 时，(3)的左边趋于(1)的左边. 因此从(3)得到(1).

1.29 定理 假定 $f: X\to[0,\infty]$ 是可测的，且

(1) $$\varphi(E)=\int_E f d\mu\qquad(E\in\mathfrak{M})$$

则 φ 是 $\mathfrak{M}$ 上的一个测度. 且

(2) $$\int_X g d\varphi=\int_X g f d\mu$$

对其值域在 $[0,\infty]$ 内的 X 上的每一个可测函数 g 成立.

证明 设 $E_1, E_2, E_3, \cdots$ 是 $\mathfrak{M}$ 的不相交的元，他们的并是 E. 考察

(3) $$\chi_E f=\sum_{j=1}^{\infty}\chi_{E_j}f$$

和

(4) $$\varphi(E)=\int_X \chi_E f d\mu;\ \varphi(E_j)=\int_X \chi_{E_j} f d\mu$$

于是从定理 1.27 得到

(5) $$\varphi(E)=\sum_{j=1}^{\infty}\varphi(E_j)$$

因为 $\varphi(\varnothing)=0$,(5)证明了 φ 是一个测度.

其次,(1)指出对 $E\in\mathfrak{M}$, 当 $g=\chi_E$ 时(2)总是成立的. 因此(2)对每一个简单函数 g 成立, 而一般情况则从单调收敛定理可以得出.

评注 定理 1.29 的第二个断言有时写成

(6) $$d\varphi=f d\mu$$

的形式. 我们不给符号 $d\varphi$ 和 $d\mu$ 以独立意义, (6)仅仅意味着(2)对每一个可测的 $g\geqslant 0$ 成立.

定理 1.29 有一个非常重要的逆定理, 即 Radon-Nikodym 定理, 将在第六章内证明.

复函数的积分

象以前一样, 在这节里 μ 是在任意可测空间 X 上的正测度.

1.30 定义 我们定义 $L^1(\mu)$ 是所有使得

$$\int_X |f| d\mu<\infty$$

的、X 上的复可测函数 f 的集族.

注意, 象在命题 1.9(b)所见到的一样, f 的可测性蕴涵着 $|f|$ 的可测性. 因此上面的积分是确定了的.

$L^1(\mu)$ 的元素称为(关于 μ 的)Lebesgue 可积函数或称为可求和函数. 指数 1 的含义将在第三章阐明.

1.31 定义 若 $f=u+iv$, 这里 u 和 v 是 X 上实可测函数,

若 $f\in L^1(\mu)$，对每一个可测集 E 定义

$$\int_E f d\mu=\int_E u^+ d\mu-\int_E u^- d\mu+i\int_E v^+ d\mu-i\int_E v^- d\mu \tag{1}$$

象在 1.15 节内定义的一样，这里 u^+ 和 u^- 是 u 的正部和负部. v^+ 和 v^- 从 v 类似地得到. 这四个函数都是实的、可测的和非负的. 因此，根据定义 1.23，(1)右边的四个积分存在. 而且有 $u^+\leqslant|u|\leqslant|f|$ 等等，所以这四个积分的每一个都是有限的. 于是(1)定义了左边的积分为一个复数.

有时候，定义一个值域在$[-\infty,\infty]$中的可测函数 f 的积分为

$$\int_E f d\mu=\int_E f^+ d\mu-\int_E f^- d\mu \tag{2}$$

也是合情合理的. 如果(2)的右边的积分至少有一个是有限的，这时(2)的左边则是在$[-\infty,\infty]$内的一个数.

1.32 定理 假定 f 和 $g\in L^1(\mu)$，且 α 和 β 是复数，则 $\alpha f+\beta g\in L^1(\mu)$，且

$$\int_X(\alpha f+\beta g)d\mu=\alpha\int_X f d\mu+\beta\int_X g d\mu \tag{1}$$

证明 $\alpha f+\beta g$ 的可测性从命题 1.9(c)得出. 根据 1.24节和定理 1.27，有

$$\begin{aligned}\int_X|\alpha f+\beta g|d\mu&\leqslant\int_X(|\alpha||f|+|\beta||g|)d\mu\\&=|\alpha|\int_X|f|d\mu+|\beta|\int_X|g|d\mu<\infty\end{aligned}$$

于是 $\alpha f+\beta g\in L^1(\mu)$.

为证明(1)，显然只要证明

$$\int_X(f+g)d\mu=\int_X f d\mu+\int_X g d\mu \tag{2}$$

和

$$\int_X(\alpha f)d\mu=\alpha\int_X f d\mu \tag{3}$$

就够了. 并且若(2)对 $L^1(\mu)$内实的 f 和 g 成立，(2)的一般场合也即得出.

承认这一点之后，令 $h=f+g$，就有

$$h^+-h^-=f^+-f^-+g^+-g^-$$

或

$$h^++f^-+g^-=f^++g^++h^- \tag{4}$$

根据定理 1.27，有

$$\int h^++\int f^-+\int g^-=\int f^++\int g^++\int h^- \tag{5}$$

因为这些积分的每一个是有限的，通过移项就可以得到(2).

当 $\alpha\geqslant 0$ 时，从命题 1.24(c)可得到(3). 利用关系式 $(-u)^+=u^-$，容易验证当 $\alpha=-1$ 时，(3)也成立. 对 $\alpha=i$ 的情况也是容易证明的：若 $f=u+iv$，则

$$\int(if)=\int(iu-v)=\int(-v)+i\int u=-\int v+i\int u=i\left(\int u+i\int v\right)$$

$$=i\int f$$

这些情况同(2)合在一起，我们得到对任意复数 α，(3)成立.

1.33 定理 若 $f\in L^1(\mu)$，则

$$\left|\int_X f\,d\mu\right|\leqslant\int_X|f|\,d\mu$$

证明 令 $z=\int_X f\,d\mu$，因为 z 是一个复数，存在一个复数 α，$|\alpha|=1$，使得 $\alpha z=|z|$. 设 u 是 αf 的实部. 则 $u\leqslant|\alpha f|=|f|$. 因此

$$\left|\int_X f\,d\mu\right|=\alpha\int_X f\,d\mu=\int_X\alpha f\,d\mu=\int_X u\,d\mu\leqslant\int_X|f|\,d\mu$$

因为前面已指出 $\int\alpha f\,d\mu$ 是实数，故上面等式的第三个等号成立.

我们用另一个重要的收敛定理结束这一节.

1.34 Lebesgue控制收敛定理 假定$\{f_n\}$是一个X上的复可测函数序列,使得

(1) $$f(x)=\lim_{n\to\infty}f_n(x)$$

对每一个$x\in X$存在. 若存在一个函数$g\in L^1(\mu)$使得

(2) $$|f_n(x)|\leqslant g(x) \qquad (n=1,2,3,\cdots;x\in X)$$

则$f\in L^1(\mu)$,

(3) $$\lim_{n\to\infty}\int_X|f_n-f|d\mu=0$$

并且

(4) $$\lim_{n\to\infty}\int_X f_n d\mu=\int_X f d\mu$$

证明 因为$|f|\leqslant g$且f是可测的,故$f\in L^1(\mu)$. 因为$|f_n-f|\leqslant 2g$,把Fatou引理应用于函数$2g-|f_n-f|$得到

$$\begin{aligned}\int_X 2g d\mu &\leqslant \liminf_{n\to\infty}\int_X(2g-|f_n-f|)d\mu\\ &=\int_X 2g d\mu+\liminf_{n\to\infty}\left(-\int_X|f_n-f|d\mu\right)\\ &=\int_X 2g d\mu-\limsup_{n\to\infty}\int_X|f_n-f|d\mu\end{aligned}$$

因为$\int 2g d\mu$是有限的,从两边减去它,便得

(5) $$\limsup_{n\to\infty}\int_X|f_n-f|d\mu\leqslant 0$$

若一个非负的实数序列不收敛于0,则它的上极限是正的,于是(5)蕴涵着(3). 把定理1.33应用于f_n-f,(3)便蕴涵着(4).

零测集所起的作用

1.35 定义 设P是对于点x可以具有或者不具有的一种性

质. 譬如，若 f 是一个给定的函数，P 可以是性质“$f(x)>0$”，若 $\{f_n\}$ 是给定的函数序列，P 可以是性质“$\{f_n(x)\}$ 收敛”.

如果 μ 是一个 σ-代数 $\mathfrak{M}$ 上的测度，$E\in\mathfrak{M}$，“P 在 E 上几乎处处成立”(简记为“P. a. e. 于 E)这句话意味着：存在一个 $N\in\mathfrak{M}$，使得 $\mu(N)=0$，$N\subset E$，并且 P 在 $E-N$ 的每一点上成立. 当然“几乎处处”这个概念非常强烈地依赖于所给定的测度. 当明确要求指出测度的时候，我们将记作“a. e. $[\mu]$”.

例如，如果 f 和 g 是可测函数，并且当

$$\mu(\{x:f(x)\neq g(x)\})=0 \tag{1}$$

时，我们说 $f=g$ a. e. $[\mu]$ 于 X 上，并记为 $f\sim g$. 容易看出这是一个等价关系. 传递性($f\sim g$ 和 $g\sim h$ 蕴涵着 $f\sim h$)是“两个零测集的并是一个零测集”这一事实的推论.

注意，若 $f\sim g$，则对每一个 $E\in\mathfrak{M}$，

$$\int_E f\,d\mu=\int_E g\,d\mu \tag{2}$$

为了看出这点，设 N 是满足(1)式的集；则 E 是 $E-N$ 和 $E\cap N$ 这两个不相交的集的并，在 $E-N$ 上 $f=g$，而 $\mu(E\cap N)=0$.

于是，一般地说，零测集在积分中是可以忽略的. “可忽略集的每一个子集是可忽略的”这个结论应当是正确的. 但是可能遇到一些集 $N\in\mathfrak{M}$，$\mu(N)=0$，而它有子集 E 不是 $\mathfrak{M}$ 的元素. 当然在这种情况下，可以定义 $\mu(E)=0$，但是 μ 的这种扩充还能是一个测度吗？即，它还是定义在一个 σ-代数上吗？所幸的是，回答是肯定的.

1.36 定理 设 $(X,\mathfrak{M},\mu)$ 是一个测度空间，$\mathfrak{M}^*$ 是所有这样的 $E\subset X$ 的集族，对于 E 存在 A 和 $B\in\mathfrak{M}$，使得 $A\subset E\subset B$，且 $\mu(B-A)=0$，在这种情况下，定义 $\mu(E)=\mu(A)$，则 $\mathfrak{M}^*$ 是一个 σ-代数，且 μ 是 $\mathfrak{M}^*$ 上的一个测度.

因为零测集的所有子集现在都是可测的，这个扩张了的测度 μ 称为完备测度，σ-代数$\mathfrak{M}^*$ 称为$\mathfrak{M}$的 μ 完备化．这个定理说每一个测度都能完备化．所以为了方便起见，总可以假定任何给定的测度是完备的；这恰好给出更多的可测集，因此，给出更多的可测函数．通常讨论中所遇到的大多数测度都已经是完备化的．但也有些例外；其中之一将在第七章的 Fubini 定理的证明中遇到．

证明 验证σ-代数的三个规定性质．(i)$X\in\mathfrak{M}$，因此$X\in\mathfrak{M}^*$．(ii)若 $A\subset E\subset B$，则 $B^c\subset E^c\subset A^c$，且 $A^c-B^c=B-A$．(iii)若 $A_i\subset E_i\subset B_i$，$A=\bigcup A_i$，$E=\bigcup E_i$，和 $B=\bigcup B_i$，则 $A\subset E\subset B$．并且

$$B-A\subset\bigcup_1^\infty (B_i-A_i)$$

故当 $\mu(B_i-A_i)=0, i=1,2,3,\cdots$成立时，$\mu(B-A)=0$．

其次，我们来验证 μ 在$\mathfrak{M}^*$上是完全确定的．假定 $A\subset E\subset B$，$A_1\subset E\subset B_1$，且 $\mu(B-A)=\mu(B_1-A_1)=0$．则

$$A-A_1\subset B_1-A_1$$

所以 $\mu(A-A_1)=0$，类似地 $\mu(A_1-A)=0$，因此

$$\mu(A)=\mu(A_1\cap A)=\mu(A_1)$$

在$\mathfrak{M}^*$上 μ 的可数可加性是明显的．

I. 37 就积分来说，a. e. 相等的函数可以不加区别，这个事实启发我们，扩大可测函数的定义是有益的．如果 $\mu(E^c)=0$，并且对每一个开集 V，$f^{-1}(V)\cap E$ 是可测的，我们称定义在 $E\in\mathfrak{M}$上的函数 f 为在 X 上可测．若对 $x\in E^c$，定义 $f(x)=0$，就得到在旧的意义下的 X 的可测函数．假如测度是完备的，在 E^c 上依照完全任意的方式定义 f，仍可得到一个可测函数．在任何 $A\in\mathfrak{M}$ 上 f 的积分与 E^c 上 f 的定义无关，所以 f 在 E^c 上的定义甚至完全不需要指定．

有许多自然会遇到的情况，譬如，在实直线上的一个函数 f 可

以仅仅是几乎处处可微的(关于Lebesgue 测度),但是在一定的条件下,f 是它的导数的积分仍然正确,这将在第八章内讨论. 或者 X 上的可测函数序列$\{f_n\}$ 可以仅仅是几乎处处收敛; 用可测性的新定义,这个极限仍是 X 上的可测函数,我们没有必要非得把它缩减到实际收敛的点集上不可.

为了说明问题,我们叙述 Lebesgue 控制收敛定理的一个推论,其叙述方式容许一些测度为零的例外集.

1.38 定理 假定$\{f_n\}$是一个在 X 上几乎处处有定义的复可测函数序列,满足

$$\sum_{n=1}^{\infty}\int_X |f_n|d\mu<\infty \tag{1}$$

则级数

$$f(x)=\sum_{n=1}^{\infty}f_n(x) \tag{2}$$

对几乎所有的 x 收敛,$f\in L^1(\mu)$,并且

$$\int_X f d\mu=\sum_{n=1}^{\infty}\int_X f_n d\mu \tag{3}$$

证明 设 S_n 是 f_n 有定义的点集,因此 $\mu(S_n^c)=0$. 对$x\in S=\cap S_n$,令 $\varphi(x)=\Sigma|f_n(x)|$. 则 $\mu(S^c)=0$. 根据(1)和定理 1.27

$$\int_S \varphi d\mu<\infty \tag{4}$$

若 $E=\{x\in S:\varphi(x)<\infty\}$,从(4)得出 $\mu(E^c)=0$. 对于每一个 $x\in E$,级数(2)绝对收敛. 如果对 $x\in E$,$f(x)$用(2)定义,则在 E 上 $|f(x)|\leqslant\varphi(x)$. 所以根据(4),在 E 上 $f\in L^1(\mu)$. 如果 $g_n=f_1+\cdots+f_n$,则$|g_n|\leqslant\varphi$,对所有的 $x\in E$,$g_n(x)\to f(x)$,并且定理 1.34 给出了用 E 代替 X 时的(3) 式. 因为 $\mu(E^c)=0$,此结果等价于

(3).

注意，即使当 f_n 在 X 的每一点均有定义时，由(1)也只能得出(2)几乎处处收敛. 下面是另外一些情况，在这些情况中，我们只能引出几乎处处成立的结论.

1.39 定理

(*a*) 假设 $f: X\to[0,\infty]$ 是可测的，$E\in\mathfrak{M}$ 并且 $\int_E f\,d\mu=0$，则 $f=0$ a. e. 于 E 上.

(*b*) 假设 $f\in L^1(\mu)$ 并且对每一个 $E\in\mathfrak{M}$，有 $\int_E f\,d\mu=0$，则 $f=0$ a. e. 于 E 上.

(*c*) 假设 $f\in L^1(\mu)$ 并且

$$\left|\int_X f\,d\mu\right|=\int_X|f|\,d\mu$$

则存在一个常数 α，使得 $\alpha f=|f|$ a. e. 于 X 上.

注意，(*c*)叙述了这样的条件，在这个条件下，定理 1.33 内的等号成立.

证明

(*a*) 如果 $A_n=\left\{x\in E:\ f(x)>\frac{1}{n}\right\}$ $n=1,2,3,\cdots$，则

$$\frac{1}{n}\mu(A_n)\leqslant\int_{A_n} f\,d\mu\leqslant\int_E f\,d\mu=0$$

所以 $\mu(A_n)=0$，因为 $\{x\in E;f(x)>0\}=\bigcup A_n$，便得出(*a*).

(*b*) 令 $f=u+iv$，设 $E=\{x:u(x)\geqslant 0\}$，则 $\int_E f\,d\mu$ 的实部是 $\int_E u^+\,d\mu$. 因此 $\int_E u^+\,d\mu=0$，(*a*)蕴涵着 $u^+=0$ a. e. 类似地得出

$$u^-=v^+=v^-=0\qquad\text{a. e.}$$

(*c*) 考察定理 1.33 的证明，我们现在的假定蕴涵着在定理

1.33 的证明中最后一个不等号实际上必须是等号. 因此 $\int(|f|-u)d\mu=0$. 因为 $|f|-u\geqslant 0$, (a) 证明了 $|f|=u$ a.e. 这就是说, $\mathrm{Re}\alpha f=|\alpha f|$ a.e., 因此, $\alpha f=|\alpha f|=|f|$ a.e., 这就是所要的结论.

1.40 定理 假设 $\mu(X)<\infty$, $f\in L^1(\mu)$, S 是复平面上的一个闭集, 且对于每一个 $E\in\mathfrak{M}$, $\mu(E)>0$, 平均值

$$A_E(f)=\frac{1}{\mu(E)}\int_E f d\mu$$

在 S 内. 则对于几乎所有的 $x\in X$, $f(x)\in S$.

证明 设 Δ 是在 S 的余集内的闭圆盘 (譬如说中心在 α 且半径 $r>0$), 因 S^c 是可数个这样的圆盘的并, 只需证明对 $E=f^{-1}(\Delta)$, $\mu(E)=0$ 就足够了.

假设 $\mu(E)>0$ 则

$$|A_E(f)-\alpha|=\frac{1}{\mu(E)}\left|\int_E(f-\alpha)d\mu\right|$$

$$\leqslant\frac{1}{\mu(E)}\int_E|f-\alpha|d\mu\leqslant r$$

由于 $A_E(f)\in S$, 上述结果是不可能的. 因此 $\mu(E)=0$.

1.41 定理 设 $\{E_k\}$ 是在 X 内的可测集序列, 满足

(1) $$\sum_{k=1}^{\infty}\mu(E_k)<\infty$$

则几乎所有的 $x\in X$, 至多属于有限个 E_k.

证明 若 A 是所有属于无限个 E_k 的 x 的集, 我们要证明 $\mu(A)=0$. 令

(2) $$g(x)=\sum_{k=1}^{\infty}\chi_{E_k}(x)\qquad(x\in X)$$

对每一个 x, 这个级数的每一项或是 0 或是 1. 因此, $x\in A$, 当且仅当 $g(x)=\infty$. 根据定理 1.27, g 在 X 上的积分等于(1)的和. 于

是 $g\in L^1(\mu)$. 所以 $g(x)<\infty$ a. e.

习　题

1　是否存在一个仅具有可数个元素的无限 σ-代数?

2　对 n 个函数证明类似定理 1.8 的结果.

3　若 f 是在可测空间 X 上的实函数, 使得对每一个有理数 r,

$$\{x:f(x)\geqslant r\}$$

是可测集, 证明 f 是可测的.

4　设 $\{a_n\}$ 和 $\{b_n\}$ 是 $[-\infty,\infty]$ 内的序列, 证明下列结论:

(a) $\limsup\limits_{n\to\infty}(-a_n)=-\liminf\limits_{n\to\infty}a_n$

(b) $\limsup\limits_{n\to\infty}\{a_n+b_n\}\leqslant\limsup\limits_{n\to\infty}a_n+\limsup\limits_{n\to\infty}b_n$

假定这些和中没有 $\infty-\infty$ 的形式.

(c) 对所有的 n, $a_n\leqslant b_n$, 则

$$\liminf_{n\to\infty}a_n\leqslant\liminf_{n\to\infty}b_n$$

举例说明在 (b) 内严格不等式能成立.

5　证明实可测函数序列的收敛点集是可测集.

6　设 X 是一个不可数集, 设 $\mathfrak{M}$ 是所有的使得 E 或 E^c 至多是可数的集 $E\subset X$ 所成的集族. 在第一种情形下定义 $\mu(E)=0$, 在第二种情形下定义 $\mu(E)=1$. 证明 $\mathfrak{M}$ 是 X 内的 σ-代数, μ 是 $\mathfrak{M}$ 上的一个测度.

7　假设对 $n=1,2,3,\cdots$, $f_n:X\to[0,\infty]$ 是可测的. 对每一个 $x\in X$, $f_1\geqslant f_2\geqslant f_3\geqslant\cdots\geqslant 0$, 当 $n\to\infty$ 时 $f_n(x)\to f(x)$, 且 $f_1\in L^1(\mu)$. 证明

$$\lim_{n\to\infty}\int_X f_n d\mu=\int_X f d\mu$$

并且指出若省去条件"$f_1\in L^1(\mu)$", 这个结论就得不出来.

8　若 n 是奇数, 令 $f_n=\chi_E$, 若 n 是偶数, 令 $f_n=1-\chi_E$, 这个例子与Fatou引理有什么关系?

9 假设 μ 是 X 上的正测度, $f:X\to[0,\infty]$ 是可测的, $\int_X f d\mu=c$, 这里 $0<c<\infty$, α 是一个常数. 证明

$$\lim_{n\to\infty}\int_X n\log[1+(f/n)^\alpha]d\mu=\begin{cases}\infty & \text{若 } 0<\alpha<1\\ c & \text{若 } \alpha=1\\ 0 & \text{若 } 1<\alpha<\infty\end{cases}$$

提示：若 $\alpha \geqslant 1$，被积函数被 αf 控制．若 $\alpha < 1$，能应用 Fatou 引理．

10　假设 $\mu(X) < \infty$，$\{f_n\}$ 是 X 上的一个有界复可测函数序列，且 $f_n \to f$ 在 X 上是一致的．证明

$$\lim_{n\to\infty} \int_X f_n d\mu = \int_X f d\mu$$

并且指明假设"$\mu(X) < \infty$"不能省略．

11　证明在定理 1.41 中

$$A = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} E_k$$

因此，证明此定理就无需涉及积分．

12　假设 $f \in L^1(\mu)$．证明对每一个 $\varepsilon > 0$，存在一个 $\delta > 0$ 使得当 $\mu(E) < \delta$ 时，$\int_E |f| d\mu < \varepsilon$

第二章　正 Borel 测度

向量空间

2.1　定义　复向量空间(或复数域上的向量空间）是一个集 V，它的元素称为向量，在元素间定义了两种运算，一种称为加法，一种称为纯量乘法，并满足下列熟知的代数性质：

对于一对向量 x 和 y 对应着一个向量 $x+y$，并满足 $x+y=y+x$ 和 $x+(y+z)=(x+y)+z$；V 包含唯一的向量 0(零向量或 V 的原点)，使得对每个 $x\in V$，$x+0=x$；并且对每个 $x\in V$，对应着唯一的向量 $-x$，使得 $x+(-x)=0$，

对于每一对(α,x)，$x\in V$，α 是一个纯量(今后的行文中，纯量的意思就是复数)，对应着一个向量 $\alpha x\in V$，满足 $1x=x$，$\alpha(\beta x)=(\alpha\beta)x$，以及两个分配律：

(1)　　$$\alpha(x+y)=\alpha x+\alpha y,\ (\alpha+\beta)x=\alpha x+\beta x$$

向量空间 V 到向量空间 V_1 内的线性变换是 V 到 V_1 内的一个映射 A，使得对所有 $x,y\in V$ 和一切的纯量 α,β，有

(2)　　$$A(\alpha x+\beta y)=\alpha Ax+\beta Ay$$

在 V_1 是纯量域的特殊情况下（除了仅由 0 组成的平凡场合之外，这是最简单的向量空间的例子），A 称为线性泛函．因此，线性泛函是 V 上的，满足(2)的一个复函数．

注意，若 A 是线性的，我们常常用记号 Ax，而不用 $A(x)$．

当然，上述定义无论用任何一个域代替复数域都是适用的．除非明白地指出是相反的情形，本书中的向量空间均是复向量空间．然而，有个值得注意的例外：欧氏空间 R^k 是实数域上的向量空间．

2.2 作为线性泛函的积分 分析学是充满了向量空间和线性变换的，并在积分为一方、以线性泛函为另一方之间有一种特别密切的关系.

例如，定理 1.32 证明了 $L^1(\mu)$对任何正测度 μ 是一个向量空间，而映射

$$f \longrightarrow \int_X f d\mu \tag{1}$$

是 $L^1(\mu)$上的一个线性泛函. 类似地，如果 g 是任何有界可测函数，则映射

$$f \longrightarrow \int_X fg d\mu \tag{2}$$

也是 $L^1(\mu)$上的一个线性泛函；在第 6 章中，我们将看到，在某种意义上，泛函(2)是 $L^1(\mu)$上仅有的一种我们感兴趣的泛函.

另一个例子，设 C 是单位区间 $I=[0,1]$ 上一切连续复函数的集. 两个连续函数的和是连续的，一个连续函数的任何纯量积也是连续的. 因此，C 是一个向量空间. 如果

$$\Lambda f=\int_0^1 f(x)dx \qquad (f\in C) \tag{3}$$

是通常的 Riemann 积分，则 Λ 显然是 C 上的线性泛函；Λ 有一个附带的有趣的性质：它是一个正的线性泛函. 即是说，当 $f\geqslant 0$ 时，有 $\Lambda f\geqslant 0$.

我们当前的任务之一仍然是构造 Lebesgue 测度. 根据下述观察得知，此构造可以建立在线性泛函(3)的基础之上：考虑开区间 $(a,b)\subset I$ 和定义在 I 上的这样一类函数 C，当 $f\in C$ 时，$0\leqslant f\leqslant 1$，而对所有不在 (a,b) 中的 x，$f(x)=0$. 对所有这样的 f，有 $\Lambda f<b-a$，然而，我们可以选择 f，使得 Λf 要多么接近 $b-a$ 都行. 因此，(a,b)的长度(或测度)是和泛函 Λ 的值直接有关的.

从更一般的观点看，上述观察导致出著名的并且极为重要的

F. Riesz 定理:

对 C 上的每一个正线性函数 Λ, 都对应 I 上的一个有限正 Borel 测度 μ, 使得

$$\Lambda f=\int_I f d\mu \qquad (f\in C) \tag{4}$$

[逆定理是显然的: 若 μ 是 I 上的正有限 Borel 测度, Λ 由(4)式定义, 则 Λ 是 C 上的一个正线性函数.]

显然, 用 R^1 代替有界区间 I 是有意思的. 如果我们只考虑 R^1 上的那些在某些有界区间外取值为零的连续函数, 就能做到这一点(这些函数是, 举例说, Riemann 可积的). 其次, 多个变量的函数常常出现在分析中. 因此, 我们应该由 R^1 扩充到 R^n. 结果是 Reisz 定理的证明几乎不需作任何修改仍然通得过. 而且, 它表明 R^n 的欧氏性质(坐标, 正交性等等)在证明中并不起什么作用; 事实上, 如果对它们考虑得太多, 反而会造成妨碍. 对于证明来说, 实质性的东西乃是 R^n 的一些拓扑性质(这是自然的, 因为现在我们谈的是连续函数). 起决定作用的性质是局部紧性: R^n 的每一点有一个邻域, 它的闭包是紧的.

因此, 我们将在很一般的条件下建立 Riesz 定理(定理2.14). Lebesgue 测度的存在性于是就作为一种特殊情况而得到. 希望集中注意力于更加具体的情况的人, 至少对于初学者, 可以轻松地跳过下述关于拓扑预备知识的那一节 (最有趣的一段是 Urysohn 引理; 参看习题 3), 并在这一章的其余部分可以用 R^1 代替 X.

拓扑学预备知识

2.3 定义 设 X 是象 1.2 节所定义的拓扑空间.

(*a*) 集 $E\subset X$ 是*闭的*, 如果它的余集 E^c 是*开的*. (因此, $\varnothing$ 和 X 是闭集, 闭集的有限并是闭集, 闭集的任意交是闭集.).

(*b*) 集 $E\subset X$ 的闭包 $\bar{E}$ 是 X 中包含 E 的最小闭集. (下述推理证明了 $\bar{E}$ 存在: X 中包含 E 的所有闭子集的集族 Ω 是非空的(因为 $X\in\Omega$). 令 $\bar{E}$ 是 Ω 的一切元素的交.)

(*c*) 集 $K\subset X$ 是紧的, 如果 K 的每个开覆盖包含有限子覆盖. 更明确些, 此要求是: 如果 $\{V_\alpha\}$ 是开集的集族, 它们的并包含 K, 则 $\{V_\alpha\}$ 的某个有限子族的并也包含 K.

特别, 如果 X 本身是紧的, 则 X 称为紧空间.

(*d*) 点 $p\in X$ 的一个邻域是 X 的任意一个包含 p 的开子集. (使用这个词不是十分标准的; 有些人对任意包含一个含 p 的开集的集使用"p 的邻域"这个词.)

(*e*) X 是一个 Hausdorff 空间, 如果下述条件成立: 若 $p\in X$, $q\in X$, 且 $p\neq q$, 则 p 有一个邻域 U, q 有一个邻域 V, 使得 $U\cap V=\emptyset$.

(*f*) X 是局部紧的, 如果 X 的每一点有一个邻域, 它的闭包是紧的.

显然, 每个紧空间是局部紧的.

我们回顾一下 Heine-Borel 定理: 欧氏空间 R^n 的紧的子集恰好是那些闭的而且有界的集([26]†, 定理 2.41). 由此定理容易得到: R^n 是局部紧的 Hausdorff 空间. 而且, 每个度量空间也是 Hausdorff 空间.

2.4 定理 *在拓扑空间 X 中, 设 K 是紧的而 F 是闭集. 如果 $F\subset K$, 则 F 是紧的.*

证明 如果 $\{V_\alpha\}$ 是 F 的开覆盖, $W=F^c$, 则 $\{W\}\cup\{V_\alpha\}$ 覆盖 K; 因此, 存在有限集族 $\{V_{\alpha_i}\}$ 使得

$$K\subset W\cup V_{\alpha_1}\cup\cdots\cup V_{\alpha_n}$$

† 括号中的数字请看参考书目.

则 $$F\subset V_{\alpha_1}\cup\cdots\cup V_{\alpha_n}$$

系 若 $A\subset B$，而 B 有紧闭包，则 A 也有.

2.5 定理 设 X 是 Hausdorff 空间，$K\subset X$，K 是紧的，且 $p\in K^c$. 则存在开集 U 和 W，使得 $p\in U$，$K\subset W$，并且 $U\cap W=\varnothing$.

证明 设 $q\in K$，由 Hausdorff 分离公理推出存在不交的开集 U_q 和 V_q，使得 $p\in U_q$，$q\in V_q$. 因为 K 是紧的，存在点 $q_1,\cdots,q_n\in K$，使得

$$K\subset V_{q_1}\cup\cdots\cup V_{q_n}$$

通过集

$$U=U_{q_1}\cap\cdots\cap U_{q_n}\text{ 和 }W=V_{q_1}\cup\cdots\cup V_{q_n}$$

定理的要求得到满足.

系

(*a*) Hausdorff 空间的紧子集是闭集.

(*b*) 在 Hausdorff 空间中，如果 F 是闭集，K 是紧的，则 $F\cap K$ 是紧的.

由系(*a*)和定理 2.4 就得到系(*b*).

2.6 定理 设 $\{K_\alpha\}$ 是 Hausdorff 空间紧子集的集族，$\bigcap_\alpha K_\alpha=\varnothing$，则 $\{K_\alpha\}$ 族中存在有限个集它们的交也是空的.

证明 令 $V_\alpha=K_\alpha^c$. 固定 $\{K_\alpha\}$ 的一个元素 K_1. 由于 K_1 没有点属于每个 K_α，$\{V_\alpha\}$ 是 K_1 的一个开覆盖. 因此，对某个有限集族 $\{V_{\alpha_i}\}$，有 $K_1\subset V_{\alpha_1}\cup\cdots\cup V_{\alpha_n}$. 由此得到

$$K_1\cap K_{\alpha_1}\cap\cdots\cap K_{\alpha_n}=\varnothing$$

2.7 定理 设 U 是局部紧 Hausdorff 空间 X 的开集，$K\subset U$，K 是紧的. 则存在一个具有紧闭包的开集 V，使得

$$K\subset V\subset\bar{V}\subset U$$

证明 由于 K 的每个点有一个闭包是紧的邻域，而且 K 被这些邻域的有限并所覆盖，K 便位于一个闭包是紧的开集 G 内. 如

果 $U=X$，则取 $V=G$.

否则，令 C 是 U 的余集. 定理 2.5 表明，对每个 $p\in C$ 对应一个开集 W_p，使得 $K\subset W_p$ 和 $p\notin\overline{W}_p$. 因此，p 跑遍 C 时，$\{C\cap\bar{G}\cap\overline{W}_p\}$是一个紧集的集族，其交是空集. 由定理 2.6，存在点 $p_1,\cdots,p_n\in C$，使得

$$C\cap\bar{G}\cap\overline{W}_{p_1}\cap\cdots\cap\overline{W}_{p_n}=\varnothing$$

因为

$$\bar{V}\subset\bar{G}\cap\overline{W}_{p_1}\cap\cdots\cap\overline{W}_{p_n}$$

这时集

$$V=G\cap W_{p_1}\cap\cdots\cap W_{p_n}$$

具有所要求的性质.

2.8 定义 设 f 是拓扑空间上的实(或广义实数的)函数. 如果对每个实数 α，$\{x: f(x)>\alpha\}$是开的，则称 f 为下半连续的. 如果对每个实数 α，$\{x: f(x)<\alpha\}$是开的，则称 f 为上半连续的.

下述半连续函数的性质几乎是这个定义的直接推论:

(*a*) 一个实函数是连续的当且仅当它既上半连续又下半连续.

(*b*) 开集的特征函数是下半连续的；闭集的特征函数是上半连续的.

(*c*) 任意一族下半连续函数的上确界是下半连续的. 任意一族上半连续函数的下确界是上半连续的.

2.9 定义 拓扑空间 X 上的复函数 f 的支集是集

$$\{x: f(x)\neq 0\}$$

的闭包. 记 X 上支集是紧的所有连续复函数的集为 $C_c(X)$.

可以看出，$C_c(X)$是向量空间. 这是基于两个事实:

(*a*) $f+g$ 的支集位于 f 的支集和 g 的支集的并内，而紧集的任意有限并是紧的.

(*b*) 象连续函数的纯量倍数一样，两个连续复函数的和也是连续的.

(定理 1.8 的叙述和证明当用"连续函数"代替"可测函数"，用"拓扑空间"代替"可测空间"还是照样成立的；取$\Phi(s,t)=s+t$，或$\Phi(s,t)=st$，去证明连续函数的和与积是连续的.)

2.10 定理 设X和Y是拓扑空间，并设$f:X\to Y$是连续的. 如果K是X的紧子集，则$f(K)$是紧的.

证明 如果$\{V_\alpha\}$为$f(K)$的一个开覆盖，则$\{f^{-1}(V_\alpha)\}$是K的一个开覆盖. 于是，对某些$\alpha_1,\cdots,\alpha_n$，$K\subset f^{-1}(V_{\alpha_1})\cup\cdots\cup f^{-1}(V_{\alpha_n})$，因此，$f(K)\subset V_{\alpha_1}\cup\cdots\cup V_{\alpha_n}$.

系 任何$f\in C_c(X)$的值域是复平面的一个紧子集.

事实上，如果K是$f\in C_c(X)$的支集，则$f(X)\subset f(K)\cup\{0\}$. 若$X$不是紧的，则$0\in f(X)$. 但0不一定属于$f(K)$，这由简单例子即可看出.

2.11 记号 在本章，将使用下述约定. 记号

(1) $$K\prec f$$

表示K是X的紧子集，$f\in C_c(X)$，对一切$x\in X$，$0\leqslant f(x)\leqslant 1$，并对一切$x\in K$，$f(x)=1$. 记号

(2) $$f\prec V$$

表示V是开集，$f\in C_c(X)$，$0\leqslant f\leqslant 1$，并且f的支集含于V. 记号

(3) $$K\prec f\prec V$$

表示(1)和(2)都成立.

2.12 Urysohn 引理 设X是局部紧的 Hausdorff 空间，V是X中的开集，$K\subset V$，而K是紧的. 则存在一个$f\in C_c(X)$，使得

(1) $$K\prec f\prec V$$

利用特征函数的语言，这个结论断言存在一个连续函数f，满

足不等式$\chi_K \leqslant f \leqslant \chi_V$. 注意，容易找到半连续函数满足此不等式. 例如，χ_K 和 χ_V.

证明 令 $r_1=0, r_2=1$ 而 $r_3, r_4, r_5, \cdots$ 是(0, 1) 内的全体有理数的一种排列. 由定理 2.7，能找到开集 V_0 和 V_1 使得 $\overline{V}_0$ 是紧的，并且

$$K \subset V_1 \subset \overline{V}_1 \subset V_0 \subset \overline{V}_0 \subset V \tag{2}$$

设 $n \geqslant 2$，并且 $V_{r_1}, \cdots, V_{r_n}$ 已按这样一种方式选择好，使当 $r_i < r_j$ 时，有 $\overline{V}_{r_j} \subset V_{r_i}$. 则数 $r_1, \cdots, r_n$ 中的一个，例如 r_i 将是小于 r_{n+1} 的诸数中最大的那一个，另外，例如 r_j 将是大于 r_{n+1} 的诸数中最小的那一个. 再由定理 2.7，能找到 $V_{r_{n+1}}$，使得 $\overline{V}_{r_j} \subset V_{r_{n+1}} \subset \overline{V}_{r_{n+1}} \subset V_{r_i}$.

继续下去，我们得到一个开集族 $\{V_r\}$，对[0, 1]中的每个有理数 r，都有如下的性质：$K \subset V_1$，$\overline{V}_0 \subset V$，每个 $\overline{V}_r$ 是紧的，并且

$$\text{当 } s > r \text{ 时，} \overline{V}_s \subset V_r \tag{3}$$

定义

$$f_r(x)=\begin{cases} r & \text{若 } x \in V_r \\ 0 & \text{若 } x \notin V_r \end{cases} \quad g_s(x)=\begin{cases} 1 & \text{若 } x \in \overline{V}_s \\ s & \text{若 } x \notin \overline{V}_s \end{cases} \tag{4}$$

和

$$f=\sup_r f_r, \quad g=\inf_s g_s \tag{5}$$

定义 2.8 的评注表明 f 是下半连续的，而 g 是上半连续的. 显然有 $0 \leqslant f \leqslant 1$，当 $x \in K$ 时，$f(x)=1$，并且 f 的支集在 $\overline{V}_0$ 中. 今证 $f=g$ 以完成本证明.

仅当 $r>s$，$x \in V_r$ 和 $x \notin \overline{V}_s$ 时，不等式 $f_r(x) > g_s(x)$ 才可能成立. 但是 $r>s$ 蕴涵着 $V_r \subset V_s$. 因此，对一切 r 和 s，有 $f_r \leqslant g_s$，于是 $f \leqslant g$.

设对某个 x，$f(x) < g(x)$. 则存在有理数 r 和 s，使得 $f(x) <$

$r<s<g(x)$. 因为 $f(x)<r$, 我们有 $x\notin V_r$; 因为 $g(x)>s$, 我们有 $x\in\overline{V}_s$. 由(3), 这是矛盾的. 因此, $f=g$.

2.13 定理 设 $V_1,\cdots,V_n$ 是局部紧 Hausdorff 空间 X 的开子集, K 是紧的, 并且

$$K\subset V_1\cup\cdots\cup V_n$$

则存在函数 $h_i\prec V_i(i=1,2,\cdots,n)$, 使得

$$h_1(x)+\cdots+h_n(x)=1 \qquad (x\in K) \tag{1}$$

因为(1), 集族 $\{h_1,\cdots,h_n\}$ 称为 K 上的从属于覆盖 $\{V_1,\cdots,V_n\}$ 的单位划分.

证明 由定理 2.7, 每个 $x\in K$ 有一个邻域 W_x, 具有紧闭包 $\overline{W}_x\subset V_i$, 对某个 i(依赖于 x)成立. 存在点 $x_1,\cdots,x_m$, 使得 $W_{x_1}\cup\cdots\cup W_{x_m}\supset K$. 如果 $1\leqslant i\leqslant n$, 令 H_i 是含于 V_i 的 $\overline{W}_{x_j}$ 的并. 由 Urysohn 引理, 存在函数 g_i, 使得 $H_i\prec g_i\prec V_i$. 定义

$$\begin{aligned} h_1&=g_1\\ h_2&=(1-g_1)g_2\\ &\cdots\cdots\cdots\cdots\\ h_n&=(1-g_1)(1-g_2)\cdots(1-g_{n-1})g_n \end{aligned} \tag{2}$$

则 $h_i\prec V_i$. 由归纳法, 容易验证

$$h_1+h_2+\cdots+h_n=1-(1-g_1)(1-g_2)\cdots(1-g_n) \tag{3}$$

因为 $K\subset H_1\cup\cdots\cup H_n$, 故对每一点 $x\in K$, 至少有一个 $g_i(x)=1$, 因此, (3)证明了(1)成立.

Riesz 表示定理

2.14 定理 设 X 是局部紧的 Hausdroff 空间, Λ 是 $C_c(X)$ 上的正线性泛函. 则在 X 内存在一个包含 X 的一切 Borel 集的 σ-代数 $\mathfrak{M}$, 并存在 $\mathfrak{M}$ 上的唯一一个正测度 μ, μ 在下述意义上表示了 Λ:

(a) 对每个 $f\in C_c(X)$，$\Lambda f=\int_X f\,d\mu$.

并有下述的附加性质：

(b) 对每个紧集 $K\subset X$, $\mu(K)<\infty$.

(c) 对每个 $E\in\mathfrak{M}$, 有

$$\mu(E)=\inf\{\mu(V):E\subset V, V \text{ 是开集}\}$$

(d) 对每个开集 E 和每个 $E\in\mathfrak{M}$ 而 $\mu(E)<\infty$, 有

$$\mu(E)=\sup\{\mu(K):K\subset E, K \text{ 是紧集}\}$$

(e) 如果 $E\in\mathfrak{M}$, $A\subset E$, 并且 $\mu(E)=0$, 则 $A\in\mathfrak{M}$.

当然，(a)是最感兴趣的一个性质. 定义 $\mathfrak{M}$ 和 μ 之后，证明 $\mathfrak{M}$ 是 σ-代数，μ 是可数可加的过程中，(b)到(d)也就会建立起来. 我们稍后将看到(定理 2.18)，在"合理的"空间 X 内，每一个满足(b)的 Borel 测度也满足(c)和(d)，并且在那些情况下对每个 $E\in\mathfrak{M}$，(d)一定成立. 性质(e)仅说明，在定理 1.36 的意义下，$(X,\mathfrak{M},\mu)$ 是一个完备测度空间.

在这个定理的整个证明中，字母 K 表示 X 的紧子集，V 表示 X 内的开集.

我们从证明 μ 的唯一性开始. 如果 μ 满足(c)和(d)，显然 μ 在 $\mathfrak{M}$ 上由它在紧集上的值所决定. 因此，当 μ_1 和 μ_2 是满足定理的测度时，只要对一切 K，证明 $\mu_1(K)=\mu_2(K)$ 就足够了. 于是，固定 K 和 $\varepsilon>0$. 由(b)和(c)，存在 $V\supset K$，有 $\mu_2(V)<\mu_2(K)+\varepsilon$；由 Urysohn引理，存在 f，使得 $K\prec f\prec V$；因此

$$\mu_1(K)=\int_X \chi_K\,d\mu_1\leqslant\int_X f\,d\mu_1=\Lambda f=\int_X f\,d\mu_2$$

$$\leqslant\int_X \chi_V\,d\mu_2=\mu_2(V)<\mu_2(K)+\varepsilon$$

这样，$\mu_1(K)\leqslant\mu_2(K)$. 如果改变 μ_1 和 μ_2 的地位，就会得到相反的不等式，从而证明了 μ 的唯一性.

上述计算还附带地证明了由(a)可推得(b).

μ和$\mathfrak{M}$的构造

对X中的每个开集V,定义

(1) $$\mu(V)=\sup\{Af: f\prec V\}$$

如果$V_1\subset V_2$,显然(1)蕴涵着$\mu(V_1)\leqslant\mu(V_2)$. 因此,如果$E$是一个开集,则

(2) $$\mu(E)=\inf\{\mu(V):\ E\subset V, V\text{ 是开集}\}$$

并且对每个$E\subset X$,由(2)来定义$\mu(E)$和(1)是一致的.

注意,尽管对每个$E\subset X$定义了$\mu(E)$,μ的可数可加性将仅对X中某种σ-代数$\mathfrak{M}$证明.

令$\mathfrak{M}_F$是一切$E\subset X$的类,这里E满足两个条件:$\mu(E)<\infty$,和

(3) $$\mu(E)=\sup\{\mu(K): K\subset E, K\text{ 是紧的}\}$$

最后,令$\mathfrak{M}$是一切$E\subset X$的类,这里E对每个紧集K有$E\cap K\in\mathfrak{M}_F$.

证明μ和$\mathfrak{M}$有所要求的性质

显然μ是单调的,即若$A\subset B$,有$\mu(A)\leqslant\mu(B)$,而且$\mu(E)=0$蕴涵着$E\in\mathfrak{M}_F$和$E\in\mathfrak{M}$. 因此,(e)成立,且根据定义,(c)也成立.

因为其他结论的证明相当长,把它分成几步来证比较方便.

可以看出,A的正性蕴涵着A是单调的:$f\leqslant g$蕴涵着$Af\leqslant Ag$. 这是显然的,因为$Ag=Af+A(g-f)$而$g-f\geqslant 0$. 此单调性将用于II和X.

I 如果$E_1, E_2, E_3, \cdots$是X的任意子集,则

(4) $$\mu\left(\bigcup_{i=1}^{\infty}E_i\right)\leqslant\sum_{i=1}^{\infty}\mu(E_i)$$

证明 首先证明如果V_1和V_2是开集,则

(5) $$\mu(V_1\cup V_2)\leqslant\mu(V_1)+\mu(V_2)$$

选择 $g \prec V_1 \cup V_2$. 由定理 2.13, 存在函数 h_1 和 h_2, 使得 $h_i \prec V_i$, 对 g 的支集内的一切 x, $h_1(x)+h_2(x)=1$. 因此, $h_i g \prec V_i$, $g=h_1 g+h_2 g$, 于是

$$\Lambda g=\Lambda(h_1 g)+\Lambda(h_2 g)\leqslant \mu(V_1)+\mu(V_2) \tag{6}$$

因为(6)对每个 $g \prec V_1 \cup V_2$ 都成立, 由此得到(5).

如果对某个 i, $\mu(E_i)=\infty$, 则(4)显然为真. 因此, 对每个 i, 设 $\mu(E_i)<\infty$. 取 $\varepsilon>0$. 由(2), 存在开集 $V_i \supset E_i$, 使得

$$\mu(V_i)<\mu(E_i)+2^{-i}\varepsilon \qquad (i=1,2,3,\cdots)$$

令 $V=\bigcup_1^\infty V_i$ 并选择 $f \prec V$. 因为 f 有紧的支集, 可以看出对某个 n, 有 $f \prec V_1 \cup \cdots \cup V_n$. 对(5)应用归纳法, 于是得到

$$\Lambda f \leqslant \mu(V_1 \cup \cdots \cup V_n) \leqslant \mu(V_1)+\cdots+\mu(V_n) \leqslant \sum_{i=1}^{\infty} \mu(E_i)+\varepsilon$$

因为对每个 $f \prec V$, 此不等式成立, 且 $\cup E_i \subset V$, 由此得到

$$\mu\Big(\bigcup_{i=1}^{\infty} E_i\Big) \leqslant \mu(V) \leqslant \sum_{i=1}^{\infty} \mu(E_i)+\varepsilon$$

由于 ε 是任意的, 故证明了(4).

II 如果 K 是紧的, 则 $K \in \mathfrak{M}_F$, 并且

$$\mu(K)=\inf\{\Lambda f: K \prec f\} \tag{7}$$

由此就得到定理的结论(b).

证明 如果 $K \prec f$, $0<\alpha<1$, 令 $V_\alpha=\{x: f(x)>\alpha\}$. 则 $K \subset V_\alpha$, 并且当 $g \prec V_\alpha$ 时, $\alpha g<f$. 因此

$$\mu(K) \leqslant \mu(V_\alpha)=\sup\{\Lambda g: g \prec V_\alpha\} \leqslant \alpha^{-1}\Lambda f$$

令 $\alpha \to 1$, 最终得到

$$\mu(K) \leqslant \Lambda f \tag{8}$$

因此, $\mu(K)<\infty$. 由于 K 显然满足(3), $K \in \mathfrak{M}_F$.

如果 $\varepsilon>0$, 则存在 $V \supset K$, 使 $\mu(V)<\mu(K)+\varepsilon$. 由 Urysohn

引理，存在 $f, K\prec f\prec V$. 因此

$$\Lambda f\leqslant\mu(V)\leqslant\mu(K)+\varepsilon$$

由此式并结合(8)式，就得到(7).

III 每个开集都满足(3). 因此，$\mathfrak{M}_F$ 包含每个 $\mu(V)<\infty$ 的开集 V.

证明 令 α 是一个实数，使得 $\alpha<\mu(V)$. 存在一个使 $\alpha<\Lambda f$ 的 $f\prec V$. 如果 W 是包含 f 的支集 K 的任意一个开集，则 $f\prec W$，于是 $\Lambda f\leqslant\mu(W)$. 因此，$\Lambda f\leqslant\mu(K)$. 这就表示有满足 $\alpha<\mu(K)$ 的紧集 $K\subset V$，于是对于 V，(3)式成立.

IV 设 $E=\bigcup_{i=1}^{\infty}E_i$，其中 $E_1, E_2, E_3, \cdots$ 是 $\mathfrak{M}_F$ 的互不相交的元素. 则

$$\mu(E)=\sum_{i=1}^{\infty}\mu(E_i) \tag{9}$$

并且，如果 $\mu(E)<\infty$，则 $E\in\mathfrak{M}_F$.

证明 我们首先指出，如果 K_1 和 K_2 是不相交的紧集，则

$$\mu(K_1\cup K_2)=\mu(K_1)+\mu(K_2) \tag{10}$$

取 $\varepsilon>0$. 由 Urysohn 引理，存在 $f\in C_c(X)$，使得在 K_1 上 $f(x)=1$，在 K_2 上 $f(x)=0$. 由 II，存在 g 使得

$$K_1\cup K_2\prec g \text{ 和 } \Lambda g<\mu(K_1\cup K_2)+\varepsilon$$

注意 $K_1\prec fg$ 和 $K_2\prec(1-f)g$. 因为 Λ 是线性的，由(8)式得到

$$\mu(K_1)+\mu(K_2)\leqslant\Lambda(fg)+\Lambda(g-fg)=\Lambda g<\mu(K_1\cup K_2)+\varepsilon$$

由于 ε 是任意的，因此由 I 得到(10).

如果 $\mu(E)=\infty$，则由 I 得到(9). 因此，设 $\mu(E)<\infty$，并取 $\varepsilon>0$. 因为 $E_i\in\mathfrak{M}_F$，存在紧集 $H_i\subset E_i$，有

$$\mu(H_i)>\mu(E_i)-2^{-i}\varepsilon \qquad (i=1,2,3,\cdots) \tag{11}$$

令 $K_n=H_1\cup\cdots\cup H_n$，并对(10)应用归纳法，得到

(12) $$\mu(E)\geqslant\mu(K_n)=\sum_{i=1}^{n}\mu(H_i)>\sum_{i=1}^{n}\mu(E_i)-\varepsilon$$

由于对每个 n 和每个 $\varepsilon>0$(12)成立，所以(9)的左边不小于右边. 于是从 I 得到(9).

但是，如果 $\mu(E)<\infty$ 而 $\varepsilon>0$，(9)表明，对某个 N

(13) $$\mu(E)\leqslant\sum_{i=1}^{N}\mu(E_i)+\varepsilon$$

由(12)，就得到 $\mu(E)\leqslant\mu(K_N)+2\varepsilon$，这就证明了 E 满足(3)；因此，$E\in\mathfrak{M}_F$.

V　如果 $E\in\mathfrak{M}_F$ 和 $\varepsilon>0$，则存在紧集 K 和开集 V，使得 $K\subset E\subset V$ 而 $\mu(V-K)<\varepsilon$.

证明　我们的定义表明，存在 K 和 V 使得

$$\mu(V)-\frac{\varepsilon}{2}<\mu(E)<\mu(K)+\frac{\varepsilon}{2}$$

因为 $V-K$ 是开集，由 III，$V-K\in\mathfrak{M}_F$. 于是 IV 蕴涵着

$$\mu(K)+\mu(V-K)=\mu(V)<\mu(K)+\varepsilon$$

VI　如果 $A\in\mathfrak{M}_F$ 和 $B\in\mathfrak{M}_F$，则 $A-B$，$A\cup B$ 和 $A\cap B$ 属于 $\mathfrak{M}_F$.

证明　如果 $\varepsilon>0$，V 表明存在集 K_i 和 V_i 使得 $K_1\subset A\subset V_1$，$K_2\subset B\subset V_2$，并且 $\mu(V_i-K_i)<\varepsilon$，$i=1,2$. 由于

$$A-B\subset V_1-K_2\subset(V_1-K_1)\cup(K_1-V_2)\cup(V_2-K_2)$$

I 表明

(14) $$\mu(A-B)\leqslant\varepsilon+\mu(K_1-V_2)+\varepsilon$$

因为 K_1-V_2 是 $A-B$ 的紧的子集，(14)表明 $A-B$ 满足(3)，所以 $A-B\in\mathfrak{M}_F$.

由于 $A\cup B=(A-B)\cup B$，应用 IV，可证明 $A\cup B\in\mathfrak{M}_F$. 因为 $A\cap B=A-(A-B)$，也有 $A\cap B\in\mathfrak{M}_F$.

VII　$\mathfrak{M}$ 是 X 中包含所有 Borel 集的 σ-代数.

证明　令 K 是 X 内的任一个紧集.

如果 $A\in\mathfrak{M}$, 则 $A^c\cap K=K-(A\cap K)$, 于是 $A^c\cap K$ 是 $\mathfrak{M}_F$ 的两个元素之差. 因此, $A^c\cap K\in\mathfrak{M}_F$, 于是可以得出结论: $A\in\mathfrak{M}$ 蕴涵着 $A^c\in\mathfrak{M}$.

其次, 设 $A=\bigcup_1^\infty A_i$, 这里每个 $A_i\in\mathfrak{M}$. 令 $B_1=A_1\cap K$ 且

$$B_n=(A_n\cap K)-(B_1\cup\cdots\cup B_{n-1})\qquad(n=2,3,4,\cdots)\tag{15}$$

那末, 由 VI 知 $\{B_n\}$ 是 $\mathfrak{M}_F$ 中的互不相交元素的序列, 且 $A\cap K=\bigcup_1^\infty B_n$. 由 IV 得到 $A\cap K\in\mathfrak{M}_F$. 因此, $A\in\mathfrak{M}$.

最后, 如果 C 是闭集, 则 $C\cap K$ 是紧的. 因此 $C\cap K\in\mathfrak{M}_F$, 于是 $C\in\mathfrak{M}$. 特别, $X\in\mathfrak{M}$.

因此, 我们证明了 $\mathfrak{M}$ 是 X 中的一个 σ-代数, 它含有 X 的全体闭子集. 故 $\mathfrak{M}$ 含有 X 内的全体 Borel 集.

VIII　$\mathfrak{M}_F$ 恰好由那些使 $\mu(E)<\infty$ 的 $E\in\mathfrak{M}$ 所组成.

这蕴涵着定理中的 (d).

证明　如果 $E\in\mathfrak{M}_F$, II 和 VI 蕴涵着对每个紧集 K, $E\cap K\in\mathfrak{M}_F$, 因此, $E\in\mathfrak{M}$.

反之, 设 $E\in\mathfrak{M}$, $\mu(E)<\infty$, 并取 $\varepsilon>0$. 存在开集 $V\supset E$, 而 $\mu(V)<\infty$; 由 III 和 V, 存在紧集 $K\subset V$, 而 $\mu(V-K)<\varepsilon$. 因为 $E\cap K\in\mathfrak{M}_F$, 存在紧集 $H\subset E\cap K$, 而

$$\mu(E\cap K)<\mu(H)+\varepsilon$$

因为 $E\subset(E\cap K)\cup(V-K)$, 由此得到

$$\mu(E)\leqslant\mu(E\cap K)+\mu(V-K)<\mu(H)+2\varepsilon$$

这就蕴涵着 $E\in\mathfrak{M}_F$.

IX　μ 是 $\mathfrak{M}$ 上的一个测度.

证明 直接由 IV 和 VIII 得到 μ 的可数可加性。

X 对每个 $f\in C_c(X)$, $\Lambda f=\int_X f d\mu$。

这就证明了(a)并完成此定理.

证明 显然,对实的 f 证明就够了.而且,对每个实 $f\in C_c(X)$, 只要证明不等式

$$\Lambda f\leqslant\int_X f d\mu \tag{16}$$

也就够了. 因为,一旦(16)成立, Λ 的线性表明

$$-\Lambda f=\Lambda(-f)\leqslant\int_X(-f)d\mu=-\int_X f d\mu$$

此不等式和(16)一起就说明(16)的等号成立.

令 K 是实 $f\in C_c(X)$的支集, $[a, b]$是一个包含 f 值域的区间(注意定理 2.10 的系),取 $\varepsilon>0$, 并对 $i=0,1,\cdots,n$ 选择 y_i, 使得 $y_i-y_{i-1}<\varepsilon$ 和

$$y_0<a<y_1<\cdots<y_n=b \tag{17}$$

令

$$E_i=\{x: y_{i-1}<f(x)\leqslant y_i\}\cap K \qquad (i=1,2,\cdots,n) \tag{18}$$

由于 f 连续, f 是 Borel 可测的, 因此集 E_i 是不相交的 Borel 集, 它们的并是 K. 于是存在开集 $V_i\supset E_i$, 使得

$$\mu(V_i)<\mu(E_i)+\frac{\varepsilon}{n} \qquad (i=1,2,\cdots,n) \tag{19}$$

并对一切 $x\in V_i$, 有 $f(x)<y_i+\varepsilon$. 由定理 2.13, 存在函数 $h_i\prec V_i$, 在 K 上有 $\Sigma h_i=1$. 因此 $f=\Sigma h_i f$, 并且 II 表明

$$\mu(K)\leqslant\Lambda(\Sigma h_i)=\Sigma\Lambda h_i$$

由于 $h_i f\leqslant(y_i+\varepsilon)h_i$ 和在 E_i 上 $y_i-\varepsilon<f(x)$, 有

$$\Lambda f=\sum_{i=1}^{n}\Lambda(h_i f)\leqslant\sum_{i=1}^{n}(y_i+\varepsilon)\Lambda h_i$$

$$=\sum_{i=1}^{n}(|a|+y_i+\varepsilon)\Delta h_i-|a|\sum_{i=1}^{n}\Delta h_i$$

$$\leqslant\sum_{i=1}^{n}(|a|+y_i+\varepsilon)[\mu(E_i)+\varepsilon/n]-|a|\mu(K)$$

$$=\sum_{i=1}^{n}(y_i-\varepsilon)\mu(E_i)+2\varepsilon\mu(K)+\frac{\varepsilon}{n}\sum_{i=1}^{n}(|a|+y_i+\varepsilon)$$

$$\leqslant\int_X f d\mu+\varepsilon[2\mu(K)+|a|+b+\varepsilon]$$

因为 ε 是任意的，故(16)成立. 定理证明完毕.

Borel 测度的正则性

2.15 定义 定义在局部紧的 Hausdorff 空间 X 的全体 Borel 集所成 σ-代数上的测度 μ 称为 X 上的 Borel 测度. 设 μ 是正的，并且一个 Borel 集 $E\subset X$ 具有定理 2.14 的性质(c)或(d)，我们就分别称 E 为外正则或内正则的. 如果 X 内的每个 Borel 集同时是外正则和内正则的，则称 μ 为正则的.

在 Riesz 定理的证明中，每个集 E 的外正则性是在构作的过程中建立的，而内正则性则仅就开集和 $\mu(E)<\infty$ 的 $E\in\mathfrak{M}$ 作过证明. 产生这一缺点是自然的. 在定理 2.14 的假设下，我们无法证明 μ 的正则性；习题 17 介绍了一个例子.

然而，稍强的假设能给出一个正则测度. 定理 2.17 指明了这一点. 定理 2.18 说明，要是我们把假设更特殊化一些，一切正则性的问题都会消失干净.

2.16 定义 拓扑空间一个集 E 称为 σ-紧的，如果 E 是紧集的可数并.

测度空间(测度为 μ)的一个集 E，如果 E 是集 E_i 的可数并，而 $\mu(E_i)<\infty$，那末称 E 为有 σ-有限测度.

例如，在定理 2.14 所述的情况下，每个 σ-紧集有 σ-有限测度. 也容易看出，如果 $E\in\mathfrak{M}$，E 有 σ-有限测度，则 E 是内正则的.

2.17 定理 设 X 是局部紧，σ-紧的 Hausdorff 空间. 如果 $\mathfrak{M}$ 和 μ 都象定理 2.14 所叙述的那样，则 $\mathfrak{M}$ 和 μ 有下述的性质：

(*a*) 如果 $E\in\mathfrak{M}$ 和 $\varepsilon>0$，则存在闭集 F 和开集 V 使得 $F\subset E\subset V$ 和 $\mu(V-F)<\varepsilon$.

(*b*) μ 是 X 上的一个正则 Borel 测度.

(*c*) 如果 $E\in\mathfrak{M}$，则存在集 A 和 B 使得 A 是一个 F_σ 集，B 是一个 G_δ 集，$A\subset E\subset B$ 且 $\mu(B-A)=0$.

作为(*c*)的推论，我们看出每个 $E\in\mathfrak{M}$ 是一个 F_σ 集和一个测度为 0 的集的并.

证明 令 $X=K_1\cup K_2\cup K_3\cup\cdots$，其中 K_n 是紧的. 如果 $E\in\mathfrak{M}$ 和 $\varepsilon>0$，则 $\mu(K_n\cap E)<\infty$，并存在开集 $V_n\supset K_n\cap E$，使得

$$(1)\qquad \mu(V_n-(K_n\cap E))<\frac{\varepsilon}{2^{n+1}}\qquad (n=1,2,3,\cdots)$$

如果 $V=\bigcup V_n$，则 $V-E\subset\bigcup(V_n-(K_n\cap E))$，于是

$$\mu(V-E)<\frac{\varepsilon}{2}$$

用 E^c 代替 E，并把此结果用到 E^c 上：存在开集 $W\supset E^c$ 使得 $\mu(W-E^c)<\frac{\varepsilon}{2}$. 如果 $F=W^c$，则 $F\subset E$ 并且 $E-F=W-E^c$. 于是得到(*a*).

如果 F 是闭集，则 $F=\bigcup(K_n\cap F)$，每个 $K_n\cap F$ 是紧的，并且当 $n\to\infty$ 时，

$$\mu((K_1\cup\cdots\cup K_n)\cap F)\to\mu(F)$$

因此，由(*a*)得到(*b*).

如果对 $\varepsilon=1/j(j=1,2,3,\cdots)$ 应用(*a*)，就得到闭集 F_j 和开集 V_j，使得 $F_j\subset E\subset V_j$ 并且 $\mu(V_j-F_j)<1/j$. 令 $A=\bigcup F_j$ 和 B

$=\bigcap V_j$. 则 $A\subset E\subset B$, A 是一个 F_σ 集, B 是一个 G_δ 集, 并且由于 $B-A\subset V_j-F_j, n=1,2,3,\cdots$, 故 $\mu(B-A)=0$. 这就证明了(c).

2.18 定理 设 X 是局部紧的 Hausdorff 空间. 它的每个开集是 σ-紧的. 设 λ 是 X 上的任一个正 Borel 测度, 对每个紧集 K, 有 $\lambda(K)<\infty$. 则 λ 是正则的.

注意, 每个欧氏空间 R^k 满足现在的假设, 因为 R^k 中的每个开集是闭球的可数并.

证明 对 $f\in C_c(X)$, 令 $\Lambda f=\int_X f d\lambda$. 因为对每个紧集 K, $\lambda(K)<\infty$, Λ 是 $C_c(X)$ 上的正线性泛函, 并且存在测度 μ 满足定理 2.17 的结论, 于是

$$\int_X f d\lambda=\int_X f d\mu \qquad (f\in C_c(X)) \tag{1}$$

设 V 是 X 中的开集. 则 $V=\bigcup H_i$, 而 H_i, $i=1,2,3,\cdots$ 是紧的. 选择 f_1 使得 $H_1\prec f_1\prec V$. 选择好 $f_1,\cdots,f_n$ 以后, 设它们在 V 中有支集 $K_1,\cdots,K_n$, 再选择 f_{n+1} 使得

$$H_1\cup\cdots\cup H_n\cup K_1\cup\cdots\cup K_n\prec f_{n+1}\prec V \tag{2}$$

在 X 的每点, 序列 $\{f_n\}$ 单调递增到 χ_V. 因此, (1)蕴涵着

$$\lambda(V)=\lim_{n\to\infty}\int_X f_n d\lambda=\lim_{n\to\infty}\int_X f_n d\mu=\mu(V) \tag{3}$$

设 E 是 X 中的 Borel 集, 并取 $\varepsilon>0$. 由于 μ 满足定理 2.17, 存在闭集 F 和开集 V 使得 $F\subset E\subset V$ 和 $\mu(V-F)<\varepsilon$. 但 $V-F$ 是开集. 因此(3)表明 $\lambda(V-F)<\varepsilon$, 这就如在定理 2.17 中一样证明了 λ 的正则性.

注: 对 X 中的每个 Borel 集 E, 容易得到 $\lambda(E)=\mu(E)$.

习题 18 中举出了一个紧 Hausdorff 空间, 它包含了一个不是 σ-紧的开集, 并使上述定理不真.

Lebesgue 测度

2.19 欧氏空间 k 维欧氏空间 R^k 是一切点 $x=(\xi_1, \xi_2, \cdots, \xi_k)$ 的集，它的坐标 ξ_i 是实数，并有下述代数和拓扑结构：

如果 $x=(\xi_1, \cdots, \xi_k)$，$y=(\eta_1, \cdots, \eta_k)$，$\alpha$ 为实数，定义 $x+y$ 和 αx 如下：

(1) $$x+y=(\xi_1+\eta_1, \cdots, \xi_k+\eta_k), \alpha x=(\alpha\xi_1, \cdots, \alpha\xi_k)$$

这使 R^k 成为一个实向量空间. 如果 $x\cdot y=\Sigma\xi_i\eta_i$ 和 $|x|=(x\cdot x)^{\frac{1}{2}}$，则由 Schwarz 不等式 $|x\cdot y|\leqslant|x||y|$ 导出三角不等式

(2) $$|x-y|\leqslant|x-z|+|z-y|$$

因此，得到一个由 $\rho(x, y)=|x-y|$ 定义的度量. 我们假定这些事实是熟悉的，并将在第四章中以更一般的形式加以证明.

如果 $E\subset R^k$ 和 $x\in R^k$，则 E 对于 x 的平移是集

(3) $$E+x=\{y+x: y\in E\}$$

我们称形如

(4) $$W=\{x: \alpha_i<\xi_i<\beta_i, 1\leqslant i\leqslant k\}$$

的集或者用“$\leqslant$”代替任一个或所有的“$<$”而得的集为 k-胞腔；它的体积定义为

(5) $$\mathrm{vol}(W)=\prod_{i=1}^{k}(\beta_i-\alpha_i)$$

如果 $a\in R^k$ 和 $\delta>0$，我们称集

(6) $$Q(a;\delta)=\{x: \alpha_i\leqslant\xi_i<\alpha_i+\delta, 1\leqslant i\leqslant k\}$$

为以 a 为顶点的 δ-单元. 这里 $a=(\alpha_1, \cdots, \alpha_k)$.

对 $n=1, 2, 3, \cdots$，我们令 P_n 是一切坐标是 2^{-n} 的整数倍的 $x\in R^k$ 的集，令 Ω_n 是全体以 P_n 的点为顶点的 2^{-n}-单元的集族. 我们需要 $\{\Omega_n\}$ 的下述四个性质. 第一到第三是显而易见的.

(*a*) 如果固定 n，则每个 $x\in R^k$ 属于且仅属于 Ω_n 的一个元素.

(*b*) 如果 $Q'\in\Omega_n$，$Q''\in\Omega_r$，而 $r<n$，则 $Q'\subset Q''$ 或者 $Q'\cap Q''=\varnothing$.

(*c*) 如果 $Q\in\Omega_r$，则 $\mathrm{vol}(Q)=2^{-rk}$；如果 $n>r$，则集 P_n 恰好有 $2^{(n-r)k}$ 个点属于 Q.

(*d*) R^k 中每个非空开集是属于 $\Omega_1\cup\Omega_2\cup\Omega_3\cup\cdots$ 中不相交单元的可数并.

(*d*) 的证明 如果 V 是开集，每个 $x\in V$ 属于含于 V 内的一个开球；因此，存在属于某个 Ω_n 的 Q，使 $x\in Q\subset V$. 换句话说，V 是所有那些含于 V 内且属于某个 Ω_n 的单元之并. 从此单元的集族中，选出那些属于 Ω_1 的，并从 Ω_2，Ω_3，Ω_4，$\cdots$ 中去掉包含在任何一个已选单元之内的那些单元. 从余下的集族中，选出属于 Ω_2 且含于 V 的单元，并在 $\Omega_3,\Omega_4,\Omega_5,\cdots$ 中去掉包含在已选单元之内的单元. 如果我们按这个方法继续进行下去，由(*a*)和(*b*)就证明了(*d*).

2.20 定理 在 R^k 中，存在定义在 σ-代数 $\mathfrak{M}$ 上的正完备测度 m，具有下列性质：

(*a*) $m(W)=\mathrm{vol}(W)$，对每个 k-胞腔成立.

(*b*) $\mathfrak{M}$ 包含 R^k 中的一切 Borel 集；更确切些，$E\in\mathfrak{M}$ 当且仅当存在集 $A\subset R^k$ 和 $B\subset R^k$，使得 $A\subset E\subset B$，A 是一个 F_σ 集，B 是一个 G_σ 集，并且 $m(B-A)=0$. m 也是正则的.

(*c*) m 是平移不变的，即是说，对每个 $E\in\mathfrak{M}$ 和每个 $x\in R^k$ 成立.

$$m(E+x)=m(E)$$

(*d*) 如果 μ 是 R^k 上任意一个正的平移不变 Borel 测度，使得对每个紧集 K 有 $\mu(K)<\infty$，则存在常数 c，使得对一切 Borel 集 $E\subset R^k$，$\mu(E)=cm(E)$.

$\mathfrak{M}$ 的元素是 R^k 中的 Lebesgue 可测集；m 是 Lebesgue 测度. 当需要明确这一点时，我们将记为 m_k，用以代替 m. 关于 R^k 上其他测度的描述，请参看定理 8.14.

证明 设 f 是 R^k 上任一个复函数，具有紧的支集，定义

$$\Lambda_n f = 2^{-nk}\sum_{x\in P_n} f(x) \qquad (n=1,2,3,\cdots) \tag{1}$$

这里的 P_n 与 2.19 节中的一样.

现设 $f\in C_c(R^k)$，f 是实的. W 是一个开的 k-胞腔，它包含 f 的支集，并且 $\varepsilon>0$. f 的一致连续性([26]，定理 4.19)表明存在一个整数 N 及支集位于 W 内的函数 g 和 h，使得：(i) g 和 h 在属于 Ω_N 的每个单元上是常数，(ii) $g\leqslant f\leqslant h$，(iii) $h-g<\varepsilon$. 如果 $n>N$，性质 2.19(c)表明

$$\Lambda_N g=\Lambda_n g\leqslant\Lambda_n f\leqslant\Lambda_n h=\Lambda_N h \tag{2}$$

因此，$\{\Lambda_n f\}$ 的上、下极限至多相差 $\varepsilon\mathrm{vol}(W)$，并由于 ε 的任意性，我们证明了极限

$$\Lambda f=\lim_{n\to\infty}\Lambda_n f \qquad (f\in C_c(R^k)) \tag{3}$$

的存在性.

可以直接得出 Λ 是 $C_c(R^k)$ 上的正线性泛函. (事实上，Λf 正好是 R^k 上的 Riemann 积分. 为了不致于依赖多元 Riemann 积分的任何一个定理，我们才讨论上述的构造)我们定义 m 和 $\mathfrak{M}$ 为定理 2.14 中与 Λ 相对应的测度和 σ-代数.

由于定理 2.14 给出一个完备测度而 R^k 是 σ-紧的，由定理 2.17 即推出定理 2.20 的断言(b).

为证明(a)，令 W 是 2.19(4)的开胞腔，E_r 是属于 Ω_r 而闭包含于 W 的单元的并，并取 f 使得 $\overline{E}_r\prec f\prec W$. 于是 Λf 的构造表明

(4) $$\Lambda f \geqslant \prod_{i=1}^{k} (\beta_i - \alpha_i - 2^{1-r})$$

令 $r \to \infty$，并回顾定理 2.14 的构造，有

(5) $$m(W) = \sup\{\Lambda f : f \to W\}$$

因此，对每个开胞腔 W，$m(W) = \mathrm{vol}(W)$，又由于每个胞腔是开胞腔递减序列的交，由此得到(a).

由于 $\mathrm{vol}(W+x) = \mathrm{vol}(W)$，由此推出

(6) $$m(E+x) = m(E) \qquad (x \in R^k)$$

对每个胞腔 E 成立；特别，对每个单元 E，(6)成立；因此，由性质 2.19(d)推出，对每个开集 E，(6)成立；这样一来，由于

$$m(E) = \inf\{m(V) : E \subset V, V \text{ 是开集}\}$$

(6)对每个 $E \in \mathfrak{M}$ 也成立. 这就证明了(c).

最后，设 μ 是关于 R^k 上的平移不变的 Borel 测度. 令 $c = \mu(Q_0)$，这里 Q_0 是 1-单元. 由于 Q_0 是 2^{nk} 个不相交的 2^{-n}-单元的并，而这些单元彼此的每一个可由另一个平移得出，又由于 $m(Q_0) = 1$，所以对每个 2^{-n}-单元 Q 有

(7) $$2^{nk}\mu(Q) = \mu(Q_0) = cm(Q_0) = 2^{nk}cm(Q)$$

由性质 2.19(d)推知，对每个开集 E 有 $\mu(E) = cm(E)$. m 和 μ 的正则性表明(定理 2.18)，上面的那个方程对每个 Borel 集 E 都是成立的. 这就完成了证明.

2.21 评注 如果 m 是 R^k 上的 Lebesgue 测度，习惯上用记号 $L^1(R^k)$ 来代替 $L^1(m)$. 如果 E 是 R^k 的 Lebesgue 可测子集，并把 m 限制在 E 的可测子集上，显然就得到一个新的测度空间. 术语“在 E 上，$f \in L^1$”或“$f \in L^1(E)$”是用来指明 f 在这个测度空间上是可积的.

如果 $k=1$，I 是集 (a,b)，$(a,b]$，$[a,b)$，$[a,b]$ 中任一个，如果 $f \in L^1(I)$，习惯上用记号

$$\int_a^b f(x)dx \text{ 来代替 } \int_I f dm$$

由于任意单个点的 Lebesgue 测度为 0，这就使在上述四个集上进行积分都没有什么差别.

如果 f 是$[a,b]$上的连续复函数，则 f 在$[a,b]$上的 Riemann 积分和 Lebesgue 积分一致. 当 $f(a)=0, f(b)=0$，并且对 $x<a$ 和 $x>b$，定义 $f(x)$是 0 时，这是很明显的. 推广到一般的情形也没有什么困难. 实际上，对$[a,b]$上每个 Riemann 可积的 f，同一事实依然成立. 由于以后没有机会讨论 Riemann 可积函数，我们省略了证明，请参阅[26]的定理 10.33.

至此，某些读者可能会产生两个自然的问题：每个 Lebesgue 可测集是 Borel 集吗？ R^k 的每个子集是 Lebesgue 可测吗？ 纵然 $k=1$，这两种情形的答案都是否定的.

通过一种势的推理能够解决第一个问题. 我们简述如下. 令 c 是连续统(实直线，或者等价地说，整数的一切子集族)的势. 我们知道，R^k 有可数基(中心在 R^k 的某个可数稠密子集，带有有理数半径的开球)，$\mathscr{B}_k$(R^k 的一切 Borel 集的集族)是由这个基生成的 σ-代数. 由此得到(我们略去证明)，$\mathscr{B}_k$ 有势 c. 另一方面，存在 Cantor 集 $E\subset R^1$，而 $m(E)=0$(习题 5). m 的完备性蕴涵着 E 的 2^c 个子集中的每一个是 Lebesgue 可测的. 由于 $2^c>c$，大部分 E 的子集不是 Borel 集.

第二个问题的否定答案是选择公理的一个推论. 下述例子将指明这点.

2.22 例 对实数 x 和 y，当且仅当 $x-y$ 是有理数的时候，记为 $x\sim y$. 显然，$x\sim x$，若 $x\sim y$ 则 $y\sim x$，并且 $x\sim y$，$y\sim z$ 则 $x\sim z$. 因此"$\sim$"是个等价关系(用代数术语来说，令 Q 是有理数加法群，每个等价类是 Q 在 R^1 中的一个陪集). 令 E 是(0,1)内的一

个集，它刚好含每个等价类的一个点(这样一个集E的存在是选择公理的直接应用的结果)．我们断言E不是Lebesgue可测的．

象在2.19节中一样，令$E+r=\{x+r: x\in E\}$．我们需要E的两个性质：

(a) 如果$x\in(0,1)$，则对某个有理数$r\in(-1,1)$，有$x\in E+r$．

(b) 如果r和s是不同的有理数，则$(E+r)\cap(E+s)=\varnothing$．

证明(a)，注意，每个$x\in(0,1)$对应有一个$y\in E$，使得$x\sim y$．如果$r=x-y$，则$x=y+r\in E+r$．

证明(b)，设$x\in(E+r)\cap(E+s)$．则对某个$y\in E$，$z\in E$，$x=y+r=z+s$．由于$y-z=s-r\neq 0$，得出$y\sim z$，从而E包含两个等价的点．而这是与E的选择相矛盾的．

现在，设E是Lebesgue可测的，并且令$\alpha=m(E)$．规定$S=\bigcup(E+r)$是这样的并，这里r取遍$(-1,1)$中的一切有理数．由(b)，集$E+r$是两两不相交的；由于m是平移不变的，对每个r，$m(E+r)=\alpha$．由于$S\subset(-1,2)$，$m(S)\leqslant 3$．于是，由m的可数可加性，不得不有$\alpha=0$．因此，$m(S)=0$．但是(a)可推出$(0,1)\subset S$，故$1\leqslant m(S)$．这就得到矛盾．

可测函数的连续性

由于连续函数在Borel测度、特别在Lebesgue测度的结构中占有如此突出的地位，似乎有理由相信在连续函数和可测函数之间会存在着某些有趣的联系．本节将给出两个这种类型的定理．

在这两个定理中，我们将设μ是局部紧Hausdorff空间X上的一个测度，它有定理2.14中所述及的性质．特别，μ可以是某个R^k上的Lebesgue测度．

2.23 Lusin定理 设f是X上的复可测函数，$\mu(A)<\infty$，

如果 $x\notin A, f(x)=0$，并且 $\varepsilon>0$. 则存在一个 $g\in C_c(X)$，使得

$$\mu(\{x: f(x)\neq g(x)\})<\varepsilon \tag{1}$$

并且，还可以做到

$$\sup_{x\in X}|g(x)|\leqslant\sup_{x\in X}|f(x)| \tag{2}$$

证明 首先设 $0\leqslant f<1$，并且 A 是紧的. 如定理 1.17 中的证明一样，作出一个收敛于 f 的序列 $\{s_n\}$. 并且令 $t_1=s_1$ 和对 $n=2,3,4,\cdots$, $t_n=s_n-s_{n-1}$. 则 $2^n t_n$ 是集 $T_n\subset A$ 的特征函数，而且

$$f(x)=\sum_{n=1}^{\infty}t_n(x)\qquad(x\in X) \tag{3}$$

固定开集 V 使得 $A\subset V$，而且 $\overline{V}$ 是紧的. 则存在紧集 K_n 和开集 V_n 使得 $K_n\subset T_n\subset V_n\subset V$ 且 $\mu(V_n-K_n)<2^{-n}\varepsilon$. 由 Urysohn 引理，存在函数 h_n 使得 $K_n\prec h_n\prec V_n$. 定义

$$g(x)=\sum_{n=1}^{\infty}2^{-n}h_n(x)\qquad(x\in X) \tag{4}$$

这个级数在 X 上一致收敛，因而 g 是连续的. 并且，g 的支集含于 $\overline{V}$ 中. 由于除开在 V_n-K_n 中以外，$2^{-n}h_n(x)=t_n(x)$，所以除开在 $\bigcup(V_n-K_n)$ 中以外，$g(x)=f(x)$，并且后一个集有比 ε 小的测度. 这样，当 A 是紧的，$0\leqslant f<1$ 时，(1)式成立.

由此得到，当 A 是紧的，f 是有界可测函数时，(1)式成立. A 的紧性容易去掉，因为，如果 $\mu(A)<\infty$，则 A 包含一个紧集 K，使 $\mu(A-K)$ 小于事先给定的任意正数. 再者，如果 f 是个复可测函数，$B_n=\{x: |f(x)|>n\}$，则 $\bigcap B_n=\varnothing$，于是由定理 1.19(e)，$\mu(B_n)\to 0$. 由于除开在 B_n 上以外，f 同有界函数 $(1-\chi_{B_n})\cdot f$ 一致，故在一般情形下(1)也成立.

最后，令 $R=\sup\{|f(x)|: x\in X\}$，并定义，若 $|z|\leqslant R$，$\varphi(z)=z$，若 $|z|>R$，$\varphi(z)=Rz/|z|$. 则 φ 是复平面到半径为 R 的圆盘上

的连续映射. 如果 g 满足(1), $g_1=\varphi\circ g$, 则 g_1 满足(1)和(2).

系 设满足 Lusin 定理的条件, 且 $|f|\leqslant 1$. 则存在序列 $\{g_n\}$, 使得 $g_n\in C_c(X)$, $|g_n|\leqslant 1$, 且

(5) $$f(x)=\lim_{n\to\infty} g_n(x) \quad \text{a. e.}$$

证明 由定理得出, 对每个 n 对应有一个 $g_n\in C_c(X)$, 而 $|g_n|\leqslant 1$, 使得 $\mu(E_n)\leqslant 2^{-n}$. 这里的 E_n 是一切使 $f(x)\neq g_n(x)$ 的 x 的集. 对几乎每个 x, 它至多属于有限多个集 E_n (定理 1.41). 对任意一个这样的 x 和充分大的 n 都有 $f(x)=g_n(x)$. 这就得到(5).

2.24 Vitali-Carathéodory定理 设 $f\in L^1(\mu)$, f 是实值的, 并且 $\varepsilon>0$. 则在 X 上存在函数 u 和 v 使得 $u\leqslant f\leqslant v$, u 是上半连续且有上界的, v 是下半连续且有下界的, 且

(1) $$\int_X (v-u)d\mu<\varepsilon$$

证明 首先假定 $f\geqslant 0$, 并且 f 不恒等于 0. 由于 f 是一个简单函数 s_n 的递增序列的点态极限, f 就是简单函数 $t_n=s_n-s_{n-1}$ (取 $s_0=0$)的和. 再由于 t_n 是特征函数的线性组合, 就存在可测集 E_i (不一定要互不相交)和常数 $c_i>0$, 使得

(2) $$f(x)=\sum_{i=1}^{\infty} c_i\chi_{E_i}(x) \qquad (x\in X)$$

由于

(3) $$\int_X f d\mu=\sum_{i=1}^{\infty} c_i\mu(E_i)$$

级数(3)是收敛的. 存在紧集 K_i 和开集 V_i 使得 $K_i\subset E_i\subset V_i$, 且

(4) $$c_i\mu(V_i-K_i)<2^{-i-1}\varepsilon \quad (i=1,2,3,\cdots)$$

令

(5) $$v=\sum_{i=1}^{\infty} c_i\chi_{V_i},\ u=\sum_{i=1}^{N} c_i\chi_{K_i}$$

N是这样选择的，使得

$$\sum_{N+1}^{\infty} c_i \mu(E_i) < \frac{\varepsilon}{2} \tag{6}$$

因此，v 是下半连续的，u 是上半连续的，$u \leqslant f \leqslant v$，并且

$$v-u = \sum_{i=1}^{N} c_i(\chi_{V_i} - \chi_{K_i}) + \sum_{N+1}^{\infty} c_i \chi_{V_i}$$

$$\leqslant \sum_{i=1}^{\infty} c_i(\chi_{V_i} - \chi_{K_i}) + \sum_{N+1}^{\infty} c_i \chi_{E_i}$$

于是由(4)和(6)得到(1).

在一般情况下，记 $f = f^+ - f^-$，如上一样，对 f^+ 作出 u_1 和 v_1，对 f^- 作出 u_2 和 v_2，并令 $u = u_1 - v_2$，$v = v_1 - u_2$. 由于 $-v_2$ 是上半连续的，而两个上半连续函数之和是上半连续的(对下半连续也有类似的情况；我们把证明留作习题)，故 u 和 v 有所要求的性质.

习　题

1　令 $\{f_n\}$ 是 R^1 上的非负实函数的序列，考虑下述四个命题：

(a) 如果 f_1 和 f_2 是上半连续的，则 f_1+f_2 是上半连续的.

(b) 如果 f_1 和 f_2 是下半连续的，则 f_1+f_2 是下半连续的.

(c) 如果每个 f_n 是上半连续的，则 $\sum_1^{\infty} f_n$ 是上半连续的.

(d) 如果每个 f_n 是下半连续的，则 $\sum_1^{\infty} f_n$ 是下半连续的.

证明其中的三个是正确的，但有一个是错误的. 如果省去"非负"一词会产生什么结果？如果用一般的拓扑空间代替 R^1 会影响命题的真实性吗？

2　设 f 是 R^1 上的任意复函数，规定

$$\varphi(x,\delta) = \sup\{|f(s)-f(t)|: s, t \in (x-\delta, x+\delta)\}$$

$$\varphi(x) = \inf\{\varphi(x,\delta): \delta > 0\}$$

证明 φ 是上半连续的，并且当且仅当 $\varphi(x)=0$ 时 f 在点 x 处连续，因此任何复函数连续点的集都是一个 G_δ 集.

用一般拓扑空间代替 R^1 后，表述并证明类似的命题.

3　设 X 是一个度量空间，其度量为 ρ. 对任意非空集 $E\subset X$，规定

$$\rho_E(x)=\inf\{\rho(x,y):y\in E\}$$

证明 ρ_E 是 X 上的一致连续函数. 如果 A 和 B 是 X 的不相交非空闭子集，检验函数

$$f(x)=\frac{\rho_A(x)}{\rho_A(x)+\rho_B(x)}$$

是否满足 Urysohn 引理.

4　检验 Riesz 定理的证明，并证明下述两个命题：

(*a*) 如果 $E_1\subset V_1$ 和 $E_2\subset V_2$，而 V_1、V_2 是不相交的开集，则 $\mu(E_1\cup E_2)=\mu(E_1)+\mu(E_2)$ 即使 E_1 和 E_2 不属于 $\mathfrak{M}$ 时也成立.

(*b*) 如果 $E\in\mathfrak{M}_F$，则 $E=N\cup K_1\cup K_2\cup\cdots$，其中 $\{K_i\}$ 是不相交的可数紧集族，并且 $\mu(N)=0$.

在习题 5 到 8 中，m 表示 R^1 上的 Lebesgue 测度.

5　令 E 是熟知的 Cantor 三分 (middle thirds) 集. 证明 $m(E)=0$，即使 E 和 R^1 有相同的势.

6　构造一个全不连通的紧集 $K\subset R^1$，使得 $m(K)>0$（K 没有多于一点的连通子集）.

如果 V 是下半连续的，$v\leqslant\chi_K$，证明确实有 $v\leqslant 0$. 因此 χ_K 不能在 Vitali-Carathéodory 定理的意义下，用下半连续函数从下面逼近.

7　给定 $\varepsilon>0$，构造一个开集 $E\subset[0,1]$，它在 $[0,1]$ 中是稠密的，使得 $m(E)=\varepsilon$（A 在 B 中稠密意思是说 A 的闭包包含 B）.

8　构造一个 Borel 集 $E\subset R^1$，使得对每个非空开区间 I 有

$$0<m(E\cap I)<m(I)$$

对这样一个集 E，有 $m(E)<\infty$ 的可能吗？

9　在 $[0,1]$ 上构造一个连续函数序列 f_n，使得 $0\leqslant f_n\leqslant 1$，且

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx=0$$

然而，却没有一个 $x\in[0,1]$ 能使序列 $\{f_n(x)\}$ 收敛.

10　如果 $\{f_n\}$ 是 $[0,1]$ 上连续函数序列，使得 $0\leqslant f_n\leqslant 1$，且当 $n\to\infty$ 时，对每个 $x\in[0,1]$，$f_n(x)\to 0$，则

$$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx=0$$

试一试，不用任何测度理论和有关 Lebesgue 积分的定理来证明它．（这是为了使你对 Lebesgue 积分的力量有个印象．W. F. Eberlein 在 1957年 Communications on Pure and Applied Mathematics vol. X, pp. 357–360 中给出了一个漂亮的证明．）

11 设 μ 是紧 Hausdorff 空间 X 上的一个正则 Borel 测度；假定 $\mu(X)=1$．证明存在一个紧集 $K\subset X$（μ 的承载集或支集）使得 $\mu(K)=1$．但对 K 的每个紧的真子集 H 有 $\mu(H)<1$．

提示：令 K 是一切紧集 K_α 的交，而 $\mu(K_\alpha)=1$；证明每个包含 K 的开集 V 也包含某个 K_α．μ 的正则性是需要的；比较习题 18．证明 K^c 是 X 中测度为 0 的最大开集．

12 证明 R^1 的每个紧子集是一个 Borel 测度的支集．

13 R^1 的每个紧子集是一个连续函数的支集，对吗？如果不对，你能把 R^1 中是连续函数支集的一切紧集的类刻划出来吗？在其他拓扑空间你的刻划也正确吗？

14 如果 f 是 R^1 上 Lebesgue 可测复函数，证明在 R^1 上存在一个 Borel 函数 g，使得 $f=g$ a. e. $[m]$．

15 当 $n\to\infty$ 时，很容易推测出

$$\int_0^n\left(1-\frac{x}{n}\right)^n e^{x/2}dx \quad 和 \quad \int_0^n\left(1+\frac{x}{n}\right)^n e^{-2x}dx$$

的极限．证明你的推测是正确的．

16 设 m 是关于 R^k 的 Lebesgue 测度，证明 $m(-E)=m(E)$，这里

$$-E=\{-x: x\in E\}$$

因此，对一切 $f\in L^1(R^k)$ 有

$$\int_{R^k} f(x)dx=\int_{R^k} f(-x)dx$$

17 设 X 是带下述拓扑的平面：一个集是开集当且仅当它与每条铅垂线的交，对 R^1 的通常拓扑而言是该线的开子集．证明 X 是局部紧的 Hausdorff 空间．如果 $f\in C_c(X)$，令 x_1，…，x_n 是 x 的这样的值，对至少一个 y，$f(x,y)\neq 0$（这样的 x 仅有有限多个！），并定义

$$Af=\sum_{j=1}^{n}\int_{-\infty}^{+\infty} f(x_j,y)dy$$

根据定理 2.14，令 μ 是与 A 对应的测度．如果 E 是 x 轴，证明即使对每个紧

集 $K\subset E$, $\mu(K)=0$, 仍然有 $\mu(E)=\infty$.

18 此题较前面任一题都需要较多的集论技巧. 令 X 是良序的不可数集, 有最后元素 ω_1, 使得 ω_1 的每个先行元素至多有可数多个先行元素("构造": 取任何良序集, 使它有元素, 其先行元素是不可数的, 令 ω_1 是这些元素的第一个; 称 ω_1 为第一个不可数序数). 对 $\alpha\in X$, 令 $P_\alpha[S_\alpha]$ 是 α 的一切先行元素(后继元素)的集. 如果 X 的一个子集是 P_α 或 S_β 或 $P_\alpha\cap S_\beta$, 或这种子集的并, 则称它为开集. 证明 X 因此是一个紧 Hausdorff 空间 (提示: 没有一个良序集含有一个无限递减序列).

证明点 ω_1 的余集是一个开集, 但不是 σ-紧集.

证明对每个 $f\in C(X)$ 对应一个 $\alpha\neq\omega_1$, 使得 f 在 S_α 上是常数.

证明 X 的不可数紧子集的每个可数集族 $\{K_n\}$ 的交是不可数的(提示: 在 X 内考虑递增可数序列的极限, 它与每个 K_n 相交于无限多点).

令 $\mathfrak{M}$ 是一切 $E\subset X$ 的集族, 使得或者 $E\cup\{\omega_1\}$ 或者 $E^c\cup\{\omega_1\}$ 包含一个不可数紧集; 在第一种情况下, 定义 $\lambda(E)=1$; 在第二种情况下, 定义 $\lambda(E)=0$. 证明 $\mathfrak{M}$ 是一个包含 X 内全体 Borel 集的 σ-代数, λ 是 $\mathfrak{M}$ 上的测度, 但不是正则的(ω_1 的每个邻域有测度 1), 而且对每个 $f\in C(X)$

$$f(\omega_1)=\int_X f\,d\lambda$$

试刻划定理 2.14 中与这个线性泛函对应的正则的 μ.

19 如果 μ 是任意一个正测度并且 $f\in L^1(\mu)$, 证明 $\{x:f(x)=0\}$ 有 σ-有限测度.

20 如果 X 是集 X_t 的可数并, 而 $\mu(X_t)<\infty$, 则称集 X 上的正测度 μ 为 σ-有限. 证明当且仅当存在 $f\in L^1(\mu)$ 使得对每个 $x\in X$ 有 $f(x)>0$ 时, μ 为 σ-有限.

第三章　L^p-空　间

凸函数和不等式

分析中许多最普通的不等式都有其凸性概念方面的起源.

3.1　定义　定义在开区间(a,b)上的实函数φ，这里$-\infty\leqslant a<b\leqslant\infty$，如果不等式

(1)　　$$\varphi((1-\lambda)x+\lambda y)\leqslant(1-\lambda)\varphi(x)+\lambda\varphi(y)$$

当$a<x<b, a<y<b$，和$0\leqslant\lambda\leqslant 1$时成立，$\varphi$就称为凸的.

从图形上看，条件就是"当$x<t<y$时，点$(t,\varphi(t))$应该在平面内的两点$(x,\varphi(x))$和$(y,\varphi(y))$的联线上或者在它的下方."同时(1)等价于要求

(2)　　$$\frac{\varphi(t)-\varphi(s)}{t-s}\leqslant\frac{\varphi(u)-\varphi(t)}{u-t}$$

对$a<s<t<u<b$成立.

微分中值定理和(2)一起可直接证明"实的可微函数φ是在(a,b)上凸的充要条件是，当$a<s<t<b$时，$\varphi'(s)\leqslant\varphi'(t)$"，即当且仅当导函数$\varphi'$是单调递增函数.

例如，指数函数在$(-\infty,\infty)$上就是凸的.

3.2　定理　如果φ是在(a,b)上凸的，则φ在(a,b)上连续.

证明　证明的思想很容易用几何的语言表达. 担心不"严格"的人们请把它转译为ε和δ的说法.

假定$a<s<x<y<t<b$. 把平面内的点$(s,\varphi(s))$记为S，类似地处理x,y,t. 则X在SY直线上或它的下方，Y在过S和X的直线上或它的上方；同时Y在XT上或它的下方. 当$y\to x$时，

得出 $Y\to X$，即 $\varphi(y)\to\varphi(x)$. 用同样的方法处理左极限便得出 φ 的连续性.

注意，这个定理依赖于我们的讨论是在开区间上进行的这一事实，比如说，若在$[0,1)$上 $\varphi(x)=0$ 且 $\varphi(1)=1$，则 φ 在$[0,1]$上不是连续的，但它满足 3.1(1).

3.3 定理 (Jensen 不等式) 设 μ 是在集 Ω 内的 σ-代数 $\mathfrak{M}$ 上的正测度，使得 $\mu(\Omega)=1$. 如果 f 是在 $L^1(\mu)$ 内的实函数，对所有的 $x\in\Omega$，$a<f(x)<b$，且 φ 是在 (a,b) 上凸的，则

$$\varphi\left(\int_\Omega f d\mu\right)\leqslant\int_\Omega(\varphi\circ f)d\mu \tag{1}$$

注：不排除 $a=-\infty$ 和 $b=\infty$ 的情况.

证明 令 $t=\int_\Omega f d\mu$，则 $a<t<b$. 若 β 是在 3.1(2) 左边的商的上确界，这里 $a<s<t$，则对任一 $u\in(t,b)$，β 不大于在 3.1(2) 右边的商. 由此得出

$$\varphi(s)\geqslant\varphi(t)+\beta(s-t)\qquad(a<s<b) \tag{2}$$

因此

$$\varphi(f(x))-\varphi(t)-\beta(f(x)-t)\geqslant 0 \tag{3}$$

对每一个 $x\in\Omega$ 成立. 因为 φ 是连续的，$\varphi\circ f$ 便是可测的. 若在 (3)式的两边对 μ 积分，从我们关于 t 的选择和假定 $\mu(\Omega)=1$ 便得出(1)式.

为给出一个例子，取 $\varphi(x)=e^x$. 则(1)式变成

$$\exp\left\{\int_\Omega f d\mu\right\}\leqslant\int_\Omega e^f d\mu \tag{4}$$

如果 Ω 是一个有限集，比如说由点 $p_1,\cdots,p_n$ 组成，且若

$$\mu(\{p_i\})=\frac{1}{n},\ f(p_i)=x_i$$

对于实的 x_i，(4)变成

(5) $$\exp\left\{\frac{1}{n}(x_1+\cdots+x_n)\right\}\leqslant\frac{1}{n}(e^{x_1}+\cdots+e^{x_n})$$

令 $y_i=e^{x_i}$，获得 n 个正数的算术平均与几何平均之间的熟悉的不等式：

(6) $$(y_1y_2\cdots y_n)^{1/n}\leqslant\frac{1}{n}(y_1+\cdots+y_n)$$

从这里回到(4)式，那末

(7) $$\exp\left\{\int_\Omega \log g d\mu\right\}\leqslant\int_\Omega g d\mu$$

的左边和右边为什么经常地被分别称为正函数 g 的几何平均值和算术平均值应该变得清楚了.

如果取 $\mu(\{p_i\})=\alpha_i>0$，这里 $\Sigma\alpha_i=1$. 则代替(6)式，我们得到

(8) $$y_1^{\alpha_1}y_2^{\alpha_2}\cdots y_n^{\alpha_n}\leqslant\alpha_1y_1+\alpha_2y_2+\cdots+\alpha_ny_n$$

这些恰好是为定理 3.3 中所包含的少数几个范例.

其逆命题，见习题 20.

3.4 定义 如果 p 和 q 都是正实数，使得 $p+q=pq$，或等价地说

(1) $$\frac{1}{p}+\frac{1}{q}=1$$

则称 p 和 q 为一对共轭指数. 显然(1)蕴涵着 $1<p<\infty$ 和 $1<q<\infty$. 一个重要的特殊情况是 $p=q=2$.

当 $p\to1$ 时，(1) 迫使 $q\to\infty$. 因而 1 和∞也被看作一对共轭指数.

3.5 定理 设 p 和 q 是共轭指数，$1<p<\infty$. X 是一个具有测度 μ 的测度空间. 设 f 和 g 是 X 上的可测函数，其值域为 $[0,\infty]$. 则

(1) $$\int_X fgd\mu\leqslant\left\{\int_X f^pd\mu\right\}^{1/p}\left\{\int_X g^qd\mu\right\}^{1/q}$$

并且

(2) $$\left\{\int_X (f+g)^p d\mu\right\}^{1/p} \leqslant \left\{\int_X f^p d\mu\right\}^{1/p} + \left\{\int_X g^p d\mu\right\}^{1/p}$$

成立.

不等式(1)是 Hölder 不等式; (2)是 Minkowski 不等式. 若 $p=q=2$, (1)就是大家知道的 Schwarz 不等式.

证明 设 A 和 B 是(1)式右边的两个因子. 若 $A=0$, 则 $f=0$ a. e. (由定理 1.39); 因此 $fg=0$ a. e., 所以(1)成立. 若 $A>0$ 且 $B=\infty$, (1)也是平凡的. 所以只需要考虑 $0<A<\infty, 0<B<\infty$ 的情况. 令

(3) $$F=\frac{f}{A},\ G=\frac{g}{B}$$

这给出了

(4) $$\int_X F^p d\mu = \int_X G^q d\mu = 1$$

如果 $x\in X$ 是使得 $0<F(x)<\infty$ 和 $0<G(x)<\infty$ 的点, 则存在实数 s 和 t 使得 $F(x)=e^{s/p}$, $G(x)=e^{t/q}$. 因为 $1/p+1/q=1$, 指数函数的凸性蕴涵着

(5) $$e^{s/p+t/q} \leqslant p^{-1}e^s + q^{-1}e^t$$

由此得出

(6) $$F(x)G(x) \leqslant p^{-1}F(x)^p + q^{-1}G(x)^q$$

对每一个 $x\in X$ 成立. 根据(4), (6)式的积分产生

(7) $$\int_X FG d\mu \leqslant p^{-1}+q^{-1}=1$$

把(3)代入(7)式, 便得(1).

注意, (6)也能作为不等式 3.3(8)的特殊情况得出.

为了证明(2), 记

(8) $$(f+g)^p = f\cdot(f+g)^{p-1} + g\cdot(f+g)^{p-1}$$

Hölder 不等式给出

$$\int f\cdot(f+g)^{p-1}\leqslant\left\{\int f^p\right\}^{1/p}\left\{\int(f+g)^{(p-1)q}\right\}^{1/q} \tag{9}$$

设(9′)是在(9) 内交换 f 和 g 的位置而得的不等式. 因为 $(p-1)q=p$, (9)和(9′)相加给出

$$\int(f+g)^p\leqslant\left\{\int(f+g)^p\right\}^{1/q}\left[\left\{\int f^p\right\}^{1/p}+\left\{\int g^p\right\}^{1/p}\right] \tag{10}$$

显然,只要证明(2)式在左边大于 0 而右边小于∞的场合成立就足够了. 函数 t^p 在 $0<t<\infty$ 上的凸性表明

$$\left(\frac{f+g}{2}\right)^p\leqslant\frac{1}{2}(f^p+g^p)$$

因此(2)的左边小于∞, 这样, 假如用 (10) 右边的第一个因式除(10)的两边, 记住 $1-1/q=1/p$, 从(10)式就可得出(2). 于是完成了证明.

知道在不等式中等号能够成立的条件有时是有用的. 在许多情况下,这方面的信息可通过考察不等式的证明得到.

比如, 在(7)式中等号成立当且仅当(6)式的等号对几乎每一个 x 成立. 在(5)中, 当且仅当 $s=t$ 时等号成立. 因此, 若假定(4)成立, 则"$F^p=G^q$ a. e."是(7)式的等号成立的充要条件. 用原来函数 f 和 g 的话来说,则获得下列结论:

假定 $A<\infty$ 和 $B<\infty$, (1)的等号成立当且仅当存在两个不同时为 0 的常数 α 和 β, 使得 $\alpha f^p=\beta g^q$ a. e.

我们把(2)式等号成立的类似的讨论留作习题.

L^p-空间

在这一节里, X 是一个具有正测度 μ 的任意的测度空间.

3.6 定义 如果 $0<p<\infty$ 且 f 是 X 上的一个复可测函数, 定义

(1) $$\|f\|_p=\left\{\int_X|f|^p d\mu\right\}^{1/p}$$

且设 $L^p(\mu)$ 由所有满足

(2) $$\|f\|_p<\infty$$

的 f 组成. 称 $\|f\|_p$ 为 f 的 L^p-范数.

如果 μ 是 R^k 上的 Lebesque 测度，象在 2.21 节一样，记 $L^p(R^k)$ 以代替 $L^p(\mu)$. 若 μ 是集 A 上的计数测度，习惯上用 $l^p(A)$ 记对应的 L^p-空间. 当 A 可数时，便简记为 l^p. l^p 的一个元素可以看作一个复序列 $x=\{\xi_n\}$，并且

$$\|x\|_p=\left\{\sum_{n=1}^{\infty}|\xi_n|^p\right\}^{1/p}$$

3.7 定义 假定 g: $X\to[0,\infty]$ 是可测的. 设 S 是所有使得

(1) $$\mu(g^{-1}((\alpha,\infty]))=0$$

的实数 α 的集. 如果 $S=\varnothing$，令 $\beta=\infty$. 如果 $S\neq\varnothing$，令 $\beta=\inf S$. 因为

(2) $$g^{-1}((\beta,\infty])=\bigcup_{n=1}^{\infty}g^{-1}((\beta+\frac{1}{n},\infty])$$

并且因为 0 测度集的可数并有测度 0，可以看出 $\beta\in S$. 我们称 β 为 g 的本性上确界.

如果 f 是 X 上的复可测函数，定义 $\|f\|_\infty$ 为 $|f|$ 的本性上确界，并设 $L^\infty(\mu)$ 由所有满足 $\|f\|_\infty<\infty$ 的 f 所组成. $L^\infty(\mu)$ 的元素有时称为 X 上的本性有界可测函数.

从这个定义得出，当且仅当 $\lambda\geqslant\|f\|_\infty$ 时，不等式 $|f(x)|\leqslant\lambda$ 对几乎所有 x 成立.

象定义 3.6 一样，$L^\infty(R^k)$ 表示所有的在 R^k 上(关于 Lebesgue 测度)本性有界函数的类，$l^\infty(A)$ 是所有 A 上有界函数的类. (因为这里每一个非空集具有正测度，有界与本性有界的含义相同!)

3.8 定理 如果 p 和 q 是共轭指数，$1\leqslant p\leqslant\infty$，$f\in L^p(\mu)$ 和 $g\in L^q(\mu)$，则 $fg\in L^1(\mu)$，并且

(1) $$\|fg\|_1\leqslant\|f\|_p\|g\|_q$$

成立.

证明 对于 $1<p<\infty$，(1) 简单地说就是应用于 $|f|$ 和 $|g|$ 的 Hölder 不等式；$p=\infty$ 时，注意

(2) $$|f(x)g(x)|\leqslant\|f\|_\infty|g(x)|$$

对几乎所有 x 成立. 积分(2)式，即得(1)式. 若 $p=1$，则 $q=\infty$，可以采用同样的推理.

3.9 定理 假定 $1\leqslant p\leqslant\infty$，并且 $f\in L^p(\mu)$，$g\in L^p(\mu)$. 则 $f+g\in L^p(\mu)$，且

(1) $$\|f+g\|_p\leqslant\|f\|_p+\|g\|_p$$

成立.

证明 因为 $$\int_X|f+g|^p d\mu\leqslant\int_X(|f|+|g|)^p d\mu$$

对 $1\leqslant p\leqslant\infty$，从 Minkowski 不等式得出(1). 对 $p=1$ 或 $p=\infty$，(1)是不等式 $|f+g|\leqslant|f|+|g|$ 的平凡的推论.

3.10 评注 固定 p，$1\leqslant p\leqslant\infty$. 如果 $f\in L^p(\mu)$ 并且 α 是复数，显然 $\alpha f\in L^p(\mu)$. 事实上

(1) $$\|\alpha f\|_p=|\alpha|\,\|f\|_p$$

与定理 3.9 合起来说明 $L^p(\mu)$ 是一个复向量空间.

假定 f，g 和 h 都在 $L^p(\mu)$ 内，在定理 3.9 中用 $f-g$ 代替 f，用 $g-h$ 代替 g，得到

(2) $$\|f-h\|_p\leqslant\|f-g\|_p+\|g-h\|_p$$

这就启发我们，可以通过定义 f 和 g 之间的距离为 $\|f-g\|_p$，而在 $L^p(\mu)$ 内引进度量. 暂时称这个距离为 $d(f,g)$. 则 $0\leqslant d(f,g)<\infty$，$d(f,f)=0$，$d(g,f)=d(f,g)$，并且(2)式指出三角不等式

$d(f,h)\leqslant d(f,g)+d(g,h)$也成立. 确定一个度量空间的 d 所应该具备的最后一个性质是:$d(f,g)=0$ 应该蕴涵着 $f=g$. 在我们现在的情况下,这一点并非如此;确切地说,当 $f(x)=g(x)$ 对几乎所有的 x 成立时,就有 $d(f,g)=0$.

让我们记 $f\sim g$ 当且仅当 $d(f,g)=0$,显然这是一个在 $L^p(\mu)$ 内的等价关系,它把 $L^p(\mu)$分成各个等价类;每一类都由所有与给定的一个函数等价的那些函数组成. 如果 F 和 G 是两个等价类,选取 $f\in F$ 和 $g\in G$,定义 $d(F,G)=d(f,g)$;注意 $f\sim f_1$ 和 $g\sim g_1$ 蕴涵着

$$d(f,g)=d(f_1,g_1)$$

所以 $d(F,G)$是完全确定的.

在这个定义下,等价类的集现在是一个度量空间. 注意,因为 $f\sim f_1$ 和 $g\sim g_1$ 蕴涵着 $f+g\sim f_1+g_1$ 和 $\alpha f\sim\alpha f_1$,它也是一个向量空间.

当 $L^p(\mu)$被看作度量空间时,实际上所考虑的空间不是以函数为元素的空间,而是一个以函数的等价类为元素的空间. 不过为了语言的简洁起见,习惯上把这种差别降低到心照不宣的地位,并且继续说 $L^p(\mu)$是一个函数空间. 我们将遵循这种习惯.

如果$\{f_n\}$是 $L^p(\mu)$的一个序列,$f\in L^p(\mu)$,且$\lim\limits_{n\to\infty}\|f_n-f\|_p=0$,我们就说$\{f_n\}$在 $L^p(\mu)$内收敛于 f(或$\{f_n\}$ p 次平均收敛于 f,或$\{f_n\}$是 L^p-收敛于 f). 若对每一个 $\varepsilon>0$,都对应有一个整数 N,使得当 $n>N$、$m>N$ 时有 $\|f_m-f_n\|_p<\varepsilon$,我们就称$\{f_n\}$是 $L^p(\mu)$内的 Cauchy 序列. 这些定义完全象在任何一个度量空间中的定义一样.

一个极为重要的事实是,$L^p(\mu)$是一个完备的度量空间,即每一个 $L^p(\mu)$内的 Cauchy 序列收敛于 $L^p(\mu)$的一个元素.

3.11 定理 对于 $1\leqslant p\leqslant\infty$ 和每一个正测度 μ,$L^p(\mu)$是一个完备的度量空间.

证明 首先假定 $1\leqslant p<\infty$. 设 $\{f_n\}$ 是在 $L^p(\mu)$ 内的一个 Cauchy 序列, 则存在一个子序列 $\{f_{n_i}\}$, $n_1<n_2<\cdots$, 使得

(1) $$\|f_{n_{i+1}}-f_{n_i}\|<2^{-i} \qquad (i=1,2,3,\cdots)$$

令

(2) $$g_k=\sum_{i=1}^{k}|f_{n_{i+1}}-f_{n_i}|,\quad g=\sum_{i=1}^{\infty}|f_{n_{i+1}}-f_{n_i}|$$

因为(1)成立,Minkowski 不等式指出 $\|g_k\|_p<1, k=1,2,3,\cdots$. 因此将 Fatou 引理应用于 $\{g_k^p\}$, 得出 $\|g\|_p\leqslant 1$. 特别是 $g(x)<\infty$ a. e. 成立, 所以级数

(3) $$f_{n_1}(x)+\sum_{i=1}^{\infty}(f_{n_{i+1}}(x)-f_{n_i}(x))$$

对几乎所有的 $x\in X$ 是绝对收敛的. 对于在(3)式中收敛的那些点 x, 记(3)的和为 $f(x)$, 在余下的 0 测集上, 令 $f(x)=0$. 因为

(4) $$f_{n_1}+\sum_{i=1}^{k-1}(f_{n_{i+1}}-f_{n_i})=f_{n_k}$$

看出

(5) $$f(x)=\lim_{i\to\infty}f_{n_i}(x) \quad \text{a. e.}$$

在已经找到一个函数 f, 它是 $\{f_{n_i}\}$ a. e. 点态收敛的极限以后, 现在还得证明这个 f 是 $\{f_n\}$ 的 L^p-极限. 选取 $e>0$, 这时存在一个 N, 使得 $n>N$、$m>N$ 时有 $\|f_m-f_n\|_p<e$ 成立. 于是 Fatou 引理指出对每一个 $m>N$,

(6) $$\int_X|f-f_m|^p d\mu\leqslant\liminf_{i\to\infty}\int_X|f_{n_i}-f_m|^p d\mu\leqslant e^p$$

从(6)得出 $f-f_m\in L^p(\mu)$. 因此 $f\in L^p(\mu)$ [因为 $f=(f-f_m)+f_m$]. 最后, 当 $m\to\infty$ 时 $\|f-f_m\|_p\to 0$. 这就完成了对 $1\leqslant p<\infty$ 情况的证明.

在 $L^\infty(\mu)$ 中的证明容易得多. 假定 $\{f_n\}$ 是 $L^p(\mu)$ 内的Cauchy序列, 设 A_k 和 $B_{m,n}$ 是使 $|f_k(x)|>\|f_k\|_\infty$ 和 $|f_n(x)-f_m(x)|>\|f_n-f_m\|_\infty$ 的集, 并设 E 是这些 A_k 和 $B_{m,n}$ $k, m, n=1,2,3,\cdots$ 的并. 则 $\mu(E)=0$, 并且在 E 的余集上序列 $\{f_n\}$ 一致收敛于有界函数 f. 对 $x\in E$, 定义 $f(x)=0$. 则 $f\in L^\infty(\mu)$ 且当 $n\to\infty$ 时 $\|f_n-f\|_\infty\to 0$.

上面的证明包含了一个足以单独叙述的有趣的结果:

3.12 定理 如果 $1\leqslant p\leqslant\infty$, $\{f_n\}$ 是在 $L^p(\mu)$ 内的 Cauchy 序列, 它的极限为 f, 则 $\{f_n\}$ 存在一个子序列, 它几乎处处点态收敛于 $f(x)$.

简单函数在 $L^p(\mu)$ 内起着一个有趣的作用:

3.13 定理 设 S 是所有使得

$$\mu(\{x: s(x)\neq 0\})<\infty \tag{1}$$

的 X 上的复可测简单函数 s 的类. 如果 $1\leqslant p<\infty$, 则 S 在 $L^p(\mu)$ 内是稠密的.

证明 首先, $S\subset L^p(\mu)$ 是很明显的. 假定 $f\geqslant 0$, $f\in L^p(\mu)$, 并设 $\{s_n\}$ 是与在定理 1.17 内一样的序列. 因为 $0\leqslant s_n\leqslant f$, 有 $s_n\in L^p(\mu)$, 因此 $s_n\in S$. 因为 $|f-s_n|^p\leqslant f^p$, 控制收敛定理表明当 $n\to\infty$ 时 $\|f-s_n\|_p\to 0$. 于是 f 位于 S 的 L^p-闭包内. 由此也可以得出一般(f 是复的)情况.

连续函数逼近

至此我们已经讨论了在任何测度空间上的 $L^p(\mu)$. 现在设 X 是一个局部紧的 Hausdorff 空间, 并设 μ 是 X 的 σ-代数 $\mathfrak{M}$ 上的测度, 它具有定理 2.14 所述的性质. 例如 X 可以是 R^k, μ 可以是 R^k 上的 Lebesgue 测度.

在这些情况下, 我们有下列类似于定理 3.13 的结论:

3.14 定理 对 $1\leqslant p<\infty$, $C_c(X)$ 在 $L^p(\mu)$ 内稠密.

证明 象定理 3.13 一样定义 S. 若 $s\in S$ 且 $\varepsilon>0$, 则存在一个 $g\in C_c(X)$, 使得除开一个测度$<\varepsilon$ 的集外 $g(x)=s(x)$ 都成立, 并且$|g|\leqslant\|s\|_\infty$(Lusin 定理). 因此

(1) $$\|g-s\|_p\leqslant 2\varepsilon^{1/p}\|s\|_\infty$$

因为S 在 $L^p(\mu)$ 内稠密,这就完成了证明.

3.15 评注 让我们稍微详细地讨论一下 $L^p(R^k)$ (R^k 上的 Lebesgue 测度的 L^p-空间)和空间 $C_c(R^k)$ 之间的关系. 我们考虑一个固定的维数 k.

对于每一个 $p\in[1,\infty]$, 在 $C_c(R^k)$上有一个度量; f 和 g 之间的距离是 $\|f-g\|_p$. 注意这是一个真正的度量, 而不必通过等价类. 关键在于, 如果 R^k 上的两个连续函数是不相等的, 则它们在某个非空开集 V 上不同. 因为 V 包含一个 k-胞腔, $m(V)>0$. 于是若 $C_c(R^k)$ 的两元素是 a. e. 相等的, 它们就是相等的. 注意到 $C_c(R^k)$的元素的本性上确界和真实的上确界相同也是有意思的: 对于 $f\in C_c(R^k)$

(1) $$\|f\|_\infty=\sup_{x\in R^k}|f(x)|$$

若 $1\leqslant p<\infty$, 定理 3.14 说明 $C_c(R^k)$在 $L^p(R^k)$内稠密, 定理 3.11 指出 $L^p(R^k)$ 是完备的. 于是 $L^p(R^k)$ 是赋予 $C_c(R^k)$以 L^p-距离所得的度量空间的完备化.

$p=1$ 和 $p=2$ 是最有兴趣的两种情况. 让我们再一次用不同的语言来叙述上面所说的当 $p=1$ 和 $k=1$ 时的结论; 这种说法表明 Lebesgue 积分确实是 Riemann 积分的"正确"的推广:

如果在 R^1 内具有紧支集的两个连续函数 f 和 g 之间的距离定义为

(2) $$\int_{-\infty}^{\infty}|f(t)-g(t)|\,dt$$

则这样所得度量空间的完备化恰好由 R^1 上的 Lebesgue 可积函数

所组成. 这里假定任何两个几乎处处相等的函数都看成是同一的.

当然，每一个度量空间 S 有一个完备化 S^*，它的每一个元素可以抽象地看成是 S 内的 Cauchy 序列的等价类（见[26] p. 71）. 现在情况的重要之处在于 $C_c(R^k)$ 的各种 L^p- 完备化再一次得出是 R^k 上的函数空间.

在 $p=\infty$ 的情形与 $p<\infty$ 的情形不同，$C_c(R^k)$ 的 L^∞-完备化不是 $L^\infty(R^k)$，而是 $C_0(R^k)$，即所有 R^k 上"在无穷远点为 0"的连续函数的空间. 这个概念将在 3.16 节内定义. 因为(1)指出 L^∞-范数与 $C_c(R^k)$ 上的上确界范数重合，上面关于 $C_0(R^k)$ 的结论就是定理 3.17 的一个特殊情况.

3.16 定义 一个局部紧 Hausdorff 空间 X 上的复函数 f，如果对每一个 $\varepsilon>0$，存在一个紧集 $K\subset X$，使得对所有不在 K 内的 x，$|f(x)|<\varepsilon$ 成立，则称 f 在无穷远点为 0.

所有在无穷远点为 0 的 X 上的连续函数的类称为 $C_0(X)$.

显然 $C_c(X)\subset C_0(X)$，并且若 X 是紧的，这两个类就是重合的. 在这种情况下对它们中的任何一个记以 $C(X)$.

3.17 定理 如果 X 是一个局部紧Hausdorff 空间，则 $C_0(X)$ 是 $C_c(X)$ 相对于由上确界范数

$$\|f\|=\sup_{x\in X}|f(x)| \tag{1}$$

所定义的度量的完备化.

证明 如果将 f 和 g 之间的距离取作 $\|f-g\|$，初等的验证指出 $C_0(X)$ 满足度量空间的公理. 必须证明：$(a)C_c(X)$ 在 $C_0(X)$ 内稠密和 $(b)C_0(X)$ 是完备的度量空间.

给定 $f\in C_0(X)$ 和 $\varepsilon>0$，存在一个紧集 K 使得在 K 的外面 $|f(x)|<\varepsilon$，Urysohn 引理给出了一个 $g\in C_c(X)$，使得 $0\leqslant g\leqslant 1$，且在 K 上 $g(x)=1$. 令 $h=f\cdot g$. 则 $h\in C_c(X)$ 且 $\|f-h\|<\varepsilon$. 这就证明

了(a).

为了证明(b)，设$\{f_n\}$是$C_0(X)$内的Cauchy序列，即假定$\{f_n\}$一致收敛，则它的点态收敛的极限函数f是连续的. 给定$\varepsilon>0$，存在一个n，使得$\|f_n-f\|<\varepsilon/2$，并存在一个紧集K使得在K的外面$|f_n(x)|<\varepsilon/2$. 因此在K的外面$|f(x)|<\varepsilon$. 这就已经证明了f在无穷远点为0，于是$C_0(X)$是完备的.

习　题

1　证明任意一族(a,b)上的凸函数的上确界在(a,b)上是凸的，并证明凸函数序列的点态极限是凸的. 关于凸函数序列的上极限和下极限如何？

2　如果φ在(a,b)上是凸的，而ψ在φ的值域上是凸的且是不减的，证明$\psi\circ\varphi$在(a,b)上是凸的. 对于$\varphi>0$，证明$\log\varphi$的凸性蕴涵着φ的凸性，但反之不真.

3　假定φ是(a,b)上的连续实函数，使得

$$\varphi\left(\frac{x+y}{2}\right)\leqslant\frac{1}{2}\varphi(x)+\frac{1}{2}\varphi(y)$$

对所有的x及$y\in(a,b)$成立. 证明φ是凸的.（如果从假定中省略了连续性，这个结论就不能得到）.

4　假定f是X上的复可测函数，μ是X上的正测度，并且

$$\varphi(p)=\int_X|f|^p d\mu=\|f\|_p^p\qquad(0<p<\infty)$$

设$E=\{p:\varphi(p)<\infty\}$. 假定$\|f\|_\infty>0$.

(a) 若$r<p<s$，$r\in E$，$s\in E$，证明$p\in E$.

(b) 证明$\log\varphi$在E的内部是凸的且φ是在E上连续的.

(c) 根据(a)，E是连通的. E一定是开的吗？一定是闭的吗？E能由单点组成吗？E能是$(0,\infty)$的任何连通子集吗？

(d) 若$r<p<s$，证明$\|f\|_p\leqslant\max(\|f\|_r,\ \|f\|_s)$. 指出这一结果蕴涵着$L^r(\mu)\cap L^s(\mu)\subset L^p(\mu)$.

(e) 假定对某个$r<\infty$，$\|f\|_r<\infty$成立. 证明当$p\to\infty$时，

$$\|f\|_p\to\|f\|_\infty$$

5　在习题4的假设中再加上假定

$$\mu(X)=1$$

(*a*) 证明，若 $0<r<s\leqslant\infty$，则 $\|f\|_r\leqslant\|f\|_s$.

(*b*) 在什么条件之下，会出现 $0<r<s\leqslant\infty$ 而 $\|f\|_r=\|f\|_s<\infty$ 的情况？

(*c*) 若 $0<r<s$，证明 $L^r(\mu)\supset L^s(\mu)$. 在什么样条件之下，这两个空间所包含的函数相同？

(*d*) 假定对某个 $r>0$，$\|f\|_r<\infty$. 若 $\exp\{-\infty\}$ 定义为 0，证明

$$\lim_{p\to 0}\|f\|_p=\exp\left\{\int_X\log|f|\,d\mu\right\}$$

6　设 m 是 $[0,1]$ 上的 Lebesgue 测度，关于 m 所确定的 $\|f\|_p$，找出使关系式

$$\Phi\left(\lim_{p\to 0}\|f\|_p\right)=\int_0^1(\Phi\circ f)\,dm$$

对每一个有界的可测正函数 f 成立的所有在 $[0,\infty)$ 上的函数 Φ. 首先证明

$$c\Phi(x)+(1-c)\Phi(1)=\Phi(x^c)\quad(x>0,\ 0\leqslant c\leqslant 1)$$

试与习题 5(*d*) 比较.

7　对某些测度，关系式 $r<s$ 蕴涵 $L^r(\mu)\subset L^s(\mu)$；对另一些测度这个包含关系相反；还存在一些测度，当 $r\neq s$ 时 $L^r(\mu)$ 不包含 $L^s(\mu)$. 试给出这些情形的例子，并找出使得这些情形将会出现的关于 μ 的条件.

8　若 g 是在 $(0,1)$ 上的正函数，使得当 $x\to 0$ 时 $g(x)\to\infty$，则存在一个 $(0,1)$ 上的凹函数 h，使得 $h\leqslant g$ 且当 $x\to 0$ 时 $h(x)\to\infty$. 这对不对？若 $(0,1)$ 用 $(0,\infty)$ 代替，$x\to 0$ 用 $x\to\infty$ 代替，这个问题会变化吗？

9　假定 f 是 $(0,1)$ 上的 Lebesgue 可测函数，并且不是本性有界的. 根据习题 4(*e*)，当 $p\to\infty$ 时，$\|f\|_p\to\infty$. $\|f\|_p$ 能任意缓慢地趋于 ∞ 吗？更确切地说，对在 $(0,\infty)$ 上每一个满足当 $p\to\infty$ 时 $\Phi(p)\to\infty$ 的正函数 Φ，能找到一个 f，使得当 $p\to\infty$ 时 $\|f\|_p\to\infty$，但对所有充分大的 p，$\|f\|_p\leqslant\Phi(p)$ 成立. 这个结论正确吗？

10　假定 $f_n\in L^p(\mu)$，$n=1,2,3,\cdots$，当 $n\to\infty$ 时 $\|f_n-f\|_p\to 0$ 和 $f_n\to g$ a. e. 成立，问 f 和 g 之间存在什么关系？

11　假定 $\mu(\Omega)=1$，f 和 g 是 Ω 上的正可测函数，使得 $fg\geqslant 1$. 证明

$$\int_\Omega f\,d\mu\cdot\int_\Omega g\,d\mu\geqslant 1$$

12　假定 $\mu(\Omega)=1$，且 $h:\Omega\to[0,\infty]$ 是可测的. 若

$$A=\int_\Omega h\,d\mu$$

证明

$$\sqrt{1+A^2}\leqslant\int_a\sqrt{1+h^2}\,d\mu\leqslant 1+A$$

如果 μ 是[0, 1]上的 Lebesgue 测度并且 h 是连续的，$h=f'$，则上面的不等式有一个简单的几何解释，试从这点推测(对一般的 Ω)关于 h 在什么条件下，上面不等式之一的等号能够成立，并且证明你的推测.

13 关于 f 和 g 在什么条件下，定理 3.8 和3.9 的结论中的等号能够成立？你或许不得不分别处理 $p=1$ 和 $p=\infty$ 的情况.

14 假定 $1<p<\infty$，就 Lebesgue 测度而言 $f\in L^p=L^p((0,\infty))$，并且

$$F(x)=\frac{1}{x}\int_0^x f(t)\,dt \qquad (0<x<\infty)$$

(a) 证明Hardy 不等式

$$\|F\|_p\leqslant\frac{p}{p-1}\|f\|_p$$

这表明映射 $f\to F$ 映 L^p 到 L^p 内.

(b) 证明仅当 $f=0$ a. e. 时，等号成立.

(c) 证明常数 $p/(p-1)$不能用一个比它小的数代替.

(d) 若 $f>0$ 且 $f\in L^1$. 证明 $F\notin L^1$.

提示: (a)首先假定 $f\geqslant 0$ 且 $f\in C_c((0,\infty))$. 用分部积分给出

$$\int_0^\infty F^p(x)\,dx=-p\int_0^\infty F^{p-1}(x)xF'(x)\,dx$$

注意 $xF'=f-F$. 应用 Hölder 不等式于 $\int F^{p-1}f$，然后导出一般情况. (c) 对于充分大的 A，在[1, A]上取 $f(x)=x^{-1/p}$，在其他点上取 $f(x)=0$.

15 假定$\{a_n\}$是正数序列. 当 $1<p<\infty$ 时证明

$$\sum_{N=1}^{\infty}\left(\frac{1}{N}\sum_{n=1}^{N}a_n\right)^p\leqslant\left(\frac{p}{p-1}\right)^p\sum_{n=1}^{\infty}a_n^p$$

提示: 若 $a_n\geqslant a_{n+1}$，这个结果能从习题 14 得出. 这个特殊情况蕴涵着一般的情况.

16 证明 Egoroff 定理: 若 $\mu(X)<\infty$，$\{f_n\}$ 是一个复的可测函数序列，它在 X 的每一点上点态收敛，并且如果 $\varepsilon>0$，则存在一个可测集 $E\subset X$，具有 $\mu(X-E)<\varepsilon$，使得$\{f_n\}$ 在 E 上一致收敛.

(这个结论是说通过在测度任意小的集上重新定义 f_n，我们能使点态收

敛的序列转化为一致收敛序列；注意与 Lusin 定理的类似性.)

提示：令

$$S(n,k)=\bigcap_{i,j\geqslant n}\left\{x:|f_i(x)-f_j(x)|<\frac{1}{k}\right\}$$

证明当 $n\to\infty$ 时对每一个 k, $\mu(S(n,k))\to\mu(X)$，因此存在一个适当的递增的序列$\{n_k\}$使得 $E=\bigcap S(n_k,k)$ 具有所要求的性质.

证明定理不能扩张到 σ-有限的空间.

证明定理不能(用同一证法)扩张到用函数族$\{f_t\}$代替序列$\{f_n\}$ 时的情况，这里 t 在正实数内变化，并且这个假定是：对每一个 $x\in X$, 当 $t\to\infty$ 时，$f_t(x)\to f(x)$.

17 (*a*) 如果 $0<p<\infty$, 令 $\gamma_p=\max(1,2^{p-1})$，证明对任意的复数 α 和 β,

$$|\alpha-\beta|^p\leqslant\gamma_p(|\alpha|^p+|\beta|^p)$$

(*b*) 假定 μ 是 X 上的正测度，$0<p<\infty$, $f\in L^p(\mu)$, $f_n\in L^p(\mu)$. 当 $n\to\infty$ 时，$f_n(x)\to f(x)$ a. e.，并且$\|f_n\|_p\to\|f\|_p$. 通过完成略述如下的两个证明然后证明$\lim\|f-f_n\|_p=0$.

(i) 根据 Egoroff 定理，作出 $X=A\cup B$, 使得$\int_A|f|^p<\varepsilon$, $\mu(B)<\infty$ 而 $f_n\to f$ 在 B 上一致收敛，应用 Fatou 引理于 $\int_A|f_n|^p$ 上，得出

$$\limsup\int_A|f_n|^p d\mu\leqslant\varepsilon$$

(ii) 令 $h_n=\gamma_p(|f|^p+|f_n|^p)-|f-f_n|^p$，并且象在定理 1.34 的证明一样运用 Fatou 引理.

(*c*) 证明，若去掉$\|f_n\|_p\to\|f\|_p$的假设，那末甚至在 $\mu(X)<\infty$ 的情况下(*b*)的结论也不成立.

18 设 μ 是 X 上的正测度. X 上的复可测函数序列$\{f_n\}$ 称为依测度收敛于可测函数 f，如果对每一个 $\varepsilon>0$，存在一个对应的 N，使得

$$\mu(\{x:|f_n(x)-f(x)|>\varepsilon\})<\varepsilon$$

对所有 $n>N$ 成立(这个概念在概率论中是很重要的). 假定 $\mu(X)<\infty$, 证明下列命题：

(*a*) 若 $f_n(x)\to f(x)$ a. e.， 则$f_n\xrightarrow{\text{依测度}}f$.

(*b*) 若 $f_n \in L^p(\mu)$，且 $\|f_n - f\|_p \to 0$，则 $f_n \xrightarrow{\text{依测度}} f$，这里 $1 \leqslant p \leqslant \infty$.

(*c*) 若 $f_n \xrightarrow{\text{依测度}} f$，则 $\{f_n\}$ 具有一个 a. e. 收敛于 f 的子序列.

审查(*a*)和(*b*)的逆定理. 若 $\mu(X) = \infty$，比如，如果 μ 是 R^1 上的 Lebesgue 测度，(*a*)(*b*)(*c*)会发生什么变化？

19 定义函数 $f \in L^\infty(\mu)$ 的本性值域为集合 R_f，它由所有使得

$$\mu(\{x: |f(x) - w| < \varepsilon\}) > 0$$

对每一个 $\varepsilon > 0$ 成立的复数 w 所组成. 证明 R_f 是紧的. 集 R_f 和数 $\|f\|_\infty$ 之间存在什么关系？

设 A_f 是所有平均值

$$\frac{1}{\mu(E)} \int_E f d\mu$$

的集合，这里 $E \in \mathfrak{M}$，$\mu(E) > 0$. A_f 和 R_f 之间存在什么关系？A_f 总是闭的吗？是否存在这样的测度 μ 使得对每一个 $f \in L^\infty(\mu)$，A_f 是凸集？是否存在这样的测度 μ，使得对某一个 $f \in L^\infty(\mu)$，A_f 不是凸集？

比如说，用 $L^1(\mu)$ 代替 $L^\infty(\mu)$，会怎样影响这些结论？

20 假定 φ 是 R^1 上的实函数，使得

$$\varphi\left(\int_0^1 f(x)dx\right) < \int_0^1 \varphi(f)dx$$

对每一个实的有界可测函数 f 成立. 证明 φ 是凸的.

21 称度量空间 Y 是度量空间 X 的一个完备化，如果 X 在 Y 内稠密，且 Y 是完备的. 在 3.15 节中曾经提到过度量空间的这种完备化. 试叙述并证明一个说明上面术语是合理的唯一性定理.

22 假定 X 是一个度量空间，其中每一个 Cauchy 序列有收敛子序列. 能得出 X 是完备的吗(见定理 3.11 的证明)？

23 假定 μ 是 X 上的正测度，$\mu(X) < \infty$，$f \in L^\infty(\mu)$，$\|f\|_\infty > 0$，且

$$\alpha_n = \int_X |f|^n d\mu \qquad (n = 1, 2, \cdots)$$

证明

$$\lim_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty$$

24 假定 μ 是正测度，$f \in L^p(\mu)$，$g \in L^p(\mu)$.

(*a*) 若 $0 < p < 1$，证明

$$\int\left||f|^p-|g|^p\right|d\mu\leqslant\int|f-g|^p d\mu$$

并证明 $\Delta(f,g)=\int|f-g|^p d\mu$ 定义 $L^p(\mu)$ 上的一个距离.

(*b*) 若 $1\leqslant p<\infty$ 并且 $\|f\|_p\leqslant R$, $\|g\|_p\leqslant R$, 证明

$$\int||f|^p-|g|^p|d\mu\leqslant 2pR^{p-1}\|f-g\|_p$$

提示: 首先对 $x\geqslant 0$, $y\geqslant 0$, 证明

$$|x^p-y^p|\leqslant\begin{cases}|x-y|^p & 0<p<1\\ p|x-y|(x^{p-1}+y^{p-1}) & 1\leqslant p<\infty\end{cases}$$

注意(*a*)和(*b*)建立起映 $L^p(\mu)$ 入 $L^1(\mu)$ 的映射 $f\to|f|^p$ 的连续性.

第四章　Hilbert空间的初等理论

内积和线性泛函

4.1　定义　复向量空间H称为内积空间(或U空间)，如果向量x和$y\in H$的每个序对都对应一个复数(x,y)，即所谓x和y的内积(或纯量积)，使得下面的法则成立.

(a) $(y,x)=\overline{(x,y)}$(横线表示取共轭复数).

(b) 当x,y和$z\in H$时，$(x+y,z)=(x,z)+(y,z)$.

(c) 当x和$y\in H$，α是纯量时，$(\alpha x,y)=\alpha(x,y)$.

(d) 对所有$x\in H$，$(x,x)\geqslant 0$.

(e) $(x,x)=0$当且仅当$x=0$.

下面列出这些公理的某些直接推论:

(c) 蕴涵着对所有的$y\in H$，$(0,y)=0$.

(b)和(c)可以合并为下述说法：对每个$y\in H$，映射$x\to(x,y)$是H上的一个线性泛函.

(a)和(c)表明$(x,\alpha y)=\bar{\alpha}(x,y)$.

(a)和(b)蕴涵着第二分配律:

$$(z,x+y)=(z,x)+(z,y)$$

由(d)，可以定义向量$x\in H$的范数$\|x\|$为(x,x)的非负平方根.这样

(f)　　$$\|x\|^2=(x,x)$$

4.2　Schwarz不等式　性质4.1(a)—(d)蕴涵着

$$|(x,y)|\leqslant\|x\|\|y\|$$

对所有$x,y\in H$成立.

证明 令 $A=\|x\|^2, B=|(x,y)|, C=\|y\|^2$. 存在一个复数 α 使得 $|\alpha|=1$ 而 $\alpha(y,x)=B$. 对任意实数 r, 便有

(1) $(x-r\alpha y,\ x-r\alpha y)=(x,x)-r\alpha(y,x)-r\bar{\alpha}(x,y)+r^2(y,y)$

(1)式的左边是实的和非负的. 因此, 对每个实数 r

(2) $$A-2Br+Cr^2\geqslant 0$$

如果 $C=0$, 则必有 $B=0$; 否则当 r 是很大的正数时(2)式不成立. 如果 $C>0$, 在(2)式中取 $r=B/C$, 便得 $B^2\leqslant AC$.

4.3 三角不等式 对任意 x 和 $y\in H$, 都有

$$\|x+y\|\leqslant\|x\|+\|y\|$$

证明 由 Schwarz 不等式

$$\|x+y\|^2=(x+y,x+y)=(x,x)+(x,y)+(y,x)+(y,y)$$

$$\leqslant\|x\|^2+2\|x\|\|y\|+\|y\|^2=(\|x\|+\|y\|)^2$$

4.4 定义 由三角不等式得

(1) $$\|x-z\|\leqslant\|x-y\|+\|y-z\| \qquad (x,y,z\in H)$$

如果我们定义 x 与 y 之间的距离为 $\|x-y\|$, 则度量空间的全部公理都能满足. 在这里, 我们第一次用上定义 4.1 的(e).

这样, H 就成了一个度量空间. 如果这个度量空间是完备的, 就是说, H 中的每一个 Cauchy序列都收敛, 那末 H 就称为 Hilbert 空间.

在本章的余下部分, 字母 H 都表示 Hilbert 空间.

4.5 例

(a)对任一固定的 n, 所有 n 元序组

$$x=(\xi_1,\xi_2,\cdots,\xi_n)$$

的集 C^n, 这里 $\xi_1,\cdots,\xi_n$ 是复数. 当我们象通常一样按分量定义加法和纯量乘法, 并定义

$$(x, y)=\sum_{j=1}^{n}\xi_j\bar{\eta}_j \qquad (y=(\eta_1, \cdots, \eta_n))$$

就成为一个 Hilbert 空间.

(*b*)如果 μ 是一个正测度，$L^2(\mu)$就是一个具有内积

$$(f, g)=\int_X f\bar{g}d\mu$$

的 Hilbert 空间. 这里右端的被积函数由定理 3.8，是属于 $L^1(\mu)$ 的. 因此(f, g)有确定的意义. 注意

$$\|f\|=(f, f)^{1/2}=\left\{\int_X|f|^2d\mu\right\}^{1/2}=\|f\|_2$$

$L^2(\mu)$的完备性(定理 3.11)表明 $L^2(\mu)$的确是一个 Hilbert 空间[要记住 $L^2(\mu)$是看作函数的等价类的空间的；比较 3.10 节的讨论.]

当 $H=L^2(\mu)$时，不等式 4.2 和 4.3 转化为 Hölder 和 Minkowski 不等式的特款.

注意，例(*a*)是例(*b*)的特殊情况. 在(*a*)中的测度是什么？

(*c*)[0,1]上全体连续复函数的向量空间是一个内积空间，如果

$$(f, g)=\int_0^1 f(t)\overline{g(t)}dt$$

但它不是 Hilbert 空间.

4.6 定理 对任意固定的 $y\in H$，映射

$$x\to(x, y),\quad x\to(y, x),\quad x\to\|x\|$$

都是H上的连续函数.

证明 Schwarz 不等式蕴涵着

$$|(x_1, y)-(x_2, y)|=|(x_1-x_2, y)|\leqslant\|x_1-x_2\|\|y\|$$

这就证明了 $x\to(x, y)$事实上是一致连续的. 对 $x\to(y, x)$也一样是真确的. 由三角不等式$\|x_1\|\leqslant\|x_1-x_2\|+\|x_2\|$得出

$$\|x_1\|-\|x_2\| \leqslant \|x_1-x_2\|$$

若交换 x_1 与 x_2，可以看出

$$|\|x_1\|-\|x_2\|| \leqslant \|x_1-x_2\|$$

对所有 x_1 和 $x_2 \in H$ 成立. 这样 $x \to \|x\|$ 也是一致连续的.

4.7 子空间 向量空间 V 的一个子集 M 称为 V 的子空间，如果 M 本身关于 V 中所定义的加法和纯量乘法是一个向量空间. 对于集 $M \subset V$ 成为一个子空间的充要条件是当 $x, y \in M$，α 是一个纯量时 $x+y \in M$，$\alpha x \in M$.

在谈及向量空间的时候，“子空间”一词将总是赋予上述的含义. 有时为了强调，我们也可以用“线性子空间”一词来代替子空间.

例如，设 V 为集 S 上全体复函数的向量空间，S 上全体有界复函数的集就是 V 的一个子空间. 但是所有那些 $f \in V$，使对每个 $x \in S$，$|f(x)| \leqslant 1$ 的函数的集则不是一个子空间. 实向量空间 R^3 有如下的子空间，而不再有其他的了：$(a)R^3$，(b)所有通过原点的平面，(c)所有通过原点的直线，$(d)\{0\}$.

H 的一个闭子空间 M 是这样的一个子空间，它相对于由 H 的度量引出的拓扑是一个闭集.

4.8 凸集 向量空间 V 中的集 E 称为凸的，如果它有下面的几何性质：当 $x \in E$，$y \in E$，且 $0<t<1$ 时，点

$$z_t=(1-t)x+ty$$

也属于 E. 当 t 从 0 变到 1 时，我们可以设想点 z_t 在 V 中描出一条从 x 到 y 的直线段. 凸性要求 E 包含它的任意两点之间的线段.

显然，V 的每个子空间都是凸的.

同样，如果 E 是凸的，那末它的每一个平移

$$E+x=\{y+x;\ y \in E\}$$

也一定是凸的.

4.9 正交性 若对某个 x 和 $y\in H$, $(x,y)=0$, 我们就说 x 正交于 y, 有时记作 $x\perp y$. 由于 $(x,y)=0$ 蕴涵着 $(y,x)=0$, 关系"$\perp$"是对称的.

设 $x^{\perp}$ 表示所有正交于 x 的 $y\in H$ 的集; 而当 M 是 H 的子空间时, 设 $M^{\perp}$ 是所有正交于每个 $x\in M$ 的那些 $y\in H$ 的集.

注意 $x^{\perp}$ 是 H 的一个子空间, 因为 $x\perp y$ 和 $x\perp y'$ 蕴涵着 $x\perp(y+y')$ 和 $x\perp\alpha y$. $x^{\perp}$ 也恰好是那些使连续函数 $y\to(x,y)$ 等于 0 的那些点的集. 因此, $x^{\perp}$ 是 H 的闭子空间. 由于

$$M^{\perp}=\bigcap_{x\in M}x^{\perp}$$

$M^{\perp}$ 是闭子空间的交, 于是得知 $M^{\perp}$ 是 H 的闭子空间.

4.10 定理 在 Hilbert 空间 H 中, 每一个非空闭凸集 E 都包含唯一的一个具有最小范数的元素.

换句话说, 存在唯一的一个 $x_0\in E$, 使得对每个 $x\in E$, $\|x_0\|\leqslant\|x\|$.

证明 通过仅仅用到定义 4.1 所列举的性质的简单计算, 可得到恒等式

(1) $$\|x+y\|^2+\|x-y\|^2=2\|x\|^2+2\|y\|^2 \qquad (x,y\in H)$$

这就是熟知的平行四边形法则: 如果把 $\|x\|$ 解释为向量 x 的长度, (1)式就等于说一个平行四边形对角线的平方和等于各边的平方和. 这是平面几何中一个熟悉的命题.

设 $\delta=\inf\{\|x\|:\ x\in E\}$. 对任意的 x 和 $y\in E$, 应用(1)于 $\frac{1}{2}x$ 和 $\frac{1}{2}y$, 得到

(2) $$\frac{1}{4}\|x-y\|^2=\frac{1}{2}\|x\|^2+\frac{1}{2}\|y\|^2-\left\|\frac{x+y}{2}\right\|^2$$

因为E是凸的，$(x+y)/2\in E$. 因此

$$\|x-y\|^2\leqslant 2\|x\|^2+2\|y\|^2-4\delta^2 \qquad (x,y\in E) \tag{3}$$

如果还有$\|x\|=\|y\|=\delta$，则(3)蕴涵着$x=y$，这就证明了定理的唯一性断言.

δ的定义表明E中存在一个序列$\{y_n\}$，使$n\to\infty$时，$\|y_n\|\to\delta$. 在(3)中用y_n和y_m代替x和y，那末，当$n\to\infty$和$m\to\infty$时，(3)的右边趋于0. 这表明$\{y_n\}$是Cauchy序列. 由于H是完备的，存在$x_0\in H$，使得当$n\to\infty$时，$y_n\to x_0$，即$\|y_n-x_0\|\to 0$. 因$y_n\in E$而E是闭的，$x_0\in E$. 由于范数是H上的连续函数(定理4.6)，故有

$$\|x_0\|=\lim_{n\to\infty}\|y_n\|=\delta$$

4.11 定理 设M是H的闭子空间，则存在唯一的一对映射P和Q，使得P映H到M内，Q映H到$M^\perp$内，且对所有的$x\in H$，

$$x=Px+Qx \tag{1}$$

这些映射有下列进一步的性质：

(2) 如果$x\in M$，则$Px=x$，$Qx=0$；如果$x\in M^\perp$，则$Px=0$，$Qx=x$.

(3) 如果$x\in H$，则$\|x-Px\|=\inf\{\|x-y\|:y\in M\}$.

$$\|x\|^2=\|Px\|^2+\|Qx\|^2 \tag{4}$$

(5) P和Q都是线性映射.

系 如果$M\neq H$，则存在$y\in H$，$y\neq 0$使得$y\perp M$.

P和Q称为H到M上和$M^\perp$上的正交射影.

证明 对任意的$x\in H$，集$x+M=\{x+y:y\in M\}$是闭的和凸的. 定义Qx为$x+M$中具有最小范数的那个唯一的元素. 根据定理4.10，它是存在的. 定义$Px=x-Qx$，则式(1)成立. 因为$Qx\in x+M$，很明显$Px\in M$，于是P将H映入M内.

其次我们必须证明对所有的$y\in M$，$(Qx,y)=0$. 不失一般性，假定$\|y\|=1$，而令$z=Qx$. Qx的极小性质表明对每个纯量α，

$$(z,z)=\|z\|^2\leqslant\|z-\alpha y\|^2=(z-\alpha y,z-\alpha y)$$

简化之，得

$$0\leqslant -\alpha(y,z)-\bar{\alpha}(z,y)+|\alpha|^2$$

令 $\alpha=(z,y)$，即给出 $0\leqslant -|(z,y)|^2$. 从而$(z,y)=0$. 这样，Q 就映 H 入$M^\perp$内.

现在，若 $x=x_0+x_1$，其中 $x_0\in M, x_1\in M^\perp$，那末

$$x_0-Px=Qx-x_1$$

因为 $x_0-Px\in M, Qx-x_1\in M^\perp$，且 $M\cap M^\perp=\{0\}$[$(x,x)=0$蕴涵着 $x=0$ 这一事实的一个直接推论]，我们有 $x_0=Px, x_1=Qx$. 这就证明了唯一性的论断.

P 和 Q 的线性可类似地证明：将(1)式应用于 x, y 和 $\alpha x+\beta y$，我们得到

$$P(\alpha x+\beta y)-\alpha Px-\beta Py=\alpha Qx+\beta Qy-Q(\alpha x+\beta y)$$

左边属于M，右边属于$M^\perp$；因此二者都是0. 于是 P 和 Q 都是线性的.

性质(2)由(1)得到；(3)用来定义 P；因为$(Px,Qx)=0$，而(4)从(1)得到. 要证明系，因为 $x\neq Px, y\neq 0$，只须取 $x\in H$，$x\notin M$，并令 $y=Qx$ 即可.

我们已经观察到对每个 $y\in H$，$x\to(x,y)$是H上的一个连续的线性泛函. 一个极为重要的事实是：H上所有的连续线性泛函都是这种形式.

4.12 定理 如果L是H上的一个连续线性泛函，则存在唯一的一个 $y\in H$，使得

(1) $$Lx=(x,y) \qquad (x\in H)$$

证明 如果对所有 x 有 $Lx=0$，则取 $y=0$；否则的话，定义

(2) $$M=\{x: Lx=0\}$$

L 的线性表明M是个子空间. L 的连续性表明M是闭的. 因为存在 $x\in H, Lx\neq 0$. 定理 4.11 表明$M^\perp$并不由单独一个 0 组成.

因此，存在 $z\in M^{\perp}$，$\|z\|=1$. 令

$$u=(Lx)z-(Lz)x \tag{3}$$

由于 $Lu=(Lx)(Lz)-(Lz)(Lx)=0$，便有 $u\in M$. 于是$(u,z)=0$，这就给出

$$Lx=(Lx)(z,z)=(Lz)(x,z) \tag{4}$$

取 $\bar{\alpha}=Lz, y=\alpha z$，则(1)就成立了.

y 的唯一性是容易证明的. 因为，如果对所有 $x\in H$，$(x,y)=(x,y')$，令 $z=y-y'$，则对所有 $x\in H$，$(x,z)=0$，特别$(z,z)=0$. 因而 $z=0$.

规范正交集

4.13 定义 如果 V 是一个向量空间，$x_1,\cdots,x_k\in V$，而 $c_1,\cdots,c_k$ 是纯量，则 $c_1x_1+c_2x_2+\cdots+c_kx_k$ 称为 $x_1,\cdots,x_k$ 的一个线性组合. 集$\{x_1,\cdots,x_k\}$称为独立的，如果 $c_1x_1+\cdots+c_kx_k=0$ 蕴涵着 $c_1=c_2=\cdots=c_k=0$. 集 $S\subset V$ 是独立的，如果 S 的每个有限子集都是独立的. 由 S 的所有有限子集的所有的线性组合所成的集$[S]$(也称为 S 的元素的全体有限线性组合的集)，很清楚是一个向量空间. $[S]$是 V 中包含 S 的最小的子空间；$[S]$ 称为 S 的张成空间，或称由 S 张成的空间.

Hilbert 空间中向量 u_α 组成的集，这里 α 跑遍某个指标集 A，称为规范正交的，如果对所有的 $\alpha\in A$，$\beta\in A$，$\alpha\neq\beta$ 都满足正交性关系$(u_\alpha,u_\beta)=0$，并且它被规范化，使得对每个 $\alpha\in A$，$\|u_\alpha\|=1$. 换句话说，$\{u_\alpha\}$是规范正交的，即规定

$$(u_\alpha,u_\beta)=\begin{cases}1 & \alpha=\beta\\ 0 & \alpha\neq\beta\end{cases} \tag{1}$$

如果$\{u_\alpha:\alpha\in A\}$是规范正交的，对于每个 $x\in H$，我们都对应于指标 A 上的一个复函数 $\hat{x}$，它定义为

(2) $$\hat{x}(\alpha)=(x,x_\alpha)\qquad(\alpha\in A)$$

有时，称这些数 $\hat{x}(\alpha)$ 为关于集 $\{u_\alpha\}$ 的 Fourier 系数.

4.14 定理 如果 $u_1,\cdots,u_k$ 是规范正交集，而 $x=\sum_{n=1}^{k}c_nu_n$，则

(1) $$c_n=(x,u_n)\qquad(1\leqslant n\leqslant k)$$

(2) $$\|x\|^2=\sum_{n=1}^{k}|c_n|^2$$

证明 应用 4.13 的关系(1).

系 每个规范正交集都是独立的.

证明 从(2)式即可得出.

4.15 一个逼近问题 设 $v_1,\cdots,v_k$ 是 H 中一个独立向量集，假定 $x\in H$. 问题是找出一种当 $c_1,\cdots,c_k$ 取遍所有纯量时计算出

(1) $$\|x-\sum_{j=1}^{k}c_jv_j\|$$

的最小值、并确定出相应的 $c_1,\cdots,c_k$ 的值的方法.

设 M 是 $v_1,\cdots,v_k$ 的张成空间. 假如我们知道 M 是闭的，就能应用定理 4.11，并推出唯一的极小化元素 $x_0=Px$ 的存在性，这里

(2) $$x_0=\sum_{j=1}^{k}c_jv_j$$

同时也具有 $x-x_0\in M^\perp$ 的性质. 这些事实因此可以用来获得关于(2)中的系数 $c_1,\cdots,c_k$ 的信息.

因为 M 是向量的有限集的张成空间，M 是闭的这一点看来是明显的. 观察到 $\{0\}$ 肯定是闭的，并且借助于下面的引理，我们可以通过归纳法来证明它.

如果 V 是 H 的一个闭子空间，$y\in H$，$y\notin V$，而 V^* 是由 V 和 y 张成的空间，则 V^* 是闭的.

为证明这一点，假定 z 是 V^* 的极限点. 则

$$z=\lim_{n\to\infty}(x_n+\lambda_n y)$$

这里 $x_n\in V$，而 λ_n 是纯量. 因为度量空间中的收敛序列是有界的，故存在 $\eta<\infty$，使得 $\|x_n+\lambda_n y\|<\eta$ 对 $n=1,2,3,\cdots$ 都成立. 假如 $|\lambda_n|\to\infty$ 的话，我们应当有

$$\|\lambda_n^{-1}x_n+y\|<\frac{\eta}{|\lambda_n|}\longrightarrow 0.$$

因为 V 是闭的，于是 $-y\in V$. 但 $y\notin V$，因此 $\{\lambda_n\}$ 有一个收敛于某个 λ 的子序列 $\{\lambda_{n_i}\}$. 从而，作为两个 Cauchy 序列之差的 $\{x_{n_i}\}$ 本身也是 H 中的一个 Cauchy 序列，并且收敛于某个 $x\in V$. 所以 $z=x+\lambda y$. 这就证明 V^* 包含了它所有的极限点.

现在回到我们的问题. 设

$$a_{ij}=(v_j,v_i),\ b_i=(x,v_i) \tag{3}$$

那末，如果由(2)给出的 x_0 是极小化元素，就一定有

$$(x-x_0,v_i)=0$$

对 $i=1,2,\cdots,k$ 成立. 这个式子引出一组未知数 $c_1,\cdots,c_k$ 的 k 个线性方程

$$\sum_{j=1}^{k}a_{ij}c_j=b_i\qquad(1\leqslant i\leqslant k) \tag{4}$$

从定理 4.11 我们知道 x_0 存在且是唯一的. 因此 a_{ij} 的行列式不为 0，而 c_j 可以从(4)算出.

其次，设 δ 是 (1) 的极小值. 由于 $(x-x_0,\ v_i)=0$，有 $(x-x_0,x_0)=0$；因此，

$$\delta^2=(x-x_0,x-x_0)=(x,x-x_0)=(x,x-\sum_{j=1}^{k}c_jv_j)$$

从而

(5) $$\delta^2=\|x\|^2-\sum_{j=1}^{k}\bar{c}_jb_j$$

这样，利用(3)中诸量就解决了我们的问题.

现在转向一个特殊情况：用一个规范正交集 $u_1,\cdots,u_k$ 来代替 $v_1,\cdots,v_k$. 于是，当 $i=j$ 时 $a_{ij}=1$，而 $i\neq j$ 时 $a_{ij}=0$. 因而(4)给出 $c_i=b_i$. 而(5)变成为

(6) $$\delta^2=\|x\|^2-\sum_{j=1}^{k}|b_j|^2$$

我们可以总结如下：

4.16 定理 设 $u_1,\cdots,u_k$ 是 H 中的一个规范正交集，$x\in H$. 则

(1) $$\|x-\sum_{j=1}^{k}(x,u_j)u_j\|\leqslant\|x-\sum_{j=1}^{k}\lambda_ju_j\|$$

对所有纯量 $\lambda_1,\cdots,\lambda_k$ 成立. (1) 中的等号当且仅当对 $1\leqslant j\leqslant k$，$\lambda_j=(x,u_j)$ 时才成立. 向量

(2) $$\sum_{j=1}^{k}(x,u_j)u_j$$

是 x 到子空间 $[u_1,\cdots,u_k]$ 内的正射影. 并且，若 δ 是 x 到这个子空间的距离，则

(3) $$\sum_{j=1}^{k}|(x,u_j)|^2=\|x\|^2-\delta^2$$

系（Bessel 不等式） 如果 $\{u_\alpha,\alpha\in A\}$ 是 H 的任意规范正交集，$x\in H$，且 $\hat{x}(\alpha)=(x,u_\alpha)$，则

(4) $$\sum_{\alpha\in A}|\hat{x}(\alpha)|^2\leqslant\|x\|^2$$

这个系需要作一些注释. 集 A 是任意的指标集，甚至可能是不可数的，但完全没有安排顺序. 在这些条件之下，(4)式左边的

和意味着什么呢？我们定义如下：如果对每个$\alpha\in A$，$0\leqslant\varphi(\alpha)\leqslant\infty$，符号

(5) $$\sum_{\alpha\in A}\varphi(\alpha)$$

表示所有有限和$\varphi(\alpha_1)+\cdots+\varphi(\alpha_k)$的集的上确界，这里$\alpha_1,\cdots,\alpha_k$是$A$中不同元素．在这种约定之下，很清楚(4)可由(3)得出．

从矩的观点看，和式（5）恰好是φ关于A上的计数测度的Lebesgue积分．设$l^2(A)$是关于A上计数测度的L^2-空间．那末(4)断言$\hat{x}\in l^2(A)$并且$\|\hat{x}\|_2\leqslant\|x\|$．

(4)的一个直接推论应当明确地提出来：

对每个$x\in H$和H中的任意一个规范正交集$\{u_\alpha\}$，所有使$\hat{x}(\alpha)\neq 0$的α的集至多是可数的．

设F是将每个$x\in H$对应于A上的函数$\hat{x}$的映射，对每个$\alpha\in A$，$x\to(x,\ u_\alpha)$是一个线性泛函．因此F是H到$l^2(A)$内的线性变换（见定义2.1）．由于$\|\hat{x}-\hat{y}\|_2\leqslant\|x-y\|$，故$F$也不会增加距离．特别，$F$是连续的．

我们现在将要看到：H的完备性蕴涵着F映H到$l^2(A)$上，以及在关于$\{u_\alpha\}$的一定的条件下F确实是一个等距，即对所有$x\in H$，$\|\hat{x}\|_2=\|x\|$．从而F当然是一一的．

4.17 Riesz-Fischer 定理 设$\{u_\alpha:\ \alpha\in A\}$是$H$中的一个规范正交集．假设$\varphi\in l^2(A)$．则存在$x\in H$，$\varphi=\hat{x}$．

证明 对$n=1,2,3,\cdots$，设$A_n=\left\{\alpha:\ |\varphi(\alpha)|>\frac{1}{n}\right\}$．每个$A_n$都是一个有限集(事实上，容易验证$A_n$至多有$n^2\|\varphi\|_2^2$个元素)．令

(1) $$x_n=\sum_{\alpha\in A_n}\varphi(\alpha)u_\alpha\qquad(n=1,2,3,\cdots)$$

则$\hat{x}_n=\varphi\cdot\chi_{A_n}$，于是对每个$\alpha\in A$，$\hat{x}_n(\alpha)\to\varphi(\alpha)$，且$|\varphi-\hat{x}_n|^2\leqslant$

$|\varphi|^2$. 因此，由控制收敛定理的初等形式知道$\|\varphi-\hat{x}_n\|_2\to 0$. 于是得知$\{\hat{x}_n\}$是$l^2(A)$中的 Cauchy 序列. 由于集$A_n$有限，定理 4.14 表明$\|x_n-x_m\|=\|\hat{x}_n-\hat{x}_m\|_2$，于是$\{x_n\}$是$H$中的 Cauchy 序列，并由$H$的完备性，存在$x=\lim_{n\to\infty}x_n\in H$. 对任何$\alpha\in A$都有

$$\hat{x}(\alpha)=(x,u_\alpha)=\lim_{n\to\infty}(x_n,u_\alpha)=\lim_{n\to\infty}\hat{x}_n(\alpha)=\varphi(\alpha)$$

这就完成了证明.

4.18 定理 设$\{u_\alpha:\ \alpha\in A\}$是$H$中的规范正交集. 下面关于$\{u_\alpha\}$的四个条件中的每一个都蕴涵着另外三个：

(a) $\{u_\alpha\}$是H的极大规范正交集.

(b) $\{u_\alpha\}$的全体有限线性组合S在H中稠密.

(c) 对每个$x\in H$，有$\|x\|^2=\sum_{\alpha\in A}|\hat{x}(\alpha)|^2$.

(d) 若$x\in H$，$y\in H$，则$(x,y)=\sum_{\alpha\in A}\hat{x}(\alpha)\overline{\hat{y}(\alpha)}$.

最后一个公式就是熟知的 Parseval 恒等式. 注意$\hat{x}\in l^2(A)$，$\hat{y}\in l^2(A)$，所以$\hat{x}\bar{\hat{y}}\in l^1(A)$. 从而(d)中的和式的意义是明确的. 当然，(c)是(d)当$x=y$时的特款. 极大规范正交集通常叫做完备规范正交集或者规范正交基.

证明 简单说来，$\{u_\alpha\}$是极大的就是意味着H中再也没有向量能附加到$\{u_\alpha\}$上，使所得到的集还是规范正交的. 当H中不存在$x\neq 0$能与每个u_α都正交时，上述情况确实是会出现的.

我们将证明(a)→(b)→(c)→(d)→(a).

设M是S的闭包. 因为S是子空间，故M也是子空间($x_n\to x$和$y_n\to y$蕴涵着$x_n+y_n\to x+y$，$\lambda x_n\to\lambda x$)；且当S在H中不稠密时，$M\neq H$，由定理 4.11，$M^\perp$包含有一个非 0 向量. 这样，当S不稠密时，$\{u_\alpha\}$就不是极大的，于是(a)→(b).

假设(b)成立. 固定 $x\in H$, $\varepsilon>0$. 由于 S 稠密, 存在有限集 $u_{\alpha_1}, \cdots, u_{\alpha_k}$, 使这些向量的某个线性组合和 x 的距离小于 ε. 由定理 4.16, 这个逼近式只有在取 u_{α_j} 的系数为 $\hat{x}(\alpha_j)$时才能获得改善. 这样, 当

(1) $$z=\hat{x}(\alpha_1)u_{\alpha_1}+\cdots+\hat{x}(\alpha_k)u_{\alpha_k}$$

时, 我们有$\|x-z\|<\varepsilon$, 因此$\|x\|\leqslant\|z\|+\varepsilon$. 而定理 4.14 给出

(2) $$(\|x\|-\varepsilon)^2<\|z\|^2=|\hat{x}(\alpha_1)|^2+\cdots+|\hat{x}(\alpha_k)|^2$$

$$\leqslant\sum_{\alpha\in A}|\hat{x}(\alpha)|^2$$

因为 ε 是任意的, (c)便可以从(2)式及 Bessel 不等式得出.

(c)中的方程式也可以写成形式

(3) $$(x, x)=(\hat{x}, \hat{x})$$

右边的内积是象例 4.5(b) 一样的 Hilbert 空间 $l^2(A)$中的内积. 固定 $x\in H$, $y\in H$. 如果(c)式成立, 则对每个纯量 λ

(4) $$(x+\lambda y, x+\lambda y)=(\hat{x}+\lambda\hat{y}, \hat{x}+\lambda\hat{y})$$

因此

(5) $$\bar{\lambda}(x, y)+\lambda(y, x)=\bar{\lambda}(\hat{x}, \hat{y})+\lambda(\hat{y}, \hat{x})$$

取 $\lambda=1$ 和 $\lambda=i$, 那末(5)式表明(x, y)与$(\hat{x}, \hat{y})$有相同的实部和虚部, 因而是相等的. 于是(c)蕴涵着(d).

最后, 若(a)不真, 则在H内存在 $u\neq 0$, 使$(u, u_\alpha)=0$ 对所有的 $\alpha\in A$ 成立. 若 $x=y=u$, 则$(x, y)=\|u\|^2\neq 0$. 然而对所有的 $\alpha\in A$, $\hat{x}(\alpha)=0$, 所以(d)也不成立. 从而(d)蕴涵着(a). 证明完毕.

4.19 同构 非正式地说, 两个具有相同性质的代数系统被认为是同构的, 意思是指存在从一个系统到另一系统上的一一映射, 它保持全部有关性质. 比如说, 我们可以问两个群是不是同构, 或者两个域是不是同构. 两个向量空间是同构的, 如果有一个从一个空间到另一个空间上的一一的线性映射. 线性映射是那种保持

向量空间的有关概念，即加法和纯量乘法的映射.

同样地，两个 Hilbert 空间是同构的，如果存在一个从 H_1 到 H_2 上的一一线性映射 A，使它同时也保持内积：对所有的 x 与 $y\in H$，$(Ax, Ay)=(x, y)$成立. 这样的一个 A 就是 H_1 到 H_2 上的一个同构(或者更确切地说，一个 Hilbert 空间同构). 利用这一术语，定理 4.17 和 4.18 可以得出下列命题.

如果$\{u_\alpha:\ \alpha\in A\}$是 Hilbert 空间 H 中的极大规范正交集，$\hat{x}(\alpha)=(x, u_\alpha)$，则映射 $x\to\hat{x}$ 是 H 到 $l^2(A)$ 上的 Hilbert 空间同构.

可以证明(我们予以省略)，$l^2(A)$与 $l^2(B)$当且仅当 A 与 B 有相同的基数时才是同构的. 然而，我们将证明每一个非平凡的Hilbert空间(这意味着空间并非由单独一个 0 组成)都有一个极大规范正交集(定理4.22)，从而每一个这样的空间都同构于某个$l^2(A)$. 这一证明有赖于半序集的一个性质，该性质与选择公理等价.

4.20 半序集 集$\mathscr{P}$称为由二元关系"$\leqslant$"所半序化，如果

(a) $a\leqslant b$ 和 $b\leqslant c$ 蕴涵着 $a\leqslant c$.

(b) 对每个 $a\in\mathscr{P}$，$a\leqslant a$.

(c) $a\leqslant b$ 和 $b\leqslant a$ 蕴涵着 $a=b$.

半序集$\mathscr{P}$的子集$\mathscr{Q}$称为全序的(或线性序的)，如果每一对 a，$b\in\mathscr{Q}$或者满足 $a\leqslant b$、或者 $b\leqslant a$.

例如，一个给定集的每一个子集族由包含关系"$\subset$"所半序化.

为给出一个更特殊的例子，设$\mathscr{P}$是平面全体开集族，由集的包含关系所半序化. 而设$\mathscr{Q}$是所有中心在原点的开圆盘族. 则$\mathscr{Q}\subset\mathscr{P}$. $\mathscr{Q}$由"$\subset$"全序化，并且$\mathscr{Q}$是$\mathscr{P}$的一个极大的全序子集. 这意味着当把不属于$\mathscr{Q}$的任意一个$\mathscr{P}$的元加到$\mathscr{Q}$中以后，所得到的集族就再不能由"$\subset$"全序化了.

4.21 Hausdorff 极大性定理 每一个非空半序集都含有极大的全序子集.

这是选择公理的一个推论，并且事实上也是等价于选择公理的．它的另一种(很相似)形式是所谓 Zorn 引理．我们将在附录中给出证明．

现在设H是一个非平凡的 Hilbert 空间，则存在$u\in H$，$\|u\|=1$．因而H存在非空的规范正交集．极大规范正交集的存在性就成了下述定理的一个推论．

4.22 定理 Hilbert 空间H中的每一个规范正交集B都包含在H的一个极大规范正交集中．

证明 设$\mathscr{D}$是H的包含给定集B的全体规范正交集的类．由集的包含关系将$\mathscr{D}$半序化．因为$B\in\mathscr{D}$，$\mathscr{D}\neq\varnothing$．因此$\mathscr{D}$包含一个极大全序类$\Omega$．设$S$是全体$\Omega$的元素之并．显然$B\subset S$．我们断言$S$是极大规范正交集：

如果u_1和$u_2\in S$，则存在A_1和$A_2\in\Omega$，$u_1\in A_1$，$u_2\in A_2$．因为Ω是全序的，$A_1\subset A_2$(或$A_2\subset A_1$)，故$u_1\in A_2$且$u_2\in A_2$．因为A_2是规范正交的，当$u_1\neq u_2$时，$(u_1,u_2)=0$，当$u_1=u_2$时$(u_1,u_2)=1$．于是S是规范正交集．

假设S不是极大．则S是一个规范正交集S^*的真子集．显然$S^*\notin\Omega$，且S^*包含Ω的每个元素．因此我们可以把S^*加到Ω中而仍然保持全序．这却与Ω的极大性相矛盾．

三角级数

4.23 定义 设T是复平面上单位圆周，即所有绝对值为1的复数集．如果F是T上任意一个函数，而f在R^1上由

$$f(t)=F(e^{it}) \tag{1}$$

所定义，则f是一个周期为2π的周期函数．这表示对所有的t，$f(t+2\pi)=f(t)$．反之，如果f是R^1上的周期为2π的函数，则存在一个T上的函数F使(1)式成立．这样一来，就可以把T上的

函数与 R^1 上周期为 2π 的函数看作同一回事；并且我们尽管考虑 f 是定义在 T 上的，为了记号的简单，我们有时候宁愿写 $f(t)$ 而不写 $f(e^{it})$.

记住了这个约定以后，对于 $1\leqslant p<\infty$ 定义 $L^p(T)$ 为所有 R^1 上的、Lebesgue 可测的、周期为 2π 的，并使得范数

$$\|f\|_p=\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|^p dt\right\}^{1/p} \tag{2}$$

为有限的复函数类.

换言之，我们考察的是 $L^p(\mu)$，这里的 μ 是 $[0, 2\pi]$ 上（或 T 上）的 Lebesgue 测度除以 2π. $L^\infty(T)$ 是所有 $L^\infty(R^1)$ 中的周期为 2π 的元素的类，以其本性上确界为范数. 而 $C(T)$ 由 T 上全体连续复函数组成（或者，等价地，由全体 R^1 上的周期为 2π 的连续复函数组成），并具有范数

$$\|f\|_\infty=\sup_t|f(t)| \tag{3}$$

(2) 式中的因子 $\frac{1}{2\pi}$ 简化了我们讨论时所涉及的形式. 例如，常数函数 1 的 L^p-范数是 1.

三角多项式是形如

$$f(t)=a_0+\sum_{n=1}^{N}(a_n\cos nt+b_n\sin nt)\quad t\in R^1 \tag{4}$$

的有限和式，这里 $a_0, a_1, \cdots, a_N$ 和 $b_1, \cdots, b_N$ 是复数. 根据 Euler 恒等式，(4)也可以写成

$$f(t)=\sum_{n=-N}^{N}c_n e^{int} \tag{5}$$

的形式，这对大多数目的来说显得更方便些. 很清楚，每一个三角多项式都有周期 2π.

我们用 Z 表示全体整数（正数、零和负数）的集，并且令

(6) $$u_n(t)=e^{int}\quad(n\in Z)$$

如果在 $L^2(T)$ 中用

(7) $$(f,g)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\overline{g(t)}dt$$

来定义内积，[注意，这与(2)是相协调的]. 那末，简单的计算表明

(8) $$(u_n,u_m)=\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-i(n-m)t}$$

$$dt=\begin{cases}1 & n=m\\ 0 & n\neq m\end{cases}$$

这样，$\{u_n;\ n\in Z\}$ 是 $L^2(T)$ 的一个规范正交集，通常称之为三角函数系. 现在我们来证明这个系是极大的，并从而导出原先在Hilbert 空间那一部分所得出的抽象定理的具体形式.

4.24 三角函数系的完备性 定理 4.18 表明，一旦能证明全体三角多项式在 $L^2(T)$ 中稠密，就立刻可以证明三角函数系的极大性(或完备性). 因为由定理 3.14(注意 T 是紧的)，$C(T)$ 在 $L^2(T)$ 内是稠密的，所以只须证明对每个 $f\in C(T)$ 和 $e>0$，存在一个三角多项式 P 使 $\|f-P\|_2<e$. 因为对每个 $g\in C(T)$，$\|g\|_2\leqslant\|g\|_\infty$，估值 $\|f-P\|_2<e$ 可由 $\|f-P\|_\infty<e$ 得到. 而这一估值就是我们要证明的.

假设我们有具有下列性质的三角多项式 $Q_1,Q_2,Q_3,\cdots$

(a) 当 $t\in R^1$ 时，$Q_k(t)\geqslant 0$.

(b) $$\frac{1}{2\pi}\int_{-\pi}^{\pi}Q_k(t)dt=1$$

(c) 若 $\eta_k(\delta)=\sup\{Q_k(t):\ \delta\leqslant|t|\leqslant\pi\}$，则对每个 $\delta>0$，

$$\lim_{k\to\infty}\eta_k(\delta)=0$$

(c) 的另一种说法是：对每个 $\delta>0$，$Q_k(t)$ 在 $[-\pi,\ -\delta]\cup[\delta,\pi]$ 上一致地趋于 0.

对每个 $f\in C(T)$，我们对应一个由

(1) $$P_k(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t-s)Q_k(s)ds \qquad (k=1,2,3,\cdots)$$

定义的函数 P_k. 如果我们用 $-s$ 代换 s、然后再用 $s-t$ 代换，这时 f 和 Q_k 的周期性表明积分值将不会改变. 因此

(2) $$P_k(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(s)Q_k(t-s)ds \quad (k=1,2,3,\cdots)$$

因为每个 Q_k 是三角多项式，Q_k 形如

(3) $$Q_k(t)=\sum_{n=-N_k}^{N_k}a_{n,k}e^{int}$$

如果用 $t-s$ 代换(3)式中的 t，并且代入(2)，我们即可看出每个 P_k 都是三角多项式.

给定 $\varepsilon>0$，由于 f 在 T 上一致连续，存在 $\delta>0$，当 $|t-s|<\delta$ 时，$|f(t)-f(s)|<\varepsilon$. 由(b)，有

$$P_k(t)-f(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(t-s)-f(t)\}Q_k(s)ds$$

而(a)蕴涵着对所有 t，

$$\begin{aligned}|P_k(t)-f(t)|&\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t-s)-f(t)|Q_k(s)ds\\&=A_1+A_2\end{aligned}$$

这里 A_1 是$[-\delta,\delta]$上的积分而 A_2 是$[-\pi,\ -\delta]\cup[\delta,\ \pi]$上的积分. 在 A_1 中，被积函数小于 $\varepsilon Q_k(s)$，故由(b)得 $A_1<\varepsilon$. 在 A_2 中，有 $Q_k(s)\leqslant\eta_k(\delta)$，故由(c)，对充分大的 k

(4) $$A_2\leqslant 2\|f\|_\infty\cdot\eta_k(\delta)<\varepsilon$$

因为这些估值与 t 无关. 我们便证明了

(5) $$\lim_{k\to\infty}\|f-P_k\|_\infty=0$$

剩下来我们就是构造 Q_k 了. 这可由多种方法作出. 下面是简单的一种. 令

(6) $$Q_k(t)=c_k\left\{\frac{1+\cos t}{2}\right\}^k$$

这里 c_k 选得使(b)成立. 由于(a)是明显的，我们只须证明(c). 因为 Q_k 是偶函数，(b)表明

$$1=\frac{c_k}{\pi}\int_0^\pi\left\{\frac{1+\cos t}{2}\right\}^k dt>\frac{c_k}{\pi}\int_0^\pi\left\{\frac{1+\cos t}{2}\right\}^k \sin t\,dt=\frac{2c_k}{\pi(k+1)}$$

因为 Q_k 在$[0,\pi]$上递减，于是

(7) $$Q_k(t)\leqslant Q_k(\delta)\leqslant\frac{\pi(k+1)}{2}\left(\frac{1+\cos\delta}{2}\right)^k$$

$$(0<\delta\leqslant|t|\leqslant\pi)$$

由于 $0<\delta\leqslant\pi$ 时，$1+\cos\delta<2$，这就蕴涵了(c).

我们便已经证明了如下的重要结果:

4.25 定理 如果$f\in C(T)$且$\varepsilon>0$，则存在一个三角多项式P，对每个实的 t，有

$$|f(t)-P(t)|<\varepsilon$$

一个更为精细的结果曾由 Fejér(1904)所证得: 任何 $f\in C(T)$ 的 Fourier 级数部分和的算术平均值都一致收敛于 f. [26]的定理 8.15 中可看到一个(十分类似于上述证法)证明.

4.26 Fourier 级数 对任意 $f\in L^1(T)$，我们用公式

(1) $$\hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)e^{-int}dt \qquad (n\in Z)$$

定义 f 的 Fourier 系数，记住，这里的 Z 是全体整数的集. 这样对每个 $f\in L^1(T)$，都对应了 Z 上的一个函数 $\hat{f}$. f 的 Fourier 级数就是

(2) $$\sum_{-\infty}^{\infty}\hat{f}(n)e^{int}$$

而它的部分和是

$$s_N(t)=\sum_{-N}^{N}\hat{f}(n)e^{int}\qquad(N=0,1,2,\cdots)\tag{3}$$

因为 $L^2(T)\subset L^1(T)$，(1)也可以应用于每个 $f\in L^2(T)$. 比较 4.23 节及 4.13 节的定义，现在可以用具体的语言重新叙述定理 4.17 和 4.18：

Riesz-Ficsher 定理断言：当$\{c_n\}$是一个复数序列，使

$$\sum_{n=-\infty}^{\infty}|c_n|^2<\infty\tag{4}$$

时，则存在一个 $f\in L^2(T)$ 使

$$c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)e^{-int}dt\qquad(n\in Z)\tag{5}$$

Parseval 定理断言：当 $f\in L^2(T)$ 和 $g\in L^2(T)$ 时，

$$\sum_{n=-\infty}^{\infty}\hat{f}(n)\overline{\hat{g}(n)}=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)\overline{g(t)}dt\tag{6}$$

(6)式的左边的级数绝对收敛，并且若 s_N 象(3)中一样，那末

$$\lim_{N\to\infty}\|f-s_N\|_2=0\tag{7}$$

由(6)的一个特殊场合，有

$$\|f-s_N\|_2^2=\sum_{|n|>N}|\hat{f}(n)|^2\tag{8}$$

注意，(7)式说明每个 $f\in L^2(T)$ 都是它的 Fourier 级数部分和的 L^2-极限；也就是说 f 的 Fourier 级数在 L^2 意义下收敛于 f. 如象第五章将要见到的那样，点态收敛提出了一个更加带有技巧性的问题.

Riesz-Fischer 定理和 Parseval 定理可以通过指明映射 $f\to\hat{f}$ 是 $L^2(T)$ 到 $l^2(Z)$ 上的 Hilbert 空间的同构而予以综合.

在其他函数空间(例如，在 $L^1(T)$ 中)的 Fourier 级数的理论，

比在 $L^2(T)$ 中更加困难，而我们仅将在少数的几方面碰到它.

注意到在 Riesz-Fischer 定理的证明中，关键部分是这样一个事实，即 L^2 是完备的. 下面一点要牢牢记住，即对那些断言 L^2 或者甚至任何 L^p 的完备性的定理，有时也冠以“Riesz-Fischer 定理”这一名称.

习　题

在这一组习题中，H 总是表示一个 Hilbert 空间.

1　如果 M 是 H 的一个闭子空间，证明 $M=(M^\perp)^\perp$. 对于不一定闭的子空间 M，有没有一个与之类似的正确的命题？

2　对 $n=1,2,3,\cdots$，$\{v_n\}$ 是 H 中一个独立的向量集. 试给出由它生成一个规范正交集 $\{u_n\}$ 的构造过程，使 u_n 是 $v_1,\cdots,v_n$ 的线性组合. 注意，这可导致一个关于可分 Hilbert 空间中极大规范正交集存在的证明. 它不必借助于 Hausdorff 极大原理. （一个空间是可分的，如果它包含可数稠密子集.）

3　证明 $1\leqslant p<\infty$ 时，$L^p(T)$ 是可分的，但 $L^\infty(T)$ 不是可分的.

4　证明 H 是可分的，当且仅当 H 包含一个至多是可数的极大规范正交集.

5　若 $M=\{x:Lx=0\}$，这里 L 是 H 上的一个连续线性泛函，证明 $M^\perp$ 是一个维数为 1 的向量空间（除非 $M=H$）.

6　设 $\{u_n\}$ $(n=1,2,3,\cdots)$ 是 H 中的规范正交集. 说明这是一个有界闭集而非紧的例子. 设 Q 是 H 中所有形如

$$x=\sum_1^\infty c_n u_n \qquad \left(\text{这里}|c_n|\leqslant\frac{1}{n}\right)$$

的 x 的集. 证明 Q 是紧的（Q 称为 Hilbert 方体）.

更一般地，设 $\{\delta_n\}$ 是一个正数的序列而 S 是 H 中所有形如

$$x=\sum_1^\infty c_n u_n \qquad (\text{这里}|c_n|\leqslant\delta_n)$$

的 x 的集. 证明当且仅当 $\sum_1^\infty \delta_n^2<\infty$ 时 S 是紧的.

证明H不是局部紧的.

7　设$\{a_n\}$是一个正数的序列,对 $b_n\geqslant 0$ 且 $\Sigma b_n^2<\infty$ 时有 $\Sigma a_n b_n<\infty$. 求证 $\Sigma a_n^2<\infty$.

8　若 H_1 和 H_2 是两个 Hilbert 空间. 证明其中的一个一定同构于另一个的某个子空间. (注意, Hilbert 空间的每个闭子空间仍是 Hilbert 空间.)

9　若 $A\subset[0,2\pi]$而A是可测的,证明

$$\lim_{n\to\infty}\int_A \cos nx\,dx=\lim_{n\to\infty}\int_A \sin nx\,dx=0$$

10　设 $n_1<n_2<n_3<\cdots$是些正整数,而E是所有的使$\{\sin n_k x\}$ 收敛的点 $x\in[0,2\pi]$的集. 证明 $m(E)=0$. 提示: $2\sin^2\alpha=1-\cos 2\alpha$, 因此由习题 9, 在$E$上几乎处处有 $\sin n_k x\to\pm\dfrac{1}{\sqrt{2}}$.

11　证明恒等式

$$4(x,y)=\|x+y\|^2-\|x-y\|^2+i\|x+iy\|^2-i\|x-iy\|^2$$

在每个内积空间中都成立. 并说明它证实了定理 4.18 中的蕴涵关系 $(c)\to(d)$.

12　在 4.22 节中已指出常数 c_k 使 $k^{-1}c_k$ 保持有界. 更精确地估计相应的积分值并证明

$$0<\lim_{k\to\infty}k^{-1/2}c_k<\infty$$

13　设 f 是 R^1 上的连续函数,周期为 1. 证明

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} f(n\alpha)=\int_0^1 f(t)\,dt$$

对每个实无理数都成立. 提示:首先对

$$f(t)=\exp(2\pi ikt),\quad k=0,\pm1,\pm2,\cdots$$

来做这个习题.

14　计算

$$\min_{a,b,c}\int_{-1}^{1}|x^3-a-bx-cx^2|^2dx$$

并求出

$$\max\int_{-1}^{1}x^3g(x)\,dx$$

这里 g 满足下列限制

$$\int_{-1}^{1} g(x)dx=\int_{-1}^{1} xg(x)dx=\int_{-1}^{1} x^2g(x)dx=0;$$

$$\int_{-1}^{1} |g(x)|^2dx=1$$

15 计算

$$\min_{a,b,c}\int_0^{\infty} |x^3-a-bx-cx^2|^2e^{-x}dx$$

如习题14那样，叙述并解相应的极大值问题.

16 如果 $x_0\in H$ 而 M 是 H 的闭线性子空间，证明

$$\min\{\|x-x_0\|:x\in M\}=\max\{|(x_0,y)|:y\in M^{\perp},\|y\|=1\}$$

17 证明存在一个从[0, 1]到 H 中的一一映射 γ，使得当 $0\leqslant a\leqslant b\leqslant c\leqslant d\leqslant 1$ 时，$\gamma(b)-\gamma(a)$ 与 $\gamma(d)-\gamma(c)$ 正交（γ 可以称为“具有正交增量的曲线”）. 提示：取 $H=L^2$，并考虑[0, 1]的某些子集的特征函数.

18 给出定理 4.16 的一个直接的证明，即不依赖于 4.15 节的更一般的考虑的证明.

19 对所有 $s\in R^1$，$t\in R^1$，定义 $u_s(t)=e^{ist}$. 设 X 是这些函数 u_s 的全体有限线性组合所组成的复向量空间. 如果 $f\in X$，$g\in X$，证明

$$(f,g)=\lim_{A\to\infty}\frac{1}{2A}\int_{-A}^{A} f(t)\overline{g(t)}dt$$

存在. 说明这个内积使 X 成为一个 U 空间，其完备化 H 是一个不可分的 Hilbert 空间. 并证明 $\{u_s:s\in R^1\}$ 是 H 的一个极大规范正交集.

第五章　Banach 空间技巧的例

Banach 空间

5.1　在前一章中我们看到，关于三角级数的一些分析方面的结论，是怎样地通过一般 Hilbert 空间的实质上是几何方面的考虑，包括凸性，子空间，正交性，完备性等概念显示出来的．在分析中也有许多问题．当我们把它们放到一个适当选择的，抽象的格局中讨论时，解决起来就变得很容易．由于正交性是一种颇为特殊的东西，Hilbert 空间理论因而并不总是合适的．全体 Banach 空间组成的类却提供了更大的灵活性．这一章我们将探讨 Banach 空间的一些基本性质，并且将通过它对具体问题的应用来阐述它们．

5.2　定义　一个复向量空间 X 称为赋范线性空间，如果对每个 $x\in X$，对应一个非负实数 $\|x\|$，称为 x 的范数，使得

(a) 对所有的 x 和 $y\in X$，$\|x+y\|\leqslant\|x\|+\|y\|$，

(b) 若 $x\in X$ 而 α 为纯量，则 $\|\alpha x\|=|\alpha|\,\|x\|$，

(c) $\|x\|=0$ 蕴涵着 $x=0$．

由 (a)，三角不等式

$$\|x-y\|\leqslant\|x-z\|+\|z-y\|\qquad(x,y,z\in X)$$

成立．再把 (b) (取 $\alpha=0,\alpha=-1$) (c) 结合起来，就表明每个赋范线性空间都可以看成一个度量空间．x 和 y 之间的距离是 $\|x-y\|$．

Banach 空间就是一个赋范线性空间，它在由其范数所定义的度量下是完备的．

例如，每个 Hilbert 空间都是一个 Banach 空间．同样对 $1\leqslant$

$p\leqslant\infty$，每个 $L^p(\mu)$ 以 $\|f\|_p$ 为范数时(假定 a.e. 相等的函数看作是同一的)，以及 $C_0(X)$ 取上确界范数时，也都是 Banach 空间. 最简单的 Banach 空间当然就是取范数 $\|x\|=|x|$ 时的复数域本身.

我们可以同样地讨论实的 Banach 空间；除了所有的纯量都约定是实数之外，其定义刚好是相同的.

5.3 定义 考虑一个从赋范线性空间 X 到赋范线性空间 Y 的线性变换 A. 定义它的范数为

$$\|A\|=\sup\left\{\frac{\|Ax\|}{\|x\|}:x\in X,\ x\neq 0\right\} \tag{1}$$

若 $\|A\|<\infty$，则 A 称为有界线性变换.

在(1)中，$\|x\|$ 是 X 中的 x 的范数，$\|Ax\|$ 是 Y 中的 Ax 的范数；经常会发生几种范数同时出现的情况，我们从上下文将能搞清楚究竟指的是那一种.

我们发现，在(1)中可以限于考虑单位向量 x，即 $\|x\|=1$ 的那些 x 而不致于改变上确界，因为

$$\|A(\alpha x)\|=\|\alpha A(x)\|=|\alpha|\,\|Ax\| \tag{2}$$

同时可以发现，$\|A\|$ 是使不等式

$$\|Ax\|\leqslant\|A\|\,\|x\| \tag{3}$$

对每个 $x\in X$ 成立的最小的数.

下面的几何解释是有帮助的：A 将 X 中的闭单位球，即集

$$\{x\in X:\ \|x\|\leqslant 1\} \tag{4}$$

映到 Y 中的以 0 为中心，半径为 $\|A\|$ 的闭球之内.

取 Y 为复数域，我们得到一个重要的特款；在这种情况下，我们称之为有界线性泛函.

5.4 定理 对于从一个赋范线性空间 X 到赋范线性空间 Y 内的一个线性变换 A，下面三个条件的每一个都蕴涵着另外两个：

(a) A 是有界的.

(*b*) A是连续的.

(*c*) A在X的某个点连续.

证明 由于$\|A(x_1-x_2)\|\leqslant\|A\|\|x_1-x_2\|$，显然(*a*)蕴涵着(*b*)，而(*b*)蕴涵着(*c*)是不足道的. 假设 A 在 x_0 连续. 则对每个 $e>0$，可以找到一个 $\delta>0$，使$\|x-x_0\|<\delta$ 蕴涵着$\|Ax-Ax_0\|<e$. 换言之，$\|x\|<\delta$ 蕴涵着

$$\|A(x_0+x)-Ax_0\|<e$$

然而 A 的线性表明$\|Ax\|<e$. 因而$\|A\|\leqslant e/\delta$，故(*c*)蕴涵着(*a*).

Baire 定理的推论

5.5 对于 Banach 空间的完备性这一性质，通常所采用的方式与下面的一个关于完备度量空间的定理有关，这个定理在数学的其他领域也有很多应用. 它蕴涵着使 Banach 空间成为分析中的有用工具的三个最重要的定理中的两个：Banach-Steinhaus定理和开映射定理. 第三个是 Hahn-Banach 开拓定理，在这个定理中，完备性不起作用.

5.6 Baire 定理 如果X是完备度量空间，则每个由X的稠密开子集组成的可数族，其交在X中稠密.

特别(除在平凡的场合 $X=\varnothing$ 之外)，这个交是不空的. 这通常就是本定理的主要意义.

证明 假设$V_1,V_2,V_3,\cdots$是X中的稠密开集. 设W是X的任一开集. 我们必须证明，当 $W\neq\varnothing$ 时，$\bigcap V_n$在W中有一点.

设ρ是X的度量；我们记

$$S(x,r)=\{y\in X:\ \rho(x,y)<r\} \tag{1}$$

并设 $\bar{S}(x,r)$是 $S(x,r)$的闭包. [注：存在这样一种情况，$\bar{S}(x,r)$并不包含所有使 $\rho(x,y)\leqslant r$ 的点 y.]

由V_1稠密，$W\cap V_1$是非空开集，因此我们可以找出 x_1 和 r_1 使

(2) $$\bar{S}(x_1, r_1) \subset W \cap V_1, \text{ 且 } 0 < r_1 < 1$$

如果 $n \geqslant 2$ 且 x_{n-1} 和 r_{n-1} 都已选出，则 V_n 的稠密性表明 $V_n \cap S(x_{n-1}, r_{n-1})$ 是不空的，因此我们可以找出 x_n 和 r_n 使

(3) $$\bar{S}(x_n, r_n) \subset V_n \cap S(x_{n-1}, r_{n-1}) \quad \text{且 } 0 < r_n < \frac{1}{n}$$

由归纳法，这个过程产生出 X 中的一个序列 $\{x_n\}$. 当 $i > n$ 和 $j > n$ 时，构造方式表明 x_i 和 x_j 同在 $S(x_n, r_n)$ 内，故 $\rho(x_i, x_j) < 2r_n < 2/n$，从而 $\{x_n\}$ 是一个 Cauchy 序列. 因为 X 是完备的，便存在点 $x \in X$ 使 $x = \lim\limits_{n \to \infty} x_n$.

因为当 $i > n$ 时，x_i 都在闭集 $\bar{S}(x_n, r_n)$ 内，于是得知 x 位于每个 $\bar{S}(x_n, r_n)$ 内. 而(3)式表明 x 位于每个 V_n 内. 由(2)，$x \in W$. 这就完成了证明.

系 在完备度量空间中，任意可数个稠密的 G_δ 集之交还是稠密的 G_δ 集.

因为每个 G_δ 集是可数个开集之交，而可数多个可数集之并是可数的，从本定理即可得出系的结论.

5.7 Baire 定理有时也因为下述理由而称为类型定理.

集 $E \subset X$ 称为无处稠密，如果它的闭包 $\bar{E}$ 不包含 X 的非空开集. 无处稠密集的任意一个可数并称之为第一类型集；而 X 的其他的集就称为第二类型的(Baire 的说法). 定理 5.6 等价于说没有一个完备度量空间是第一类型的. 要看出这一点，只须在定理5.6的说法中取余集就够了.

5.8 Banach-Steinhaus 定理 假设 X 是 Banach 空间，Y 是赋范线性空间，而 $\{A_\alpha\}$ 是 X 到 Y 的一族有界线性变换，这里 α 取值于某个指标集 A 上，那末，或者存在 $M < \infty$，使对每个 $\alpha \in A$，

(1) $$\|A_\alpha\| \leqslant M$$

或者对所有属于 X 的某个稠密 G_δ 集的 x

(2) $$\sup\|A_\alpha x\|=\infty$$

用几何语言来说，这一命题另一种叙述如下：或者 Y 中有一个球 B(半径为 M 而中心为0)使每个 A_α 映 X 的单位球到 B 内，或者存在 $x\in X$(事实上是它们的整个稠密的 G_δ 集) 使 Y 中没有一个球能包含所有的 $A_\alpha x$.

这一定理有时也称为一致有界性原理.

证明 令

(3) $$\varphi(x)=\sup_{\alpha\in A}\|A_\alpha x\| \qquad (x\in X)$$

并令

(4) $$V_n=\{x:\varphi(x)>n\} \qquad (n=1,2,3,\cdots)$$

因为每个 A_α 都是连续的，并且 Y 的范数是 Y 上的连续函数(就象定理 4.6 的证明一样，它是三角不等式的一个直接推论)，每个函数 $x\to\|A_\alpha x\|$ 在 X 上连续. 因此 φ 是下半连续的，并且 V_n 是开集.

如果这些集中的一个，比如说 V_N 不稠密于 X，则存在一个 $x_0\in X$ 和 $r>0$，使 $\|x\|\leqslant r$ 蕴涵着 $x_0+x\notin V_N$；这就意味着 $\varphi(x_0+x)\leqslant N$，或

(5) $$\|A_\alpha(x_0+x)\|\leqslant N$$

对所有 $\alpha\in A$ 和满足 $\|x\|\leqslant r$ 的所有 x 都成立. 由 $x=(x_0+x)-x_0$，于是有

(6) $$\|A_\alpha x\|\leqslant\|A_\alpha(x_0+x)\|+\|A_\alpha x_0\|\leqslant 2N$$

并且得到(1)式对 $M=2N/r$ 成立.

另一种可能性是每个 V_n 在 X 中稠密. 在这种情况下，由 Baire 定理，$\bigcap V_n$ 是 X 中一个稠密 G_δ 集. 由于对每个 $x\in\bigcap V_n$，$\varphi(x)=\infty$. 证明完毕.

5.9 开映射定理 设 U 和 V 是Banach 空间 X 和 Y 的开单位

球. 对 X 到 Y 上的每一个有界线性变换 A, 相应地有一个 $\delta>0$, 使

(1) $$A(U)\supset\delta V$$

请注意假设中"到 Y 上"这个词! 符号 δV 表示集 $\{\delta y: y\in V\}$, 即所有 $\|y\|<\delta$ 的 $y\in Y$ 的集.

从(1)及 A 的线性得知, X 的每一个中心为 x_0 的开球的象都包含 Y 内一个中心在 Ax_0 的开球. 因此, 每个开集的象都是开的. 这就解释了本定理的名称.

下面是(1)的另一种说法: 对每个 y, $\|y\|<\delta$, 都对应有一个 x 满足 $\|x\|<1$, 使 $Ax=y$.

证明 给定 $y\in Y$. 存在 $x\in X$ 使 $Ax=y$; 若 $\|x\|<k$, 便有 $y\in A(kU)$. 因此 Y 是集 $A(kU)$ 当 $k=1, 2, 3, \cdots$ 时的并. 因为 Y 是完备的, Baire 定理蕴涵着存在一个非空开集 W 位于某个 $A(kU)$ 的闭包内. 这意味着 W 的每个点都是一个序列 $\{Ax_i\}$ 的极限点, 此处 $x_i\in kU$. 从现在起, k 和 W 都固定下来.

选取 $y_0\in W$ 和 $\eta>0$, 使得当 $\|y\|<\eta$ 时, $y_0+y\in W$. 对任一个这样的 y 都有 kU 中的序列 $\{x_i'\}$, $\{x_i''\}$ 使

(2) $$Ax_i'\to y_0,\quad Ax_i''\to y_0+y\qquad(i\to\infty)$$

取 $x_i=x_i''-x_i'$, 我们有 $\|x_i\|<2k$ 和 $Ax_i\to y$. 因为这对每一个 $\|y\|<\eta$ 的 y 成立, A 的线性表明当 $\delta=\eta/2k$ 时, 下述命题是真确的:

对每个 $y\in Y$ 和每个 $e>0$, 都对应一个 $x\in X$ 使

(3) $$\|x\|\leqslant\delta^{-1}\|y\|\quad 及\quad \|y-Ax\|<e$$

象在我们证明开始之前所说的一样, 除了在那里我们要求 $e=0$ 之外, 这几乎就是我们所要的结论.

固定 $y\in\delta V$ 和 $e>0$. 由(3)存在 x_1, 使 $\|x_1\|<1$, 且

(4) $$\|y-Ax_1\|<\frac{1}{2}\delta e$$

设 $x_1, \cdots, x_n$ 已选好, 使

(5) $$\|y-\Lambda x_1-\cdots-\Lambda x_n\|<2^{-n}\delta\varepsilon$$

利用(3)并以(5)式左边的向量代替其中的 y，就得到一个 x_{n+1}，使 $n+1$ 代替 n 时，(5)式成立，且

(6) $$\|x_{n+1}\|<2^{-n}\varepsilon \qquad (n=1,2,3,\cdots)$$

如果令 $s_n=x_1+\cdots+x_n$，(6)式表明 $\{s_n\}$ 是 X 中的Cauchy序列. 由 X 完备，存在 $x\in X$ 使 $s_n\to x$. 不等式 $\|x_1\|<1$ 和(6)式一道保证了 $\|x\|<1+\varepsilon$. 由 Λ 连续，$\Lambda s_n\to\Lambda x$. 由(5)，$\Lambda s_n\to y$. 因此

$$\Lambda x=y.$$

现在我们已经证明了，对每个 $\varepsilon>0$，

(7) $$\Lambda((1+\varepsilon)U)\supset\delta V$$

或者

(8) $$\Lambda(U)\supset(1+\varepsilon)^{-1}\delta V$$

当取遍所有 $\varepsilon>0$ 时，(8) 式右边的这些集的并就是 δV. 这就证明了(1).

5.10 定理 如果 X 和 Y 都是 Banach 空间，Λ 是 X 到 Y 上的一一的有界线性变换，则存在 $\delta>0$，使

(1) $$\|\Lambda x\|\geqslant\delta\|x\| \qquad (x\in X)$$

换句话说，Λ^{-1} 是 Y 到 X 上的一个有界线性变换.

证明 如果 δ 象定理 5.9 的叙述中那样选择，那末，该定理的结论和 Λ 是一一的这一事实结合起来就表明，$\|\Lambda x\|<\delta$ 蕴涵着 $\|x\|<1$. 因此，$\|x\|\geqslant1$ 蕴涵着 $\|\Lambda x\|\geqslant\delta$，于是(1)式得证.

变换 Λ^{-1} 在 Y 上可以这样定义：若 $y=\Lambda x$ 时，则 $\Lambda^{-1}y=x$. 容易验证 Λ^{-1} 是线性的，而(1)则蕴涵着 $\|\Lambda^{-1}\|\leqslant1/\delta$.

连续函数的 Fourier 级数

5.11 一个收敛问题 对每个 $f\in C(T)$，f 的 Fourier 级数在每一点 x 都收敛于 $f(x)$，对吗？

让我们回忆一下，f 的 Fourier 级数在点 x 的第 n 个部分和由

$$(1)\qquad s_n(f;x)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)D_n(x-t)\,dt \qquad (n=0,1,2,\cdots)$$

给出，这里

$$(2)\qquad D_n(t)=\sum_{k=-n}^{n}e^{ikt}$$

这直接从公式 4.26(1) 和 4.26(3) 得出.

问题在于判定

$$(3)\qquad \lim_{n\to\infty}s_n(f;x)=f(x)$$

是否对每个 $f\in C(T)$ 和每个实数 x 成立. 我们在 4.26 一节看到部分和按 L^2 范数收敛于 f，因此定理 3.12 蕴涵着每个 $f\in L^2(T)$ [从而对每个 $f\in C(T)$ 亦然]都是整个部分和序列的某个子序列的 a. e. 点态极限. 但这并没有回答现在提出的问题.

我们将会看到，Banach-Steinhaus 定理否定地回答了这个问题. 令

$$(4)\qquad s^*(f;x)=\sup_n|s_n(f;x)|$$

开始取 $x=0$，并定义

$$(5)\qquad \Lambda_n f=s_n(f;0) \qquad (f\in C(T);\ n=1,2,3,\cdots)$$

我们知道 $C(T)$ 对于上确界范数 $\|f\|_\infty$ 是一个 Banach 空间. 从(1)得知，每个 Λ_n 都是 $C(T)$ 上的有界线性泛函，其范数

$$(6)\qquad \|\Lambda_n\|\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}|D_n(t)|\,dt=\|D_n\|_1$$

我们断言

$$(7)\qquad \text{当 } n\to\infty \text{ 时，}\|\Lambda_n\|\to\infty$$

这可以通过指出(6)式中等式成立，以及

$$(8)\qquad \text{当 } n\to\infty \text{ 时，}\|D_n\|_1\to\infty$$

而予以证明.

将(2)式分别乘以 $e^{it/2}$ 和 $e^{-it/2}$，并从所得的两个等式中的一个减去另一个，得到

$$D_n(t)=\frac{\sin\left(n+\frac{1}{2}\right)t}{\sin(t/2)} \tag{9}$$

因为对所有实数 x，$|\sin x|\leqslant|x|$，(9)式指出

$$\|D_n\|_1>\frac{2}{\pi}\int_0^{\pi}\left|\sin\left(n+\frac{1}{2}\right)t\right|\frac{dt}{t}=\frac{2}{\pi}\int_0^{(n+1/2)\pi}|\sin t|\frac{dt}{t}$$

$$>\frac{2}{\pi}\sum_{k=1}^{n}\frac{1}{k\pi}\int_{(k-1)\pi}^{k\pi}|\sin t|\,dt=\frac{4}{\pi^2}\sum_{k=1}^{n}\frac{1}{k}\to\infty$$

这就证明了(8).

其次，固定 n，并且若 $D_n(t)\geqslant0$，令 $g(t)=1$；若 $D_n(t)<0$，令 $g(t)=-1$. 存在 $f_j\in C(T)$ 使 $-1\leqslant f_j\leqslant1$，且当 $j\to\infty$ 时，对每个 t，$f_j(t)\to g(t)$. 由控制收敛定理，

$$\lim_{j\to\infty}A_n(f_j)=\lim_{j\to\infty}\frac{1}{2\pi}\int_{-\pi}^{\pi}f_j(-t)D_n(t)dt$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}g(-t)D_n(t)dt=\|D_n\|_1$$

这样，(6)式中的等式成立，并且证明了(7)式.

由(7)成立，Banach-Steinhaus 定理现在就断言 $s^*(f;0)=\infty$ 对 $C(T)$ 中某个稠密 G_δ 集中的每个 f 成立.

我们选 $x=0$ 只是为了方便. 很明显，同样的结论对每个 x 都是成立的.

对每个实数 x 都对应一个集 $E_x\subset C(T)$，它是 $C(T)$ 中一个稠密 G_δ 集，使每个 $f\in E_x$，$s^*(f;x)=\infty$.

特别，每个 $f\in E_x$ 的 Fourier 级数在 x 是发散的. 因此得到了我们提出的那个问题的否定回答.

注意到下述之点是很有趣的，即上面的结论还可以通过 Baire

定理的另外一个应用而得到强化. 假如我们取可数多个点 x_i, 并且令 E 为相应的集

$$E_{x_i}\subset C(T)$$

的交. 由 Baire 定理, E 是 $C(T)$ 中的一个稠密 G_δ 集. 对于每个 $f\in E$, 在每个点 x_i 都有

$$s^*(f;x_i)=\infty$$

对每个 f, $s^*(f;x)$ 是 x 的下半连续函数. 这是因为(4)式表示了它是一族连续函数的上确界. 因此对每个 f, $\{x: s^*(f;x)=\infty\}$ 是 R^1 中的 G_δ 集. 如果取上述点 x_i 使它们的并在 $(-\pi, \pi)$ 上稠密. 我们便得如下结果:

5.12 定理 存在一个集 $E\subset C(T)$, 它在 $C(T)$ 中是稠密的 G_δ 集, 并具有下列性质: 对每个 $f\in E$, 集

$$Q_f=\{x: s^*(f;x)=\infty\}$$

是 R^1 上的稠密 G_δ 集.

如果我们证实了 E 以及每个 Q_f 都是不可数集, 那将是很有趣的.

5.13 定理 在不存在孤立点的完备度量空间中, 没有一个可数稠密集是 G_δ 集.

证明 设 x_k 是 X 的可数稠密集 E 的点. 假定 E 是 G_δ 集. 那末, $E=\bigcap V_n$, 这里 V_n 是稠密的开集. 设

$$W_n=V_n-\bigcup_{k=1}^{n}\{x_k\}$$

则每个 W_n 还是稠密开集. 但 $\bigcap W_n=\varnothing$. 这与 Baire 定理矛盾.

注: 稍微修改一下 Baire 定理的证明, 就可证明当 X 如上假设时, 其每一个稠密 G_δ 集都包含一个完全集.

L^1-函数的 Fourier 系数

5.14 象 4.26 节一样, 对每个 $f\in L^1(T)$ 我们对应 Z 上的一个

函数 $\hat{f}$，由

$$(1)\qquad \hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt \qquad (n\in Z)$$

定义．容易证明，对每个 $f\in L^1$，当 $|n|\to\infty$ 时，$\hat{f}(n)\to 0$．因为我们知道 $C(T)$ 是稠于 $L^1(T)$（定理 3.14）而三角多项式是稠于 $C(T)$ 的（定理 4.25）．这就是说如果 $\varepsilon>0$，$f\in L^1(T)$，就存在 $g\in C(T)$ 和三角多项式 P，使 $\|f-g\|_1<\varepsilon$ 和 $\|g-P\|_\infty<\varepsilon$．

由于

$$\|g-P\|_1\leqslant\|g-P\|_\infty$$

便得到 $\|f-P\|_1<2\varepsilon$，而当 $|n|$ 足够大（依赖于 P）时，

$$(2)\qquad |\hat{f}(n)|=\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}\{f(t)-P(t)\}e^{-int}dt\right|\leqslant\|f-P\|_1<2\varepsilon$$

这样，当 $n\to\pm\infty$ 时，$\hat{f}(n)\to 0$．此即所谓的 Riemann-Lebesgue 引理．

我们想提出的问题是，它的逆命题是否是正确呢？也就是说，如果 $\{a_n\}$ 是一个复数序列，当 $n\to\pm\infty$ 时，$a_n\to 0$，那末，有没有一个 $f\in L^1(T)$，使得对所有 $n\in Z$，$\hat{f}(n)=a_n$？换言之，有没有一个类似于Riesz-Fischer 定理的结论在现在的情况下保持正确？

借助于开映射定理容易对此作出（否定的）回答．

设 φ 是使得 $n\to+\infty$ 时 $\varphi(n)\to 0$ 的 Z 上的复函数，c_0 是由全体这样的 φ 所组成的空间，并赋予上确界范数

$$(3)\qquad \|\varphi\|_\infty=\sup\{|\varphi(n)|:n\in Z\}$$

则容易看出 c_0 是一个 Banach 空间．事实上，如果我们规定 Z 的每个子集都是开集，那末，Z 是一个局部紧 Hausdorff 空间，而 c_0 就是 $C_0(Z)$．

下述定理包含了我们的问题的答案：

5.15 定理 映射 $f\to\hat{f}$ 是 $L^1(T)$ 到 c_0 内（不是到上）的一个

一一有界线性变换.

证明 用 $Af=\hat{f}$ 定义 A. A 显然是线性的. 我们刚才已证明了 A 映 $L^1(T)$ 到 c_0 内，而公式 5.14(1) 表明 $|\hat{f}(n)|\leqslant\|f\|_1$，所以 $\|A\|\leqslant 1$(实际上，$\|A\|=1$，这只须取 $f=1$ 即可看出). 现在我们来证明 A 是一一的. 假设 $f\in L^1(T)$，而对每个 $n\in Z$，$\hat{f}(n)=0$. 那末，当 g 是任意一个三角多项式时，

$$\int_{-\pi}^{\pi} f(t)g(t)dt=0 \tag{1}$$

由定理 4.25 和控制收敛定理，对每个 $g\in C(T)$，(1)式也成立. 与 Lusin 定理的系联系起来，再一次应用控制收敛定理，得出(1)式对 T 中任意可测集的特征函数 g 也成立的结论. 现在，定理1.39 (b)表明 $f=0$ a. e.

假如 A 的值域是整个 c_0. 定理 5.10 将蕴涵着存在一个 $\delta>0$，使对每个 $f\in L^1(T)$

$$\|\hat{f}\|_\infty\geqslant\delta\|f\|_1 \tag{2}$$

但是，若 $D_n(t)$ 如 5.11 节一样定义，这时，对 $n=1,2,3,\cdots$，$D_n\in L^1(T)$，$\|\hat{D}_n\|_\infty=1$. 而 $n\to\infty$ 时，$\|D_n\|_1\to\infty$. 因此不存在 $\delta>0$ 使不等式

$$\|\hat{D}_n\|_\infty\geqslant\delta\|D_n\|_1 \tag{3}$$

对每个 n 成立.

定理证毕.

Hahn-Banach 定理

5.16 定理 如果 M 是赋范线性空间 X 的一个子空间，而 f 是 M 上的一个有界线性泛函，则 f 可以开拓为 X 上的一个有界线性泛函 F，使 $\|F\|=\|f\|$.

注意，M 不一定要是闭的.

在着手证明之前，看来要作些说明. 首先，我们说 F 是 f 的一个开拓(在最一般的情况下)是指 F 的定义域包含了 f 的定义域，而且对于属于 f 的定义域的所有 x, $F(x)=f(x)$. 其次，范数 $\|F\|$ 和 $\|f\|$ 是各自按 F 和 f 的定义域来计算的；显然

$$\|f\|=\sup\left\{\frac{|f(x)|}{\|x\|}:x\in M\right\},\qquad \|F\|=\sup\left\{\frac{|F(x)|}{\|x\|}:x\in X\right\}$$

第三点说明是与纯量域有关的. 迄今，所有的内容都是就复数域来讲的，但完全可以用实数域取代复数域而不必改变陈述和证明. Hahn-Banach 定理在两种场合都正确；不过，它主要是以"实"形式的定理出现. 当 Banach 写出他的经典著作"Opérations Linéares"时对复的情况没有证明，其主要原因可能是在他的著作中只考虑实数情况.

在这里引进一些临时性的名词术语是有益的. 回忆一下，V 是一个复(实)向量空间，如果当 x 和 $y\in V$ 时 $x+y\in V$，且对所有复(实)数 α, $\alpha x\in V$. 很明显，每个复向量空间同时也是一个实向量空间. 复向量空间 V 上的复函数 φ 是复线性泛函，如果对所有 x 和 $y\in V$ 及所有复数 α,

(1) $$\varphi(x+y)=\varphi(x)+\varphi(y)\quad 和\quad \varphi(\alpha x)=\alpha\varphi(x)$$

成立. 在复(实)向量空间 V 上的一个实值函数 φ 是一个实线性泛函，如果(1)式对所有实数 α 成立.

如果 u 是复线性泛函 f 的实部，即如果对所有 $x\in V$, $u(x)$ 是复数 $f(x)$ 的实部，则容易看出 u 是一个实线性泛函. 下述 f 与 u 之间的关系是成立的：

5.17 命题 设 V 是复向量空间.

(*a*) 如果 u 是 V 上的复线性泛函 f 的实部，则

(1) $$f(x)=u(x)-iu(ix)\qquad (x\in V)$$

(*b*) 如果 u 是 V 上的一个实线性泛函，而 f 由(1)式定义，则

f 是 V 上的一个复线性泛函。

(c) 如果 V 是赋范线性空间，而 f 与 u 有(1)式的关系，则 $\|f\| = \|u\|$.

证明 若 α 与 β 是实数而 $z=\alpha+i\beta$，则 iz 的实部是 $-\beta$. 对所有复数 z 这就得出了等式

$$z = \mathrm{Re}\,z - i\,\mathrm{Re}(iz) \tag{2}$$

因为

$$\mathrm{Re}(if(x)) = \mathrm{Re}\,f(ix) = u(ix) \tag{3}$$

令 $z=f(x)$，(1)便由(2)式得出.

在(b)的假设下，显然，$f(x+y)=f(x)+f(y)$，且对所有的实数 α，$f(\alpha x)=\alpha f(x)$. 但同样也有

$$f(ix) = u(ix) - iu(-x) = u(ix) + iu(x) = if(x) \tag{4}$$

这就证明了 f 是复线性的.

由于 $|u(x)| \leqslant |f(x)|$，我们有 $\|u\| \leqslant \|f\|$. 另一方面，对每个 $x \in V$ 都对应有一个复数 α，$|\alpha|=1$，使 $\alpha f(x) = |f(x)|$. 因此，

$$|f(x)| = f(\alpha x) = u(\alpha x) \leqslant \|u\| \cdot \|\alpha x\| = \|u\| \cdot \|x\| \tag{5}$$

这就证明了 $\|f\| \leqslant \|u\|$.

5.18 定理 5.16的证明 我们首先假定 X 是实赋范线性空间，从而 f 是 M 上的一个实有界线性泛函. 若 $\|f\|=0$，则所求的开拓就是 $F=0$. 去掉这种情况，不失一般性，可假定 $\|f\|=1$.

选 $x_0 \in X$，$x_0 \notin M$，并设 M_1 是由 M 和 x_0 张成的向量空间，于是 M_1 由所有形如 $x+\lambda x_0$ 的向量组成，这里 $x \in M$ 而 λ 是实数. 如果定义 $f_1(x+\lambda x_0) = f(x) + \lambda\alpha$，这里 α 是任意固定的一个实数，容易验证，这样得到从 f 到 M_1 上的一个线性泛函开拓. 问题是要选择 α 使开拓后的线性泛函仍然有范数 1. 这也就是要求

$$|f(x+\lambda x_0)| \leqslant \|x+\lambda x_0\| \qquad (x \in M, \lambda \text{ 是实数}) \tag{1}$$

用 $-\lambda x$ 代替 x，并以 $|\lambda|$ 除(1)式的两边. 这个要求于是变为

(2) $$|f(x)-\alpha|\leqslant\|x-x_0\| \qquad (x\in M)$$

即，对所有 $x\in M$，$A_x\leqslant\alpha\leqslant B_x$，这里

(3) $$A_x=f(x)-\|x-x_0\| \quad 而 \quad B_x=f(x)+\|x-x_0\|$$

这样的 α 当且仅当所有的区间$[A_x, B_x]$有公共点，也就是说，当且仅当对所有的 x 和 $y\in M$，

(4) $$A_x\leqslant B_y$$

时才会存在．但是

(5) $$f(x)-f(y)=f(x-y)\leqslant\|x-y\|\leqslant\|x-x_0\|+\|y-x_0\|$$

因此由(3)可得出(4).

现在我们已经证明了存在一个 f 的到 M_1 上的保范开拓 f_1.

设 $\mathscr{P}$ 是全体序对(M', f')的集族，其中 M' 是 X中的包含M的一个子空间，而 f' 是 f 的到 M' 上的实线性开拓，具有$\|f'\|=1$．我们声称$(M', f')\leqslant(M'', f'')$意味着 $M'\subset M''$，且对所有的 $x\in M'$，$f'(x)=f''(x)$，而将 $\mathscr{P}$ 半序化．关于半序的公理显然是满足的．由于 $\mathscr{P}$ 包含(M, f)，所以不是空集，于是 Hausdorff 极大性定理断言 $\mathscr{P}$ 有一个极大的全序子集族 Ω.

设 Φ 是使$(M', f')\in\Omega$ 的所有 M' 的集族．那末，按照集的包含关系，Φ是全序的．因此Φ 的所有元素之并 $\widetilde{M}$ 是 X的一个子空间．(注意，一般地说两个子空间之并不是一个子空间．R^3中两个通过原点的平面就是一例．)如果 $x\in\widetilde{M}$，则对某个 $M'\in\Phi$，有 $x\in M'$；定义 $F(x)=f'(x)$，这里 f' 是序对$(M', f')\in\Omega$ 中相应的函数．Ω 中半序的定义表明，只要 M' 包含 x'，究竟选择那一个 M' 去确定 $F(x)$是无关紧要的.

现在容易验证 F 是 $\widetilde{M}$ 上的线性泛函，且$\|F\|=1$．假如 $\widetilde{M}$ 是 X的一个真子空间，证明的第一部分就给出一个 F 的进一步的开拓，这将与 Ω 的极大性矛盾．因此，$\widetilde{M}=X$．这就完成了实纯量场合的证明.

现在，如果 f 是复赋范线性空间 X 的子空间 M 上的一个复线性泛函，设 u 是 f 的实部. 运用实 Hahn-Banach 定理将 u 开拓为 X 上的实线性泛函 U，使 $\|U\|=\|u\|$，并定义

$$F(x)=U(x)-iU(ix) \qquad (x\in X) \tag{6}$$

由命题 5.17，F 是 f 的一个复线性开拓，且

$$\|F\|=\|U\|=\|u\|=\|f\|$$

这就完成了证明.

让我们提出 Hahn-Banach 定理的两个重要的推论：

5.19 定理 设 M 是赋范线性空间 X 的一个线性子空间，$x_0\in X$. 则 x_0 属于 M 的闭包 $\overline{M}$，当且仅当不存在 X 上的有界线性泛函 f 使所有 $x\in M$，$f(x)=0$ 而 $f(x_0)\neq 0$.

证明 如果 $x_0\in\overline{M}$，f 是 X 上的有界线性泛函，且对所有 $x\in M$，$f(x)=0$. 则 f 的连续性表明同样有 $f(x_0)=0$.

反之，设 $x_0\notin\overline{M}$. 则存在 $\delta>0$，使所有 $x\in M$，$\|x-x_0\|>\delta$. 设 M' 是由 M 和 x_0 生成的子空间，并且，当 $x\in M$，λ 是纯量时，定义 $f(x+\lambda x_0)=\lambda$. 因为

$$\delta|\lambda|\leqslant|\lambda|\,\|x_0+\lambda^{-1}x\|=\|\lambda x_0+x\|$$

可以看出 f 是 M' 上的线性泛函，其范数至多是 δ^{-1}. 同时在 M 上，$f(x)=0$，$f(x_0)=1$. Hahn-Banach 定理允许我们将这个 f 从 M' 开拓到 X 上.

5.20 定理 如果 X 是一个赋范线性空间，且 $x_0\in X$，$x_0\neq 0$，则存在一个 X 上的范数为 1 的有界线性泛函 f 使 $f(x_0)=\|x_0\|$.

证明 设 $M=\{\lambda x_0\}$，并定义 $f(\lambda x_0)=\lambda\|x_0\|$. 则 f 是 M 上的范数为 1 的线性泛函. 这时可以再次应用 Hahn-Banach 定理.

5.21 评注 如果 X 是一个赋范线性空间，设 X^* 是 X 上全体有界线性泛函的集族. 如果线性泛函的加法和纯量乘法以明显的方式予以定义，容易看出 X^* 仍是一个赋范线性空间. 事实上，X^*

是一个 Banach 空间. 这只要从纯量域是一个完备度量空间这一事实就可以得出. 我们把 X^*的这些性质的验证留作习题.

定理 5.20 的推论之一是, 当 X 是非平凡的向量空间(即 X 有非 0 向量)时, X^*也是非平凡的. 事实上, X^*在 X 上是分离两点的. 这意味着, 如果在 X 中 $x_1 \neq x_2$, 则存在一个 $f \in X^*$ 使 $f(x_1) \neq f(x_2)$. 为了证明这点, 只须在定理 5.20 中取 $x_0 = x_2 - x_1$.

另外一个推论是, 对 $x \in X$,

$$\|x\| = \sup\{|f(x)| : f \in X^*, \|f\| = 1\}$$

因此, 对固定的 $x \in X$, 映射 $f \to f(x)$ 是 X^*上的一个范数为 $\|x\|$ 的有界线性泛函.

X 与 X^*(所谓 X 的"对偶空间")之间的这一相互对应, 构成了人们熟知的泛函分析这一数学分支的一个很大部分的基础.

Poisson 积分的一种抽象处理

5.22 要将 Hahn-Banach 定理成功地应用于具体问题, 当然有赖于所论及的赋范线性空间上的有界线性泛函的知识. 迄今为止, 我们还只是确定了一个 Hilbert 空间上的有界线性泛函. (在那种情况下, Hahn-Banach 定理的一种更简单的证明方法是存在的, 参看习题 6), 同时我们也知道 $C_c(X)$ 上的正线性泛函.

现在我们将描述一种一般场合, 在这种场合下, 前面提到的那种泛函很自然地就会出现.

设 K 是一个紧 Hausdorff 空间, H 是 K 的紧子集, 并设 A 是 $C(K)$ 的一个子空间, 使 $1 \in A$(1 表示对每个 $x \in X$ 对应于数 1 的函数), 并且使

(1) $$\|f\|_K = \|f\|_H \qquad (f \in A)$$

这里我们采用记号

(2) $$\|f\|_E = \sup\{|f(x)| : x \in E\}$$

由于 5.23 节讨论过的例子，H 有时称为对应于空间 A 的 K 的边界.

如果 $f\in A, x\in K$, (1)说明

$$|f(x)|\leqslant\|f\|_H \tag{3}$$

特别，对每个 $y\in H$，若 $f(y)=0$，则对所有 $x\in K, f(x)=0$. 因此，若 $f_1, f_2\in A$，且对每个 $y\in H, f_1(y)=f_2(y)$，则 $f_1=f_2$. 要看出这点，只须取 $f=f_1-f_2$ 即可.

设 M 是 H 上的，由所有的 A 的元素限制在 H 上而成的函数的集. 显然，M 是 $C(H)$ 的一个子空间. 前面的评注表明 M 的每一个元素都有唯一的一个开拓使它是 A 的一个元素. 这样我们就得到 M 和 A 之间的一个自然的一一对应，而且由(1)，它是保范的. 因此，当我们用同一字母来表示 A 的元素以及它在 H 上的限制时，是决不会发生混淆的.

固定一点 $x\in K$. 不等式(3)表明映射 $f\to f(x)$ 是 M 上的范数为 1 的有界线性泛函[因为 $f=1$ 时(3)式成为等式]. 由 Hahn-Banach 定理，存在一个 $C(H)$ 上的，范数为 1 的线性泛函 Λ，使

$$\Lambda f=f(x)\qquad(f\in M) \tag{4}$$

我们断言性质

$$\Lambda 1=1,\quad \|\Lambda\|=1 \tag{5}$$

蕴涵着 Λ 是 $C(H)$ 上的正线性泛函.

为了证明这一点，假设 $f\in C(H)$，$0\leqslant f\leqslant 1$. 令 $g=2f-1$ 并令 $\Lambda g=\alpha+i\beta$. 这里 α, β 是实数. 注意 $-1\leqslant g\leqslant 1$，故 $|g+ir|^2\leqslant 1+r^2$ 对每个实常数 r 成立. 因此(5)蕴涵着

$$(\beta+r)^2\leqslant|\alpha+i(\beta+r)|^2=|\Lambda(g+ir)|^2\leqslant 1+r^2 \tag{6}$$

这样，对每个 r 有 $\beta^2+2r\beta\leqslant 1$，从而必然有 $\beta=0$. 因为 $\|g\|_H\leqslant 1$，我们便有 $|\alpha|\leqslant 1$；所以

(7) $$Af=\frac{1}{2}A(1+g)=\frac{1}{2}(1+\alpha)\geqslant 0$$

现在可以应用定理 2.14. 它表明在H上存在一个正则的、正的 Borel 测度 μ_x, 使

(8) $$Af=\int_H f d\mu_x \qquad (f\in C(H))$$

特别, 我们得到表示式

(9) $$f(x)=\int_H f d\mu_x \qquad (f\in A)$$

这样, 我们已经证明了, 对每个 $x\in K$, 在"边界"H上都有一个正的测度 μ_x, 它在(9)式对每个 $f\in A$ 都成立这种意义下"表示"了 x.

注意, A 唯一地决定了 μ_x; 然而没有理由期望 Hahn-Banach 开拓是唯一的. 因此一般说来, 关于这个表示测度的唯一性, 我们没有更多的话可说. 但是, 象即将看到的那样, 在某种特殊的场合下是能够得出唯一性的.

5.23 为了看出前面那种场合的例子, 设 $U=\{z:|z|<1\}$是复平面上的开单位圆盘. 令 $K=\bar{U}$(闭单位圆盘), 并取H为U的边界 T. 我们断言每个多项式 f, 即每个形如

(1) $$f(z)=\sum_{n=0}^{N}a_n z^n$$

的函数, 这里 $a_0, \cdots, a_N$ 是复数, 都满足关系式

(2) $$\|f|_U\|=\|f\|_T$$

(注意, f 的连续性表明$|f|$在U上的上确界与在$\bar{U}$上的上确界相同.)

因为 $\bar{U}$ 是紧的, 存在 $z_0\in\bar{U}$ 使 $|f(z_0)|\geqslant|f(z)|$对所有的 $z\in\bar{U}$ 成立. 假设 $z_0\in U$, 于是

(3) $$f(z)=\sum_{n=0}^{N}b_n(z-z_0)^n$$

且当 $0<r<1-|z_0|$ 时，我们得到

$$\sum_{n=0}^{N}|b_n|^2r^{2n}=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(z_0+re^{i\theta})|^2d\theta$$

$$\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(z_0)|^2d\theta=|b_0|^2$$

从而 $b_1=b_2=\cdots=b_N=0$；即 f 是常数．这样，对每个非常数的多项式 f，$z_0\in T$．这就证明了(2)．

（我们刚才证明了最大模定理的一种特殊情况；稍后我们将看到这是所有全纯函数的一个重要性质．）

5.24 Poisson 积分 设 A 是 $C(\bar{U})$（象上面一样，这里的 $\bar{U}$ 是闭单位圆盘）的任意一个子空间，使 A 包含全体多项式，并且使得

(1) $$\|f\|_U=\|f\|_T$$

对每个 $f\in A$ 成立．我们并不排除 A 刚好由多项式组成的可能性，但 A 也可以更大一些．

将 5.22 节的所得的一般结论应用于 A，证明对每个 $z\in U$，都对应于 T 上的正 Borel 测度 μ_z，使得

(2) $$f(z)=\int_T f\,d\mu_z \qquad (f\in A)$$

（对于 $z\in T$ 也同样成立，但这是平凡的：μ_z 简单地说就是集中于点 z 的单位质量．）

现在我们固定 $z\in U$，并记 $z=re^{i\theta}$，$0\leqslant r<1$，θ 是实数．

若对 $n=0,1,2,\cdots$，$u_n(w)=w^n$，则 $u_n\in A$．因此(2)表明

(3) $$r^ne^{in\theta}=\int_T u_n\,d\mu_z \qquad (n=0,1,2,\cdots)$$

因为在 T 上 $u_{-n}=\bar{u}_n$，(3)式导出

$$(4)\qquad \int_T u_n d\mu_z = r^{|n|}e^{in\theta} \qquad (n=0,\ \pm 1,\ \pm 2,\ \cdots)$$

这就提示我们注意实函数

$$(5)\qquad P_r(\theta-t)=\sum_{n=-\infty}^{\infty} r^{|n|}e^{in(\theta-t)} \qquad (t\text{ 是实数})$$

因此

$$(6)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t)\,e^{int}dt = r^{|n|}e^{in\theta} \qquad (n=0,\ \pm 1,\ \pm 2,\ \cdots)$$

注意，级数(5)由收敛的几何级数 $\Sigma r^{|n|}$ 所控制，所以将该级数代入(6)并且逐项积分以给出(6)，这是合法的．比较(4)和(6)，当 $f=u_n$ 时，给出

$$(7)\qquad \int_T f d\mu_z = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})P_r(\theta-t)dt$$

因此对每个三角多项式 f 也成立．现在定理 4.25 蕴涵着(7)式对每个 $f\in C(T)$ 成立．[这表明 μ_z 是由(2)式唯一决定的，为什么？]

特别，当 $f\in A$ 时(7)式成立．因此(2)式也给出表达式

$$(8)\qquad f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(e^{it})P_r(\theta-t)dt \qquad (f\in A)$$

级数(5)明显地可以求和，因为它是

$$1+2\sum_{1}^{\infty}(ze^{-it})^n=\frac{e^{it}+z}{e^{it}-z}=\frac{1-r^2+2ir\sin(\theta-t)}{|1-ze^{-it}|^2}$$

的实部．这样

$$(9)\qquad P_r(\theta-t)=\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}$$

这就是所谓的"Poisson 核"．注意，当 $0\leqslant r<1$ 时 $P_r(\theta-t)\geqslant 0$.

现在对所证明的予以总结：

5.25 定理 假设 A 是闭单位圆盘 $\overline{U}$ 上的连续复函数的一个向量空间，如果 A 包含全体多项式，并且对每个 $f\in A$

(1) $$\sup_{z\in U}|f(z)|=\sup_{z\in T}|f(z)|$$

(这里 T 是单位圆周，即 U 的边界)，则 Poisson 积分表达式

(2) $$f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}f(e^{it})dt \qquad (z=re^{i\theta})$$

对每个 $f\in A$, $z\in U$ 都真确.

习　题

1　设 X 由 a 和 b 两个点组成，令 $\mu(\{a\})=\mu(\{b\})=\frac{1}{2}$，并设 $L^p(\mu)$ 是所得的实 L^p 空间．指定 X 上的每个实函数恒等于平面中的点 $(f(a), f(b))$，并对 $0<p\leqslant\infty$ 画出 $L^p(\mu)$ 的单位球的草图．注意，当且仅当 $1\leqslant p\leqslant\infty$ 时它们是凸的．对哪个 p 其单位球是一个正方形？是一个圆形？如果 $\mu(\{a\})\neq\mu(\{b\})$，这时的情况与前者有何不同？

2　证明在每个赋范线性空间中的(开或闭)单位球都是凸的．

3　如果 $1<p<\infty$，证明 $L^p(\mu)$ 的单位球是严格凸的；这表示当

$$\|f\|_p=\|g\|_p=1,\ f\neq g,\ h=\frac{1}{2}(f+g)$$

时，$\|h\|_p<1$．(从几何角度看，这个球的表面决不包含直线．)说明这对每一个 $L^1(\mu)$, $L^\infty(\mu)$ 和 $C(X)$ 是不成立的．(除开仅由一点组成的空间之类的不足道的场合．)

4　假设 C 是 $[0,1]$ 上全体连续函数的空间，赋以上确界范数．设 M 由所有使

$$\int_0^{\frac{1}{2}}f(t)dt-\int_{\frac{1}{2}}^1 f(t)dt=1$$

的 $f\in C$ 组成．证明 M 是 C 的一个闭凸子集，它不包含最小范数的元素．

5　设 M 是所有就 Lebesgue 测度而言的，使得

$$\int_0^1 f(t)dt=1$$

的 $f\in L^1([0,1])$ 的集．证明 M 是 $L^1([0,1])$ 的闭凸子集，它包含无限多个最小范数的元素．(试与习题 4 以及定理 4.10 相比较．)

6　设 f 是 Hilbert 空间 H 的子空间 M 上的有界线性泛函．证明 f 有一

个到H上的有界线性泛函的唯一的保范开拓，而且这个开拓在 $M^{\perp}$ 上等于零.

7　在某个 $L^1(\mu)$的某个子空间上构造一个有界线性泛函，使它有两个(从而是无限多个)不同的到 $L^1(\mu)$的保范线性开拓.

8　设 X是一个赋范线性空间，如在 5.21 节中那样，X^* 是它的对偶空间，具有范数

$$\|f\|=\sup\{|f(x)|:\|x\|\leqslant 1\}$$

(*a*) 证明 X^* 是 Banach 空间.

(*b*) 证明映射 $f\to f(x)$对每个 $x\in X$ 都是 X^* 上的有界线性泛函，其范数为$\|x\|$. (这给出了一个X到它的"第二对偶"X^{**}，即 X^* 的对偶空间的一个自然的嵌入.)

(*c*) 当$\{x_n\}$是X中的一个序列，使对每个 $f\in X^*$，$\{f(x_n)\}$都有界时，证明$\{\|x_n\|\}$也是有界的.

9　设 c_0, l^1, l^∞ 是全体复数序列 $x=\{\xi_i\}$, $i=1,2,3,\cdots$, 组成的Banach空间，定义如下：

$$x\in l^1 \text{ 当且仅当} \|x\|_1=\Sigma|\xi_i|<\infty.$$

$$x\in l^\infty \text{ 当且仅当} \|x\|_\infty=\sup|\xi_i|<\infty.$$

c_0 是使 $i\to\infty$ 时 $\xi_i\to 0$ 的所有 $x\in l^\infty$ 组成的 l^∞ 的子空间.

证明下列四个命题.

(*a*) 如果 $y=\{\eta_i\}\in l^1$，而对每个 $x\in c_0$，$Ax=\Sigma\xi_i\eta_i$. 则 A 是 c_0 上的一个有界线性泛函，并且$\|A\|=\|y\|_1$. 进而言之，每个 $A\in(c_0)^*$ 都可用此法得到. 简略地说，$(c_0)^*=l^1$.

(说得更精确些，这两个空间并不是相等的；前面的命题只是给出了它们之间的一个等距的向量空间的同构.)

(*b*) 在同一意义下，$(l^1)^*=l^\infty$.

(*c*) 每个 $y\in l^1$ 都象(*a*)中一样引出一个 l^∞ 上的有界线性泛函. 然而这并没有给出 $(l^\infty)^*$ 的全部. 因为$(l^\infty)^*$ 还包含有在整个 c_0 上为零的非平凡泛函.

(*d*) c_0 和 l^1 是可分的，但 l^∞ 不是.

10　如果 $\Sigma a_i\xi_i$ 对每个当 $i\to\infty$ 时 $\xi_i\to 0$ 的序列 $\{\xi_i\}$ 都收敛. 证明 $\Sigma|a_i|<\infty$.

11　对 $0<\alpha\leqslant 1$，设 Lipα 表示所有$[a,b]$上使

$$M_f=\sup_{s\neq t}\frac{|f(s)-f(t)|}{|s-t|^{\alpha}}<\infty$$

的复函数组成的空间.

证明当取$\|f\|=|f(a)|+M_f$时, Lipα是一个 Banach 空间; 如果取

$$\|f\|=M_f+\sup_x|f(x)|$$

时, 结论也是一样.

(我们说 Lipα 的元素满足 α 阶 Lipschitz 条件.)

12 设K是平面上的一个三角形 (二维图形), H是由K的顶点组成的集, A是K上的形如

$$f(x,y)=\alpha x+\beta y+\gamma \quad (\alpha,\beta\text{ 和 }\gamma\text{ 是实数})$$

的全体实函数的集.

证明对于每个$(x_0,y_0)\in K$都对应H上的唯一一个测度μ, 使

$$f(x_0,y_0)=\int_H f d\mu$$

(与 5.22节相比较.)

用正方形取代 K, H 还表示其顶点的集. A仍同上假设. 证明对K的每个点仍然存在一个H上的测度, 它具有上述性质. 不过现在却失去了唯一性.

你能不能推广成为一个更广泛的定理? (考虑其它的图形, 高维空间.)

13 设$\{f_n\}$是(非空)完备度量空间X上的连续复函数序列, 使$f(x)=\lim f_n(x)$(作为一个复数)对每个$x\in X$都存在.

(a) 证明存在一个开集$V\neq\varnothing$和一个数$M<\infty$, 对所有的$x\in V$及$n=1,2,3,\cdots$, $|f_n(x)|<M$.

(b) 若$\varepsilon>0$, 证明存在一个开集$V\neq\varnothing$和一个整数N, 当$x\in V$及$n\geqslant N$时, $|f(x)-f_n(x)|<\varepsilon$.

(b) 的提示: 对 $N=1,2,3,\cdots$, 令

$$A_N=\{x:|f_m(x)-f_n(x)|\leqslant\varepsilon,\ \text{当 } m\geqslant N \text{ 和 } n\geqslant N \text{ 时}\}$$

因为$X=\bigcup A_N$, 于是某个A_N有非空内部.

14 设C是$I=[0,1]$上全体实连续函数的空间, 赋以上确界范数. 设X_n是由f组成的C的子集, 对这些f都存在一个$t\in I$, 使对所有的$s\in I$, $|f(s)-f(t)|\leqslant n|s-t|$成立. 固定$n$并证明$C$的每个开集都包含一个与$X_n$不相交的开集. (每个$f\in C$都可以由一个斜率非常大的锯齿形函数$g$一

致逼近，并且当 $\|g-h\|$ 很小时，$h\notin X_n$.）说明这蕴涵着在 C 中存在一个完全由无处可微的函数组成的稠密的 G_δ 集.

15 设 $A=(a_{ij})$ 是以复数作为元素的无限矩阵，$i,j=0,1,2,\cdots$. A 将每个序列 $\{s_j\}$ 对应于一个序列 $\{\sigma_i\}$. 它定义为

$$\sigma_i=\sum_{j=0}^{\infty}a_{ij}s_j \qquad (i=1,2,3,\cdots)$$

假定这些级数都是收敛的.

证明当且仅当下列条件

(a) $$\lim_{i\to\infty}a_{ij}=0 \quad \text{对每个 } j,$$

(b) $$\sup\sum_{j=0}^{\infty}|a_{ij}|<\infty$$

(c) $$\lim_{i\to\infty}\sum_{j=1}^{\infty}a_{ij}=1$$

满足时，A 将每个收敛序列 $\{s_j\}$ 变为收敛于同一极限的序列 $\{\sigma_i\}$.

由 $\{s_j\}$ 过渡到 $\{\sigma_i\}$ 的过程称为求和法. 两个例子是

$$a_{ij}=\begin{cases}\dfrac{1}{i+1}, & \text{当 } 0\leqslant j\leqslant i \text{ 时}\\ 0, & \text{当 } i<j \text{ 时}\end{cases}$$

及 $$a_{ij}=(1-r_i)r_i^j \quad 0<r_i<1,\ r_i\to 1$$

证明其中的每一个都能将某个发散序列（甚至某个无界序列）$\{s_j\}$ 变换成收敛序列 $\{\sigma_i\}$.

16 设 X 和 Y 是 Banach 空间，而 A 是 X 到 Y 内的一个线性映射，具有如下性质：对 X 中的每一个序列 $\{x_n\}$，当 $x=\lim x_n$ 及 $y=\lim Ax_n$ 存在时，$y=Ax$ 成立. 证明 A 是连续的.

这就是所谓的“闭图象定理”. 提示：设 $X\oplus Y$ 是 $x\in X$，$y\in Y$ 的全部有序对 (x,y) 的集. 赋以按分量进行运算而定义的加法和纯量乘法. 当 $\|(x,y)\|=\|x\|+\|y\|$ 时，证明 $X\oplus Y$ 是一个 Banach 空间. A 的图象 G 是 $X\oplus Y$ 中由序对 (x,Ax) $x\in X$ 组成的子集. 注意，我们的假设说明 G 是闭的，因此 G 是一个 Banach 空间. 注意 $(x,Ax)\to x$ 是连续的，一一的，而且是线性的并将 G 映到 X 上.

可以看到存在非线性映射（例如 R^1 到 R^1 上的），其图象虽然是闭的，但它

们并不是连续的：若 $x\neq 0, f(x)=1/x, x=0$ 时，$f(0)=0$.

17 如果 μ 是一个正测度，每个 $f\in L^\infty(\mu)$ 都定义了一个 $L^2(\mu)$ 到 $L^2(\mu)$ 内的乘法算子 M_f，使 $M_f(g)=fg$. 证明 $\|M_f\|\leqslant\|f\|_\infty$. 哪些测度 μ 能使所有的 $f\in L^\infty(\mu)$ 都有 $\|M_f\|=\|f\|_\infty$ 呢？哪些 $f\in L^\infty(\mu)$ 能使 M_f 将 $L^2(\mu)$ 映到 $L^2(\mu)$ 上？

18 设 $\{A_n\}$ 是一个赋范线性空间 X 到 Banach 空间 Y 的有界线性变换的序列. 对所有 $n, \|A_n\|\leqslant M<\infty$，并设有一个稠密集 $E\subset X$，对每个 $x\in E$，$\{A_nx\}$ 都收敛. 证明对每个 $x\in E$，$\{A_nx\}$ 收敛.

19 如果 s_n 是函数 $f\in C(T)$ 的 Fourier 级数的前 n 项部分和，证明当 $n\to\infty$ 时，对每个 $f\in C(T)$，$s_n/\log n$ 一致地趋于 0. 也就是说，证明

$$\lim_{n\to\infty}\frac{\|s_n\|_\infty}{\log n}=0$$

另一方面，如果 $\lambda_n/\log n\to 0$，证明存在一个 $f\in C(T)$，使序列 $\{s_n(f;0)/\lambda_n\}$ 成为无界的. 提示：应用习题 18 及 5.11 节的推理，不过要用一个比原来采用的更好一些的 $\|D_n\|_1$ 的估值.

20 第 4.15 节的引理是否对每个 Banach 空间都真确？对每个赋范空间呢？

21 (a) 有没有 R^1 上一个连续正函数 f_n 的序列，使 $\{f_n(x)\}$ 当且仅当 x 是有理数时是无界的？

(b) 在 (a) 中以“无理数”代替“有理数”，回答所提出的问题.

(c) 以“当 $n\to\infty$ 时 $f_n(x)\to\infty$”代替“$\{f_n(x)\}$ 无界”，回答所产生的与 (a)，(b) 相类似的问题.

第六章　复　测　度

全 变 差

6.1　引言　设 $\mathfrak{M}$ 是集 X 内的 σ-代数. $\mathfrak{M}$ 的元素的可数集族 $\{E_i\}$ 称为 E 的一个划分，若 $E=\bigcup E_i$，且当 $i\neq j$ 时，$E_i\cap E_j=\varnothing$. $\mathfrak{M}$ 上的复测度 μ 是 $\mathfrak{M}$ 上的一个复函数，使得

$$\mu(E)=\sum_{i=1}^{\infty}\mu(E_i)\qquad(E\in\mathfrak{M}) \tag{1}$$

对 E 的每一个划分 $\{E_i\}$ 成立.

注意，级数(1)的收敛性现在是必需条件的一部分.（不象对正测度那样，在那里级数或者收敛或者发散于∞）. 因为若下标改变顺序，集 E_i 的并不变，所以级数(1)的每个重新排列也必须收敛. 因此([26], 定理 3.56)级数实际上绝对收敛.

我们考虑找一个正测度 λ 的问题，这个 λ 在 $|\mu(E)|<\lambda(E)$ 对每一个 $E\in\mathfrak{M}$ 成立的意义下，控制 $\mathfrak{M}$ 上给定的复测度 μ，并且试图使 λ 保持尽可能地小. 我们问题的每一个解（要是它肯定存在），对任一个 $E\in\mathfrak{M}$ 的每一个划分，都必须满足

$$\lambda(E)=\sum_{i=1}^{\infty}\lambda(E_i)\geqslant\sum_{i=1}^{\infty}|\mu(E_i)| \tag{2}$$

因此，$\lambda(E)$ 至少要等于(2)右边的和式对 E 所有的划分所取的上确界. 这启发我们定义 $\mathfrak{M}$ 上的一个集函数 $|\mu|$ 为

$$|\mu|(E)=\sup\sum_{i=1}^{\infty}|\mu(E_i)|\qquad E\in\mathfrak{M} \tag{3}$$

这里的上确界是对 E 的所有的划分 $\{E_i\}$ 取的. 这个记号也许不是最好的，但它是习惯的用法. 注意，$|\mu|(E)\geqslant|\mu(E)|$ 但 $|\mu|(E)$ 一般地不等于 $|\mu(E)|$.

这表明，如下面将要证明的那样，$|\mu|$ 确实是一个测度. 这样我们的问题就有了一个解. 导出(3)式所作的讨论清楚地表明，在任一其他解 λ 都具有性质"对所有 $E\in\mathfrak{M}$，$\lambda(E)\geqslant|\mu|(E)$"这个意义下，$|\mu|$ 是最小的解.

集函数 $|\mu|$ 称为 μ 的全变差，有时为避免误解，称为全变差测度. "μ 的全变差"一词也经常用于表示数 $|\mu|(X)$.

如果 μ 是正测度，自然 $|\mu|=\mu$.

$|\mu|$ 除了是一个测度之外，还有另一个意料不到的性质. $|\mu|(X)<\infty$. 因为 $|\mu(E)|\leqslant|\mu|(E)\leqslant|\mu|(X)$，这蕴涵了任一 σ-代数上的每一个复测度是有界的：如果 μ 的值域在复平面内，则它实际是在某一个有限半径的圆盘内. 这个性质(在定理 6.4 证明)有时用 μ 是有界变差的这种说法来表达.

6.2 定理 $\mathfrak{M}$ 上的复测度 μ 的全变差 $|\mu|$ 是 $\mathfrak{M}$ 上的一个正测度.

证明 设 $\{E_i\}$ 是 $E\in\mathfrak{M}$ 的一个划分，t_i 是使得 $t_i<|\mu|(E_i)$ 的实数. 则每一个 E_i 有一个划分 $\{A_{ij}\}$，使得

$$\sum_j|\mu(A_{ij})|>t_i \qquad (i=1,2,3,\cdots) \tag{1}$$

由 $\{A_{ij}\}(i,j=1,2,3,\cdots)$ 是 E 的划分，得出

$$\sum_i t_i\leqslant\sum_{i,j}|\mu(A_{ij})|\leqslant|\mu|(E) \tag{2}$$

(2)的左边对 $\{t_i\}$ 的所有允许的选择取上确界，我们看出

$$\sum_i|\mu|(E_i)\leqslant|\mu|(E) \tag{3}$$

为了证明相反的不等式，设 $\{A_j\}$ 是 E 的任一划分，则对任一

固定的 j，$\{A_j \cap E_i\}$是 A_j 的一个划分．并且对于任一固定的 i，$\{A_j \cap E_i\}$是 E_i 的一个划分．因此

$$(4)\qquad \sum_j |\mu(A_j)| = \sum_j \Big| \sum_i \mu(A_j \cap E_i) \Big|$$

$$\leqslant \sum_j \sum_i |\mu(A_j \cap E_i)|$$

$$= \sum_i \sum_j |\mu(A_j \cap E_i)|$$

$$\leqslant \sum_i |\mu|(E_i)$$

因为(4)对 E 的每一个划分$\{A_j\}$成立，故有

$$(5)\qquad |\mu|(E) \leqslant \sum_i |\mu|(E_i)$$

根据(3)和(5)，$|\mu|$是可数可加的．

注意，在(2)和(4)内用到了定理 1.27 的系．

$|\mu|$不恒等于∞是定理 6.4 的平凡的推论，但因为$|\mu|(\varnothing) = 0$，现在也能看出这个结论是对的．

6.3 引理 如果$z_1, z_2, \cdots, z_n$是复数，则存在$\{1, \cdots, n\}$的子集 S，使得

$$(1)\qquad \Big| \sum_{j \in B} z_j \Big| \geqslant \frac{1}{6} \sum_{j=1}^{n} |z_j|$$

证明 令 $w = |z_1| + \cdots + |z_n|$．复平面是用直线 $y = \pm x$ 所围成的四个闭象限的并，且这些象限中至少有一个 Q（不失一般性，假定由$|y| \leqslant x$ 所确定的那个）具有性质：$z_j \in Q$ 的那些$|z_j|$的和至少是 $w/4$．对于 $z \in Q$，有

$$\mathrm{Re}\, z \geqslant |z| / \sqrt{2}$$

若 S 是所有使得 $z_j \in Q$ 的 j 的集，便有

$$\left|\sum_{j\in S} z_j\right| \geqslant \sum_{j\in S} \mathrm{Re}\, z_j \geqslant \frac{1}{\sqrt{2}} \sum_{j\in S} |z_j| \geqslant \frac{w}{4\sqrt{2}} \geqslant \frac{w}{6}$$

6.4 定理 如果 μ 是 X 上的复测度，则

(1) $$|\mu|(X) < \infty$$

证明 首先指出：如果 $E\in\mathfrak{M}$，$|\mu|(E)=\infty$，则 $E=A\cup B$，这里 A 和 $B\in\mathfrak{M}$，$A\cap B=\varnothing$，且

(2) $$|\mu(A)|>1,\ |\mu|(B)=\infty$$

事实上，$|\mu|$ 的定义表明，对每一个 $t<\infty$，对应有一个 E 的划分 $\{E_j\}$，使得 $\Sigma|\mu(E_j)|>t$. 我们取 $t=6(1+|\mu(E)|)$，则

(3) $$\sum_{j=1}^{n} |\mu(E_j)| > t$$

对某一个 n 成立；对于 $z_j=\mu(E_j)$，应用引理 6.3，且令

(4) $$A=\bigcup_{j\in B} E_j$$

得出，$A\subset E$，且 $|\mu(A)|>\frac{t}{6}\geqslant 1$. 如果 $B=E-A$，则

(5) $$|\mu(B)| = |\mu(E)-\mu(A)| \geqslant |\mu(A)| - |\mu(E)|$$
$$> \frac{t}{6} - |\mu(E)| = 1$$

因为根据定理 6.2，$|\mu|(E)=|\mu|(A)+|\mu|(B)$，我们有 $|\mu|(A)=\infty$ 或 $|\mu|(B)=\infty$（或两个同时成立）. 如有必要的话，通过交换 A 和 B，我们就得出(2).

现在假定 $|\mu|(X)=\infty$. 令 $B_0=X$. 假定 $n\geqslant 0$. 选择 B_n，使得 $|\mu|(B_n)=\infty$. 则用 B_n 代替 E 而应用(2)，看出 B_n 是两个不相交的集 A_{n+1} 和 B_{n+1} 的并，使得 $|\mu(A_{n+1})|>1$，$|\mu|(B_{n+1})=\infty$. 于是归纳地得出，不相交集 $A_1, A_2, A_3, \cdots$，具有 $|\mu(A_n)|>1$. 若 $C=\cup A_n$，μ 的可数可加性指出，

(6) $$\mu(C)=\sum_{1}^{\infty}\mu(A_n)$$

但是这个级数不能收敛，因为当 $n\to\infty$ 时，$\mu(A_n)$ 不趋于0. 这个矛盾证明了(1)必须成立.

6.5 如果 μ 和 λ 是同一个 σ-代数 $\mathfrak{M}$ 上的复测度. 依通常的方式用

(1) $$\begin{aligned}(\mu+\lambda)E&=\mu(E)+\lambda(E)\\(c\mu)E&=c\mu(E)\end{aligned}\qquad(E\in\mathfrak{M})$$

定义 $\mu+\lambda$ 和对任一纯量 c 定义 $c\mu$. 容易验证 $\mu+\lambda$ 和 $c\mu$ 均是复测度，于是所有 $\mathfrak{M}$ 上的复测度的集族是一个向量空间. 若令

(2) $$\|\mu\|=|\mu|(X)$$

容易验证赋范线性空间的全部公理均满足.

6.6 正变差和负变差 现在我们特地考虑一个 σ-代数 $\mathfrak{M}$ 上的实测度(这样的测度经常被称为广义测度). 象前面一样定义 $|\mu|$. 并定义

(1) $$\mu^+=\frac{1}{2}(|\mu|+\mu),\quad \mu^-=\frac{1}{2}(|\mu|-\mu)$$

则 μ^+ 和 μ^- 均是 $\mathfrak{M}$ 上的正测度. 根据定理 6.4，它们是有界的，同时

(2) $$\mu=\mu^+-\mu^-,\quad |\mu|=\mu^++\mu^-$$

测度 μ^+ 和 μ^- 分别称为 μ 的正变差和负变差. μ 表示为正测度 μ^+ 和 μ^- 的差是著名的 μ 的 Jordan 分解. 在所有将 μ 分为两个正测度之差的表示中，Jordan 分解有一种最小性质，它将作为定理 6.14 的系而确定下来.

绝对连续性

6.7 定义 设 μ 是 σ-代数 $\mathfrak{M}$ 上的正测度，λ 是 $\mathfrak{M}$ 上的一个

任意测度，λ 可以是正测度或复测度(记住复测度以复平面为其值域，但"正测度"一词在我们的用法中包含了∞作为其允许值．这样正测度并不能构成复测度的子类)．

如果 $\lambda(E)=0$ 对每一个 $\mu(E)=0$ 的 $E\in\mathfrak{M}$ 成立，我们就说 λ 关于 μ 绝对连续，记作

(1) $$\lambda\ll\mu$$

如果存在一个集 $A\in\mathfrak{M}$，使得 $\lambda(E)=\lambda(A\cap E)$ 对每一个 $E\in\mathfrak{M}$ 成立．我们就说 λ 集中于 A 上．这等价于假定当 $E\cap A=\varnothing$ 时 $\lambda(E)=0$．

假定 λ_1 和 λ_2 都是 $\mathfrak{M}$ 上的测度，且假定存在一对不相交的集 A 和 B，使得 λ_1 集中于 A 上而 λ_2 集中于 B 上，则我们说 λ_1 和 λ_2 是互相奇异的．记为

(2) $$\lambda_1\perp\lambda_2$$

这些概念的一些初等性质列述如下：

6.8 命题 假定 μ,λ,λ_1 和 λ_2 都是 σ-代数 $\mathfrak{M}$ 上的测度，且 μ 是正测度．

(a) 若 λ 集中于 A 上，则 $|\lambda|$ 也是如此．

(b) 若 $\lambda_1\perp\lambda_2$，则 $|\lambda_1|\perp|\lambda_2|$．

(c) 若 $\lambda_1\perp\mu,\lambda_2\perp\mu$，则 $\lambda_1+\lambda_2\perp\mu$．

(d) 若 $\lambda_1\ll\mu,\lambda_2\ll\mu$，则 $\lambda_1+\lambda_2\ll\mu$．

(e) 若 $\lambda\ll\mu$，则 $|\lambda|\ll\mu$．

(f) 若 $\lambda_1\ll\mu,\lambda_2\perp\mu$，则 $\lambda_1\perp\lambda_2$．

(g) 若 $\lambda\ll\mu,\lambda\perp\mu$，则 $\lambda=0$．

证明

(a) 若 $E\cap A=\varnothing$，且 $\{E_j\}$ 是 E 的任一划分，则对所有的 j，$\lambda(E_j)=0$，因此，$|\lambda|(E)=0$．

(b) 从(a)直接得出．

(c) 存在不相交的集 A_1 和 B_1，使得 λ_1 集中于 A_1 上，μ 集中于 B_1 上，并存在不相交集 A_2 和 B_2，使得 λ_2 集中于 A_2 上，μ 集中于 B_2 上. 因此 $\lambda_1+\lambda_2$ 集中于 $A=A_1\cup A_2$ 上，μ 集中于 $B=B_1\cap B_2$ 上，且 $A\cap B=\varnothing$.

(d) 这是显然的.

(e) 假定 $\mu(E)=0$，$\{E_j\}$ 是 E 的一个划分. 则 $\mu(E_j)=0$；因为 $\lambda\ll\mu$，$\lambda(E_j)=0$ 对所有的 j 成立，因此，$\Sigma|\lambda(E_j)|=0$. 这蕴涵着 $|\lambda|(E)=0$.

(f) 因为 $\lambda_2\perp\mu$，存在一个具有 $\mu(A)=0$ 的集 A，λ_2 集中于 A 上. 因为 $\lambda_1\ll\mu$，故 $\lambda_1(E)=0$ 对每一个 $E\subset A$ 成立. 所以 λ_1 集中在 A 的补集上.

(g) 根据(f)，(g)的假定蕴涵着 $\lambda\perp\lambda$，这显然迫使 $\lambda=0$.

现在转到与绝对连续有关的主要定理. 事实上，它可能是测度论中最主要的定理. 我们首先对正的有限测度来叙述它.

6.9 Lebesgue-Radon-Nikodym 定理 设 μ，λ 是集 X 的 σ-代数 $\mathfrak{M}$ 上的正有界测度.

(a) 在 $\mathfrak{M}$ 上存在唯一的一对测度 λ_a 和 λ_s，使得

$$\lambda=\lambda_a+\lambda_s,\quad \lambda_a\ll\mu,\quad \lambda_s\perp\mu \tag{1}$$

这些测度都是正的，且 $\lambda_a\perp\lambda_s$.

(b) 存在唯一的一个 $h\in L^1(\mu)$，使得

$$\lambda_a(E)=\int_E h\,d\mu \qquad (E\in\mathfrak{M}) \tag{2}$$

这对 (λ_a,λ_s) 称为 λ 关于 μ 的 Lebesgue 分解. 这个分解的唯一性是容易看出的. 因为若 (λ_a',λ_s') 是满足(1)的另一对，则

$$\lambda_a'-\lambda_a=\lambda_s-\lambda_s' \tag{3}$$

$\lambda_a'-\lambda_a\ll\mu$，且 $\lambda_s-\lambda_s'\perp\mu$，因此(3)的两边均是 0；在这里我们已经用到了 6.8(c)，6.8(d)，和 6.8(g).

这个分解的存在性是(a)的重要的部分.

断言(b)是著名的 Radon-Nikodym 定理. 再者，h 的唯一性从定理 1.39(b)立即得出. 同时，若 h 是 $L^1(\mu)$的任一元素，(2)的积分定义 $\mathfrak{M}$ 上的一个测度(定理 1.29)，显然它关于 μ 绝对连续. Radon-Nikodym 定理的重点是逆命题：每一个 $\lambda \ll \mu$(在这种情况下，$\lambda_a = \lambda$)都能用这种方式得到.

(2)中出现的函数 h 被称为 λ_a 关于 μ 的 Radon-Nikodym 导数. 象定理 1.29 后面的注一样，我们可以用 $d\lambda_a = h d\mu$ 的形式表示(2)，或甚至用 $h = d\lambda_a / d\mu$ 的形式表示(2).

下面的一举同时产生出(a)和(b)的证明方法，其思想是 Von Neumann 提出的.

证明 令 $\phi = \lambda + \mu$，则 ϕ 是 $\mathfrak{M}$ 上的正有界测度. 两个测度和的定义指出，对 $f = \chi_E$，

$$\int_X f d\phi = \int_X f d\lambda + \int_X f d\mu \tag{4}$$

因此对简单函数 f，进而对非负的可测函数 f 也成立. 如果 $f \in L^2(\phi)$，Schwarz 不等式给出

$$\left|\int_X f d\lambda\right| \leqslant \int_X |f| d\lambda \leqslant \int_X |f| d\phi \leqslant \left\{\int_X |f|^2 d\phi\right\}^{1/2} \left\{\phi(X)\right\}^{1/2}$$

因为 $\phi(X) < \infty$，映射

$$f \to \int_X f d\lambda \tag{5}$$

可看成是 $L^2(\phi)$上的一个有界线性泛函.

我们知道在 Hilbert 空间 H 上的每一个有界线性泛函可以用与 H 的某个元素的内积给出来. 因此存在一个 $g \in L^2(\phi)$，使得

$$\int_X f d\lambda = \int_X f g d\phi \qquad (f \in L^2(\phi)) \tag{6}$$

要注意，$L^2(\phi)$的完备性怎样用来保证函数 g 的存在性. 同时

要注意，虽然 g 作为 $L^2(\phi)$ 的元素唯一地确定，但作为 X 的点函数只是 a. e. $[\phi]$ 确定的.

在(6)中对任一具有 $\phi(E)>0$ 的 $E\in\mathfrak{M}$，令 $f=\chi_E$. 那末(6)的左边就是 $\lambda(E)$；因为 $0\leqslant\lambda\leqslant\phi$，我们有

$$0\leqslant\frac{1}{\phi(E)}\int_E g\,d\phi\leqslant 1 \tag{7}$$

所以根据定理 1.40，$g(x)\in[0,1]$（关于 ϕ）对几乎所有的 x 成立. 因此可以假定对每一个 $x\in X$，$0\leqslant g(x)\leqslant 1$ 而不致影响(6). 把(6)重新写成

$$\int_X(1-g)f\,d\lambda=\int_X f g\,d\mu \qquad (f\in L^2(\phi)) \tag{8}$$

的形式. 令

$$A=\{x:\,0\leqslant g(x)<1\},\quad B=\{x:\,g(x)=1\} \tag{9}$$

并且定义

$$\lambda_a(E)=\lambda(A\cap E),\ \lambda_s(E)=\lambda(B\cap E) \qquad (E\in\mathfrak{M}) \tag{10}$$

如果在(8)中取 $f=\chi_B$，可看出 $\mu(B)=0$. 于是 $\lambda_s\perp\mu$. 因为 g 是有界的，若对 $n=1,2,3,\cdots$，$E\in\mathfrak{M}$ 用

$$(1+g+\cdots+g^n)\chi_E$$

代替 f，(8)式也成立. 因此有

$$\int_E(1-g^{n+1})\,d\lambda=\int_E g(1+g+\cdots+g^n)\,d\mu \tag{11}$$

在 B 的每一点，$g(x)=1$，因此 $1-g^{n+1}(x)=0$. 在 A 的每一点上，$g^{n+1}(x)$ 单调地趋于 0，所以(11)的左边当 $n\to\infty$ 时，收敛于 $\lambda(A\cap E)=\lambda_a(E)$.

(11)式右边的被积函数单调增加趋于一个非负的可测的极限函数 h. 单调收敛定理指出，当 $n\to\infty$ 时，(11)的右边趋于 $\int_E h\,d\mu$.

于是我们证明了(2)对每一个 $E\in\mathfrak{M}$ 成立. 取 $E=X$, 由 $\lambda_a(X)<\infty$, 看出 $h\in L^1(\mu)$.

最后, (2)表明 $\lambda_a\ll\mu$. 证明完毕.

6.10 定理6.9的推广 定理6.9的证明强烈地依赖于 $\phi(X)<\infty$的假定, 即 μ 和 λ 同时是有界测度.

如果 μ 是 σ-有限的, 则(根据定义) X 是使得 $\mu(X_n)<\infty$的可数多个 X_n 的并. 我们可以假定 X_n 是不相交的. 倘若不然, 我们就用 $\{Y_n\}$ 代替 $\{X_n\}$, 这里 $Y_1=X_1$, 并且对 $n\geqslant 2$,

$$Y_n=X_n-(Y_1\cup\cdots\cup Y_{n-1})$$

现在若 $\lambda(X)<\infty$, 对每一个 X_n 就能应用定理 6.9. 测度 $\lambda(E\cap X_n)$的 Lebesgue 分解的总和是 λ 的 Lebesgue 分解. 在 X_n 上得出 h_n, 当 $x\in X_n$ 时, 令 $h(x)=h_n(x)$, 通过它们定义出一个 X 上的函数 h; 因为 $\lambda(X)<\infty$, 容易得出 $h\in L^1(\mu)$.

其次, 如果保持 μ 为 σ-有限的假定而设 λ 是 $\mathfrak{M}$ 上的复测度, 则 $\lambda=\lambda_1+i\lambda_2$, λ_1, λ_2 是实的, 并且能够应用上面的结果于 λ_1 和 λ_2 的正变差和负变差(见 6.6 节).

我们可以总结如下:

如果 μ 是 $\mathfrak{M}$ 上的正的 σ-有限测度, λ 是 $\mathfrak{M}$ 上的复测度. 则 Lebesgue 分解定理和 Radom-Nikodym 定理也都是正确的.

如果 μ 和 λ 都是正的和 σ-有限的, 定理 6.9 的大部分依然正确. 现在我们可以记 $X=\bigcup X_n$. 这里对 $n=1,2,3,\cdots\mu(X_n)<\infty$. 并且 $\lambda(X_n)<\infty$. 测度 $\lambda(E\cap X_n)$的 Lebesgue 分解继续给出一个 λ 的 Lebesgue 分解, 并继续得出满足方程 6.9(2)的函数 h. 尽管 h 是"局部地在 L^1 内"即 $\int_{X_n}h\,d\mu<\infty$对每一个 n 成立, 然而 $h\in L^1(\mu)$不再正确了.

最后, 如果超出 σ-有限性的范围, 我们就会遇到所考虑的两

个定理确实不成立的情况。例如，设μ是$(0,1)$上的Lebesgue测度，并设λ是$(0,1)$内所有Lebesgue可测集的σ-代数上的计数测度。则λ没有关于μ的Lebesgue分解，并且尽管$\mu \ll \lambda$和μ是有界的，也不存在$h \in L^1(\lambda)$，使得$d\mu = h d\lambda$。我们省略这个容易的证明。

下面的定理可以说明，在论及$\lambda \ll \mu$这个关系时，为什么采用"连续"这个词[也可见习题9(*b*)]。

6.11 定理 假定μ和λ是σ-代数$\mathfrak{M}$上的测度，μ是正的，且λ是复的，则下面两个条件是等价的：

(*a*) $\lambda \ll \mu$。

(*b*) 对每一个$\varepsilon > 0$，对应有一个$\delta > 0$，对于所有的$E \in \mathfrak{M}$，$\mu(E) < \delta$时，有$|\lambda(E)| < \varepsilon$。

性质(*b*)有时用作绝对连续的定义。然而当λ是一个正的无界测度时，(*a*)并不蕴涵(*b*)。例如：设μ是$(0,1)$上的Lebesgue测度，且对于每一个Lebesgue可测集$E \subset (0,1)$，令

$$\lambda(E) = \int_E t^{-1} dt$$

证明 假定(*b*)成立。若$\mu(E) = 0$，则对每一个$\delta > 0$，$\mu(E) < \delta$，因此对每一个$\varepsilon > 0$，$|\lambda(E)| < \varepsilon$。所以$\lambda(E) = 0$。于是(*b*)蕴涵着(*a*)。

假定(*b*)不成立。则存在一个$\varepsilon > 0$，并且存在集$E_n \in \mathfrak{M}$ $(n = 1, 2, 3, \cdots)$，使得$\mu(E_n) < 2^{-n}$但$|\lambda(E_n)| \geqslant \varepsilon$。因此$|\lambda|(E_n)| \geqslant \varepsilon$，令

$$(1) \qquad A_n = \bigcup_{i=n}^{\infty} E_i, \qquad A = \bigcap_{n=1}^{\infty} A_n$$

则$\mu(A_n) < 2^{-n+1}$，$A_n \supset A_{n+1}$，所以定理1.19(*e*)指出$\mu(A) = 0$，并且由$|\lambda|(A_n) \geqslant |\lambda|(E_n)$，得

$$|\lambda|(A) = \lim_{n \to \infty} |\lambda|(A_n) \geqslant \varepsilon > 0$$

于是得出$|\lambda| \ll \mu$不成立. 因此根据命题6.8(e), (a)不成立.

Radon-Nikodym 定理的推论

6.12 定理 设μ是X的σ-代数$\mathfrak{M}$上的复测度, 则存在一个可测函数h, 使得$|h(x)|=1$对所有的$x\in X$成立, 并且使得

(1) $$d\mu = h\,d|\mu|$$

与表示一个复数为它的绝对值和一个绝对值为1的乘积类比, 方程(1)有时叫做μ的极表示(或极分解).

证明 $\mu \ll |\mu|$是明显的, 所以Radon-Nikodym定理保证了满足(1)的某个$h\in L^1(|\mu|)$的存在性.

设$A_r=\{x: |h(x)|<r\}$, 这里r是某个正数. 设$\{E_j\}$是A_r的一个划分, 则

$$\sum_j |\mu(E_j)| = \sum_j \left|\int_{E_j} h\,d|\mu|\right| \leqslant \sum_j r|\mu|(E_j) = r|\mu|(A_r)$$

所以$|\mu|(A_r)\leqslant r|\mu|(A_r)$. 若$r<1$, 这迫使$|\mu|(A_r)=0$, 于是$|h|\geqslant 1$a. e.

另一方面, 若$|\mu|(E)>0$, (1)指出

$$\left|\frac{1}{|\mu|(E)}\int_E h\,d|\mu|\right| = \frac{|\mu(E)|}{|\mu|(E)} \leqslant 1$$

现在应用定理1.40(用闭单位圆盘代替S), 便得出$|h|\leqslant 1$a. e. 成立.

设$B=\{x\in X: |h(x)|\neq 1\}$, 我们已经证明$|\mu|(B)=0$. 若在$B$上重新定义$h$, 使$h(x)=1$在$B$上成立. 我们就获得一个具有所要求的性质的函数.

6.13 定理 假定μ是$\mathfrak{M}$上的正测度, $g\in L^1(\mu)$, 且

(1) $$\lambda(E) = \int_E g\,d\mu \qquad (E\in\mathfrak{M})$$

则

$$|\lambda|(E)=\int_E |g|d\mu \qquad (E\in\mathfrak{M}) \tag{2}$$

证明 根据定理 6.12，存在一个绝对值为 1 的函数 h，使得 $d\lambda=hd|\lambda|$. 根据假定，$d\lambda=gd\mu$. 因此

$$hd|\lambda|=gd\mu$$

这给出 $d|\lambda|=\bar{h}gd\mu$（与定理 1.29 比较）.

因为 $|\lambda|\geqslant 0$，且 $\mu\geqslant 0$，所以 $\bar{h}g\geqslant 0$，a. e. $[\mu]$. 于是 $\bar{h}g=|g|$，a. e. $[\mu]$.

6.14 Hahn 分解定理 设 μ 是集 X 的 σ-代数 $\mathfrak{M}$ 上的实测度，则存在集 A 和 $B\in\mathfrak{M}$，使得 $A\cup B=X$，$A\cap B=\varnothing$，并且使 μ 的正变差 μ^+ 和负变差 μ^- 满足

$$\mu^+(E)=\mu(A\cap E),\quad \mu^-(E)=-\mu(B\cap E) \qquad (E\in\mathfrak{M}) \tag{1}$$

换句话说，X 是两个不相交的可测集 A 和 B 的并，使得“A 携带 μ 的全部正质量.”[因为若 $E\subset A$，(1)蕴涵了 $\mu(E)\geqslant 0$]. 并且“B 携带 μ 的全部负质量.”[因为当 $E\subset B$ 时，$\mu(E)\leqslant 0$]. 这对 (A, B) 称为由 μ 所诱导的 X 的 Hahn 分解.

证明 根据定理 6.12，$d\mu=hd|\mu|$，这里 $|h|=1$. 因为 μ 是实的，得出 h 是实的（a. e.，并且通过在零测集上重新予以定义，所以是处处）. 因此，$h=\pm 1$. 令

$$A=\{x: h(x)=1\},\qquad B=\{x: h=-1\} \tag{2}$$

因为 $\mu^+=\frac{1}{2}(|\mu|+\mu)$，并且因为

$$\frac{1}{2}(1+h)=\begin{cases} h & \text{在 } A \text{ 上},\\ 0 & \text{在 } B \text{ 上}, \end{cases} \tag{3}$$

所以对任何 $E\in\mathfrak{M}$，有

$$\mu^+(E)=\frac{1}{2}\int_E (1+h)d|\mu|=\int_{E\cap A} hd|\mu|=\mu(E\cap A) \tag{4}$$

因为 $\mu(E)=\mu(E\cap A)+\mu(E\cap B)$ 和 $\mu=\mu^+-\mu^-$，(1)的后一半从前一半得出.

系 如果 $\mu=\lambda_1-\lambda_2$，这里 λ_1 和 λ_2 是正测度，则 $\lambda_1\geqslant\mu^+$ 和 $\lambda_2\geqslant\mu^-$.

这就是在 6.6 节所提及的 Jordan 分解的最小性质.

证明 因为 $\mu\leqslant\lambda_1$，故有

$$\mu^+(E)=\mu(E\cap A)\leqslant\lambda_1(E\cap A)\leqslant\lambda_1(E)$$

L^p 上的有界线性泛函

6.15 设 μ 是一个正测度，假定 $1\leqslant p\leqslant\infty$. 并设 q 是共轭于 p 的指数. Hölder 不等式(定理 3.8)指出，若 $g\in L^q(\mu)$，Φ_g 由

$$\Phi_g(f)=\int_X fg\,d\mu \tag{1}$$

定义，则 Φ_g 是 $L^p(\mu)$ 上范数不超过 $\|g\|_q$ 的有界线性泛函. 自然会提出这样的问题：所有的 $L^p(\mu)$ 上的有界线性泛函是否都具有这种形式，并且这样的表示是否唯一？

对于 $p=\infty$，回答是否定的. $L^1(\mu)$ 不能提供所有的 $L^\infty(\mu)$ 上有界线性泛函. 对于 $1<p<\infty$，回答则是肯定的. 假定测度论某些病态被排除的话，对 $p=1$ 的回答也是肯定的. 对于 σ-有限的测度空间，不会出现麻烦. 我们将只限于讨论这种情况.

6.16 定理 假定 $1\leqslant p<\infty$，μ 是 X 上 σ-有限的正测度，并且 Φ 是 $L^p(\mu)$ 上的有界线性泛函. 则存在一个唯一的 $g\in L^q(\mu)$，使得

$$\Phi(f)=\int_X fg\,d\mu \qquad (f\in L^p(\mu)) \tag{1}$$

这里 q 是共轭于 p 的指数. 而且，如果 Φ 和 g 是(1)内所述及的函数，则有

(2) $$\|\Phi\| = \|g\|_q$$

换句话说，在所述条件下，$L^q(\mu)$是等距地同构于$L^p(\mu)$的对偶空间.

证明 g的唯一性是显然的. 因为若g和g'满足(1)，则$g-g'$在任何有限测度的可测集E上的积分是0(如把f取作χ_E时所看出的结果一样). 因此μ的σ-有限性蕴涵着$g-g'=0$ a. e. 成立.

其次，若(1)成立，Hölder 不等式蕴涵着

(3) $$\|\Phi\| \leqslant \|g\|_q$$

所以，只剩下证明g存在和在(3)内等号成立. 若$\|\Phi\|=0$，则取$g=0$时，(1)和(2)成立. 所以假定$\|\Phi\|>0$.

首先考虑$\mu(X)<\infty$的情况.

对于任一可测集$E\subset X$，定义

$$\lambda(E)=\Phi(\chi_E)$$

因为Φ是线性的，当A和B不相交时$\chi_{A\cup B}=\chi_A+\chi_B$，所以$\lambda$是可加的. 为证明可数可加性，假定$E$是可数个不相交的可测集$E_i$的并，令$A_k=E_1\cup\cdots\cup E_k$，并且注意到

(4) $$\|\chi_E-\chi_{A_k}\|_p=[\mu(E-A_k)]^{1/p}\to 0 \quad (k\to\infty)$$

Φ的连续性指出$\lambda(A_k)\to\lambda(E)$. 所以$\lambda$是复测度. [在(4)中，用到$p<\infty$的假定.]当$\mu(E)=0$时，因为$\|\chi_E\|_p=0$，显然$\lambda(E)=0$，于是$\lambda\ll\mu$，并且 Radon-Nikodym 定理保证，存在一个函数$g\in L^1(\mu)$，使得对每一个可测集$E\subset X$，有

(5) $$\Phi(\chi_E)=\int_E g\,d\mu=\int_X \chi_E g\,d\mu$$

根据线性，得出

(6) $$\Phi(f)=\int_X fg\,d\mu$$

对每一个简单函数f成立. 因为每一个$f\in L^\infty(\mu)$是简单函数f_i的一致极限，所以对每一个$f\in L^\infty(\mu)$成立. 注意到f_i一致收敛

于 f 蕴涵着 $\|f_i-f\|_p\to 0$. 因此当 $i\to\infty$ 时, $\Phi(f_i)\to\Phi(f)$.

要得出 $g\in L^q(\mu)$ 和(2)成立, 最好分成两种情况讨论.

情况 1 $p=1$. 此时(5)指出

$$\left|\int_E g\,d\mu\right|\leqslant\|\Phi\|\cdot\|\chi_E\|_1=\|\Phi\|\cdot\mu(E)$$

对每一个 $E\in\mathfrak{M}$ 成立. 根据定理 1.40, $|g(x)|\leqslant\|\Phi\|$ a.e. 所以 $\|g\|_\infty\leqslant\|\Phi\|$.

情况 2 $1<p<\infty$. 存在一个可测函数 α, $|\alpha|=1$, 使得 $\alpha g=|g|$[命题 1.9(e)]. 设 $E_n=\{x:|g(x)|\leqslant n\}$, 定义 $f=\chi_{E_n}|g|^{q-1}\alpha$. 则在 E_n 上 $|f|^p=|g|^q$, $f\in L^\infty(\mu)$, 并且(6)给出

$$\int_E|g|^q d\mu=\int_X fg\,d\mu=\Phi(f)\leqslant\|\Phi\|\cdot\left\{\int_{E_n}|g|^q\right\}^{1/p}$$

所以

$$\int_X\chi_{E_n}|g|^q d\mu\leqslant\|\Phi\|^q\qquad(n=1,2,3,\cdots)\tag{7}$$

若应用单调收敛定理于(7), 便得 $\|g\|_q\leqslant\|\Phi\|$.

于是(2)成立且 $g\in L^q(\mu)$. 由此得出(6)式两边都是 $L^p(\mu)$ 上的连续函数. 它们在 $L^p(\mu)$ 的稠密子集 $L^\infty(\mu)$ 上是一致的; 因此在 $L^p(\mu)$ 的全体上一致. 这就完成了 $\mu(X)<\infty$ 时的证明.

若 $\mu(X)=\infty$, 但 μ 是 σ-有限的. 这时 X 是可数个不相交集 X_n 的并, 其中 $0<\mu(X_n)<\infty$. 当 $x\in X_n$ 时, 用

$$h(x)=\frac{1}{n^2}\mu(X_n)\tag{8}$$

定义 h: $X\to(0,\infty)$. 则 $h\in L^1(\mu)$; 并且

$$\tilde{\mu}(E)=\int_E h\,d\mu\qquad(E\in\mathfrak{M})\tag{9}$$

定义了一个有限测度 $\tilde{\mu}$. 注意到 $F\to h^{1/p}F$ 是一个 $L^p(\tilde{\mu})$ 到 $L^p(\mu)$ 上的线性等距映射, 因此

$$\Psi(F)=\Phi(h^{1/p}F)\tag{10}$$

定义了 $L^p(\tilde{\mu})$ 上的一个有界线性泛函 Ψ，具有 $\|\Psi\|=\|\Phi\|$.

证明的第一部分指出，存在 $G\in L^1(\tilde{\mu})$，使得

$$\Psi(F)=\int_X FG\,d\tilde{\mu} \qquad (F\in L^p(\tilde{\mu})) \tag{11}$$

令 $g=h^{1/q}G$，(若 $p=1, g=G$)，则 $p>1$ 时，

$$\int_X |g|^q d\tilde{\mu}=\int_X |G|^q d\tilde{\mu}=\|\Psi\|^q=\|\Phi\|^q \tag{12}$$

而 $p=1$ 时，$\|g\|_\infty=\|G\|_\infty=\|\Psi\|=\|\Phi\|$. 于是(2)成立，并且因为 $Gd\tilde{\mu}=h^{1/p}gd\mu$，最后得出

$$\Phi(f)=\Psi(h^{-\frac{1}{p}}f)=\int_X h^{-\frac{1}{p}}fGd\tilde{\mu}=\int_X fgd\mu \tag{13}$$

对每一个 $f\in L^p(\mu)$ 成立.

6.17 评注 我们已经碰到过定理 6.16 当 $p=q=2$ 时的特殊情况，事实上，一般情况的证明，是以这个特殊情况为基础的，因为在 Radon-Nikodym 定理的证明中，用到了 $L^2(\mu)$ 上的有界线性泛函的知识，而后者是定理 6.16 的证明的关键. $p=2$ 的特殊情况本身又依赖于 $L^2(\mu)$ 的完备性，依赖于 $L^2(\mu)$ 因此是一个 Hilbert 空间的事实，并且依赖于 Hilbert 空间上的有界线性泛函由内积给定的事实.

我们现在转向定理 2.14 的复形式.

Riesz 表示定理

6.18 设 X 是一个局部紧的 Hausdorff 空间，定理 2.14 刻画了 $C_c(X)$ 上正的线性泛函. 现在我们有可能来刻画 $C_c(X)$ 上的有界线性泛函 Φ 了. 因为 $C_c(X)$ 是 $C_0(X)$ 关于上确界范数的稠密子空间，每一个这样的 Φ 有唯一一个到 $C_0(X)$ 上的有界线性泛函的开拓. 因此，同样可以假定从处理 Banach 空间 $C_0(X)$ 开始.

如果 μ 是复 Borel 测度，定理 6.12 断定存在一个具有 $|h|=1$

的复 Borel 函数 h, 使得 $d\mu = h d|\mu|$. 因此用公式

(1) $$\int f d\mu = \int f h d|\mu|$$

来定义关于复测度 μ 的积分是合理的.

关系式 $\int \chi_E d\mu = \mu(E)$ 是(1)的特殊情况. 于是当 μ 和 λ 是 $\mathfrak{M}$ 上的复测度和 $E \in \mathfrak{M}$ 时,

(2) $$\int_X \chi_E d(\mu+\lambda) = (\mu+\lambda)(E) = \mu(E) + \lambda(E)$$
$$= \int_X \chi_E d\mu + \int_X \chi_E d\lambda$$

这导出加法公式

(3) $$\int_X f d(\lambda+\mu) = \int_X f d\mu + \int_X f d\lambda$$

(比如说)对每一个有界可测函数 f 成立.

如果 $|\mu|$ 是在定义 2.15 的意义下是正则的, 就称 X 上的复的 Borel 测度 μ 是正则的. 如果 μ 是 X 上的复的 Borel 测度, 显然映射

(4) $$f \to \int_X f d\mu$$

是 $C_0(X)$ 上的有界线性泛函, 它的范数不大于 $|\mu|(X)$. 用这种方法得出 $C_0(X)$ 上全体有界线性泛函就是 Riesz 定理的内容.

6.19 定理 当 X 是局部紧的 Hausdroff 空间时, 对 $C_0(X)$ 上每一个有界线性泛函 Φ, 都存在唯一一个复的正则 Borel 测度 μ, 使得

(1) $$\Phi(f) = \int_X f d\mu \qquad (f \in C_0(X))$$

而且若 Φ 和 μ 如(1)所述, 则

(2) $$\|\Phi\| = |\mu|(X)$$

证明 首先解决唯一性问题. 假定 μ 是 X 上的一个正则复 Borel 测度，并且对所有的 $f\in C_0(X)$，$\int f d\mu=0$. 根据定理 6.12，存在一个 Borel 函数 h，具有 $|h|=1$，使得 $d\mu=hd|\mu|$. 对 $C_0(X)$ 内任一序列 $\{f_n\}$，因此有

$$(3)\qquad |\mu|(X)=\int_X(\bar{h}-f_n)hd|\mu|\leqslant\int_X|\bar{h}-f_n|d|\mu|$$

因为 $C_0(X)$ 在 $L^1(|\mu|)$ 内稠密(定理 3.14)，$\{f_n\}$ 能如此选择，使当 $n\to\infty$ 时，(3)的最后一个表达式趋于 0. 于是 $|\mu|(X)=0$，且 $\mu=0$. 容易看出 X 上的两个正则复 Borel 测度之差是正则的. 这就证明了每一个 Φ，至多有一个 μ 与之对应.

现在考虑在 $C_0(X)$ 上给定的一个有界线性泛函 Φ. 不失一般性，假定 $\|\Phi\|=1$. 我们将构造一个 $C_c(X)$ 上的正的线性泛函 Λ，使得

$$(4)\qquad |\Phi(f)|\leqslant\Lambda(|f|)\leqslant\|f\|\qquad(f\in C_c(X))$$

这里 $\|f\|$ 表示上确界范数.

一旦有了这个 Λ，象定理 2.14 一样，我们把它和一个正的 Borel 测度 λ 联系起来. 定理 2.14 的结论指出：当 $\lambda(X)<\infty$ 时，λ 是正则的. 因为

$$\lambda(X)=\sup\{\Lambda f:\ 0\leqslant f\leqslant 1,\ f\in C_c(X)\}$$

并且当 $\|f\|\leqslant 1$ 时，$|\Lambda f|\leqslant 1$，我们看出确实有 $\lambda(X)\leqslant 1$.

从(4)同时导出

$$(5)\qquad |\Phi(f)|\leqslant\Lambda(|f|)=\int_X|f|d\lambda=\|f\|_1\qquad(f\in C_c(X))$$

这最后一个范数出于空间 $L^1(\lambda)$. 于是 Φ 是 $C_c(X)$ 上线性泛函，它关于 $C_c(X)$ 上的 $L^1(\lambda)$ 范数而言，其范数至多是 1. 存在一个 Φ 到 $L^1(\lambda)$ 上的线性泛函的保范数开拓. 所以定理 6.16($p=1$ 的情况)给出了具有 $|g|\leqslant 1$ 的 Borel 函数 g，使得

(6) $$\Phi(f)=\int_X fgd\lambda \qquad (f\in C_c(X))$$

(6)的每一边均是$C_0(X)$上的连续泛函，而$C_c(X)$是在$C_0(X)$内稠密的. 因此(6)对所有的$f\in C_0(X)$成立. 取$d\mu=gd\lambda$，我们便得到表达式(1).

因为$\|\Phi\|=1$，(6)指出:

(7) $$\int_X |g|d\lambda\geqslant\sup\{|\Phi(f)|:\ f\in C_0(X),\ \|f\|\leqslant 1\}=1$$

我们也知道$\lambda(X)\leqslant 1$，$|g|\leqslant 1$. 这些事实仅当$\lambda(X)=1$和$|g|=1$ a. e. [λ]是相容的. 于是根据定理 6.13，$d|\mu|=|g|d\lambda=d\lambda$，且

(8) $$|\mu|(X)=\lambda(X)=1=\|\Phi\|$$

这就证明了(2).

所有这些都依赖于找出一个满足（4）的正线性泛函Λ. 若$f\in C_c^+(X)$[$C_c(X)$的所有非负实元素的类]，定义

(9) $$\Lambda f=\sup\{|\Phi(h)|:\ h\in C_c(X),\ |h|\leqslant f\}$$

则$\Lambda f\geqslant 0$，Λ满足(4). $0\leqslant f_1\leqslant f_2$蕴涵着$\Lambda f_1\leqslant\Lambda f_2$. 并且若$c$是正的常数，$\Lambda(cf)=c\Lambda f$. 我们必须证明

(10) $$\Lambda(f+g)=\Lambda f+\Lambda g \qquad (f\text{和}g\in C_c^+(X))$$

然后，还得把Λ开拓为$C_c(X)$上的线性泛函.

固定f和$g\in C_c^+(X)$. 若$\varepsilon>0$，则存在h_1和$h_2\in C_c(X)$，使得$|h_1|\leqslant f$，$|h_2|\leqslant g$，且

(11) $$\Lambda f\leqslant|\Phi(h_1)|+\varepsilon,\ \ \Lambda g\leqslant|\Phi(h_2)|+\varepsilon$$

存在复数α_i，$|\alpha_i|=1$，使$\alpha_i\Phi(h_i)=|\Phi(h_i)|$，$i=1,2$. 这时

$$\begin{aligned}\Lambda f+\Lambda g&\leqslant|\Phi(h_1)|+|\Phi(h_2)|+2\varepsilon\\&=\Phi(\alpha_1h_1+\alpha_2h_2)+2\varepsilon\\&\leqslant\Lambda(|h_1|+|h_2|)+2\varepsilon\\&\leqslant\Lambda(f+g)+2\varepsilon\end{aligned}$$

因此，在(10)内，不等号“$\geqslant$”成立.

其次，选择 $h\in C_c(X)$，只要它服从条件 $|h|\leqslant f+g$，设 $V=\{x: f(x)+g(x)>0\}$. 定义

$$h_1(x)=\frac{f(x)h(x)}{f(x)+g(x)},\quad h_2(x)=\frac{g(x)h(x)}{f(x)+g(x)}\quad x\in V,$$
$$h_1(x)=h_2(x)=0\quad (x\notin V) \tag{12}$$

显然，h_1 在 V 的每一点都连续. 若 $x_0\notin V$，则 $h(x_0)=0$；因为 h 是连续的，并且因为 $|h_1(x)|\leqslant|h(x)|$ 对所有 $x\in X$ 成立，得出 x_0 是 h_1 的连续点. 于是 $h_1\in C_c(X)$，并且对于 h_2 也同样成立.

因为 $h_1+h_2=h$ 且 $|h_1|\leqslant f$，$|h_2|\leqslant g$，我们有

$$\begin{aligned}|\Phi(h)|&=|\Phi(h_1)+\Phi(h_2)|\leqslant|\Phi(h_1)|+|\Phi(h_2)|\\&\leqslant\Lambda f+\Lambda g\end{aligned}$$

因此 $\Lambda(f+g)\leqslant\Lambda f+\Lambda g$，并因此证明了(10).

现在，若 f 是一个实函数，$f\in C_c(X)$，则 $2f^+=|f|+f$，因而 $f^+\in C_c^+(X)$；类似地，$f^-\in C_c^+(X)$；因为 $f=f^+-f^-$，自然地定义

$$\Lambda f=\Lambda f^+-\Lambda f^-\quad (f\in C_c(X), f\text{ 是实的}) \tag{13}$$

和

$$\Lambda(u+iv)=\Lambda u+i\Lambda v \tag{14}$$

就象定理 1.31 的证明中所用过的那样，通过一些简单的代数运算，现在可以证明，我们开拓的泛函在 $C_c(X)$ 上是线性的.

这就完成了证明.

习　题

1　若 μ 是 σ-代数 $\mathfrak{M}$ 上的复测度，$E\in\mathfrak{M}$. 定义

$$\lambda(E)=\sup\Sigma|\mu(E_i)|.$$

其中的上确界是对 E 的所有的有限划分 $\{E_i\}$ 取的. 能得出 $\lambda=|\mu|$ 吗？

2　证明 6.10 节末尾所给出的例子具有所述的性质.

3　证明局部紧 Hausdroff 空间 X 上的全体复正则 Borel 测度的向量

空间 $M(X)$ 取 $\|\mu\|=|\mu|(X)$ 时，构成一个 Banach 空间. 提示：与第五章习题 8 比较. [$M(X)$ 的任何两个元素之差在 $M(X)$ 内的结论，已经用于证明的第一段内，试提供这事实的证明.]

4 假定 $1\leqslant p\leqslant\infty$，$q$ 是共轭于 p 的指数. 假定 μ 是一个 σ-有限的测度，g 是可测函数，使得对每一个 $f\in L^p(\mu)$，$fg\in L^1(\mu)$. 证明 $g\in L^q(\mu)$.

5 假定 X 由两个点 a，b 组成. 定义 $\mu(\{a\})=1$，$\mu(\{b\})=\mu(X)=\infty$，$\mu(\varnothing)=0$. 对于这样的 μ，$L^\infty(\mu)$ 是 $L^1(\mu)$ 的对偶空间，对吗？

6 假定 $1<p<\infty$，证明甚至 μ 不是 σ- 有限时，$L^q(\mu)$ 仍是 $L^p(\mu)$ 的对偶空间. (象通常一样 $1/p+1/q=1$.)

7 假定 μ 是 $[0,2\pi)$ 上(或在单位圆周 T 上)的复 Borel 测度. 用

$$\hat{\mu}(n)=\int e^{-int}d\mu(t)\quad(n=0,\pm1,\pm2,\cdots)$$

定义 μ 的 Fourier 系数. 假定当 $n\to+\infty$ 时，$\hat{\mu}(n)\to0$. 证明当 $n\to-\infty$ 时，$\hat{\mu}(n)\to0$ 成立. 提示：当 f 是任一三角多项式时，用 $fd\mu$ 代替 $d\mu$，假定也是成立的. 因之当 f 是连续函数，而 f 是任一有界的 Borel 函数时，以至用 $d(\mu)$ 代替 $d\mu$ 时，假定也都成立.

8 按照习题 7 的术语，找出所有的 μ，使得 $\hat{\mu}$ 是周期为 k 的周期函数. [这意味着 $\hat{\mu}(n+k)=\hat{\mu}(n)$ 对所有的整数 n 成立. 当然 k 也假定是整数.]

9 假定 μ 是 σ-代数 $\mathfrak{M}$ 上的正有限测度. 若 $A\in\mathfrak{M}$，$B\in\mathfrak{M}$，定义

$$\rho(A,B)=\int|\chi_A-\chi_B|d\mu=\mu(A\Delta B)$$

这里的对称差 $A\Delta B$ 由 $A\Delta B=(A-B)\cup(B-A)$ 给定.

(a) 证明 ρ 是一个度量，$(\mathfrak{M},\rho)$ 是完备的度量空间.

(b) 若 λ 是 $\mathfrak{M}$ 上的复测度，证明下列三个命题的每一个都蕴涵着另外两个：

(i) λ 是 $\mathfrak{M}$ 上连续函数.

(ii) λ 在 $\varnothing$ 连续.

(iii) $\lambda\ll\mu$.

10 设 $(X,\mathfrak{M},\mu)$ 是正测度空间. 若对每一个 $\varepsilon>0$，对应一个 $\delta>0$，使得当 $f\in\Phi$ 且 $\mu(E)<\delta$ 时，有

$$\left|\int_E fd\mu\right|<\varepsilon$$

成立，就称集 $\Phi\subset L^1(\mu)$ 是一致可积的.

(*a*) 证明 $L^1(\mu)$ 的每一个有限子集是一致可积的.

(*b*) 证明下列的 Vitali 收敛定理:

若(i) $\mu(X)<\infty$, (ii) $\{f_n\}$ 是一致可积的, (iii) 当 $n\to\infty$ 时, $f_n(x)\to f(x)$, a. e. 且 (iv) $|f(x)|<\infty$, a. e., 则 $f\in L^1(\mu)$ 并且

$$\lim_{n\to\infty}\int_X |f_n-f|\,d\mu=0$$

提示: 用 Egoroff 定理.

(*c*) 证明若 μ 是 $(-\infty,\infty)$ 上的 Lebesgue 测度, 甚至当 $\{\|f_n\|_1\}$ 假定是有界时, (*b*) 也不成立. 因此, 假定(i)在(*b*)内不能省去.

(*d*) 指出对某些 μ (比如, 有限区间上的 Lebesgue 测度), 假定(iv)在(*b*)中是多余的, 但是存在有限测度, 对于这些测度(iv)的省略将使得(*b*)不成立.

(*e*) 证明对于有限测度空间 Vitali 定理蕴涵着 Lebesgue 控制收敛定理. 构造一个尽管 Lebesgue 定理的假定不成立, Vitali 定理仍能应用的例子.

(*f*) 构造一个[0, 1]上的序列 $\{f_n\}$, 使之对每一个 x, $f_n(x)\to 0$, $\int f_n\to 0$, 但是 $\{f_n\}$ (关于 Lebesgue 测度)不是一致可积的.

(*g*) 然而, 下面的 Vitali 定理的逆命题是成立的:

假定 $\mu(X)<\infty$, $f\in L^1(\mu)$, $\{f_n\}\subset L^1(\mu)$, $f_n(x)\to f(x)$, a. e. 且

$$\lim_{n\to\infty}\int_E f_n(x)\,d\mu=\int_E f\,d\mu$$

对每一个 $E\in\mathfrak{M}$ 成立, 则 $\{f_n\}$ 是一致可积的.

通过完成下面的要点, 证明这个结论.

取 $f=0$, 固定 $\varepsilon>0$. 象习题 9 那样定义 ρ. 对每个 n

$$E\to\int_E f_n\,d\mu$$

是 $(\mathfrak{M},\rho)$ 上的连续函数. 根据第五章习题 13(*b*), 存在一个 $E_0\in\mathfrak{M}$, $\delta>0$, 和 N, 使得当 $n>N$, $\rho(E,E_0)<\delta$ 时

$$\left|\int_E f_n\,d\mu\right|<\varepsilon$$

取具有 $\mu(A)<\delta$ 的 $A\in\mathfrak{M}$, 令 $B=E_0-A$, $C=E_0\cup A$. 则上面的不等式用 B 和 C 代替 E 时仍然成立. 注意 $\chi_A=\chi_C-\chi_B$. 若 $n\geq N$, 得出 $\left|\int_A f_n\,d\mu\right|<2\varepsilon$.

最后，求助于(a)．

11　假定 μ 是 X 上的正测度，$\mu(X)<\infty$，$f_n\in L^1(\mu)$，$n=1,2,3,\cdots$，$f_n(x)\to f(x)$ a. e.，并且存在 $p>1$，和 $C<\infty$，使得 $\int_X|f_n|^p d\mu<C$ 对所有的 n 成立．证明

$$\lim_{n\to\infty}\int_X|f-f_n|d\mu=0$$

提示：$\{f_n\}$ 是一致可积的．

第七章　乘积空间上的积分

本章致力于证明并讨论关于二元函数积分的 Fubiui 定理. 首先以抽象的形式提出这个定理.

笛卡儿乘积上的可测性

7.1　定义　若 X 和 Y 是两个集,它们的笛卡儿乘积 $X\times Y$ 是所有 $x\in X$, $y\in Y$ 的序对 (x, y) 的集. 若 $A\subset X$ 和 $B\subset Y$,便得出 $A\times B\subset X\times Y$. 称任一形如 $A\times B$ 的集为 $X\times Y$ 内的矩形.

现在假定 $(X,\mathscr{S})$ 和 $(Y,\mathscr{T})$ 是可测空间. 回忆一下,这无非是说 $\mathscr{S}$ 是 X 内的 σ-代数,$\mathscr{T}$ 是 Y 内的 σ-代数.

一个可测矩形是任一形如 $A\times B$ 的集,这里 $A\in\mathscr{S}$, $B\in\mathscr{T}$.

若 $Q=R_1\cup\cdots\cup R_n$,这里每个 R_i 是可测矩形,且对 $i\neq j$, $R_i\cap R_j=\varnothing$,我们就说 $Q\in\mathscr{E}$,即所有基本集的类.

$\mathscr{S}\times\mathscr{T}$ 定义为包含所有的可测矩形的 $X\times Y$ 的最小的 σ-代数.

单调类 $\mathfrak{M}$ 是具有下列性质的集族:若 $A_i\in\mathfrak{M}$, $B_i\in\mathfrak{M}$, $A_i\supset A_{i+1}$, $B_i\supset B_{i+1}$,对 $i=1,2,3,\cdots$ 成立,且若

$$(1)\qquad A=\bigcap_{i=1}^{\infty}A_i,\qquad B=\bigcap_{i=1}^{\infty}B_i$$

则 $A\in\mathfrak{M}$ 和 $B\in\mathfrak{M}$.

若 $E\subset X\times Y$, $x\in X$, $y\in Y$,定义

$$(2)\qquad E_x=\{y:\ (x,y)\in E\},\ E^y=\{x:\ (x,y)\in E\}$$

分别称 E_x 和 E^y 为 E 的 x-截口 和 y-截口. 注意 $E_x\subset Y$, $E^y\subset X$.

7.2 定理 若 $E\in\mathscr{S}\times\mathscr{T}$，则对每一个 $x\in X$ 和 $y\in Y$，$E_x\in\mathscr{T}$ 和 $E^y\in\mathscr{S}$.

证明 设 Ω 是所有 $E\in\mathscr{S}\times\mathscr{T}$ 的类，使得 $E_x\in\mathscr{T}$ 对每一个 $x\in X$ 成立. 若 $E=A\times B$，则当 $x\in A$ 时，$E_x=B$，当 $x\notin A$ 时，$E_x=\varnothing$. 因此每一个可测矩形属于 Ω. 因为 $\mathscr{T}$ 是 σ-代数，下列三个命题都是正确的. 它们证明了 Ω 是一个 σ-代数，因此 $\Omega=\mathscr{S}\times\mathscr{T}$.

(a) $X\times Y\in\Omega$.

(b) 若 $E\in\Omega$，则 $(E^c)_x=(E_x)^c$，因此，$E^c\in\Omega$.

(c) 若 $E_i\in\Omega$，$(i=1,2,3,\cdots)$，且 $E=\bigcup E_i$，则 $E_x=\bigcup(E_i)_x$. 因此 $E\in\Omega$.

对 E^y 证明是同样的.

7.3 定理 $\mathscr{S}\times\mathscr{T}$ 是包含所有基本集的最小的单调类.

证明 设 $\mathfrak{M}$ 是包含 $\mathscr{E}$ 的最小单调类；这种类的存在的证明恰好象定理 1.10 的证明那样. 因为 $\mathscr{S}\times\mathscr{T}$ 是一个单调类，有 $\mathfrak{M}\subset\mathscr{S}\times\mathscr{T}$.

恒等式

$$(A_1\times B_1)\cap(A_2\times B_2)=(A_1\cap A_2)\times(B_1\cap B_2)$$

$$\begin{aligned}&(A_1\times B_1)-(A_2\times B_2)\\&=[(A_1-A_2)\times B_1]\cup[(A_1\cap A_2)\times(B_1-B_2)]\end{aligned}$$

表明，两个可测矩形的交是可测矩形，它们的差是两个不相交的可测矩形之并，因而都是基本集. 若 $P\in\mathscr{E}$ 和 $Q\in\mathscr{E}$，容易得出

$$P\cap Q\in\mathscr{S}\quad\text{和}\quad P-Q\in\mathscr{S}.$$

因为

$$P\cup Q=(P-Q)\cup Q$$

和 $(P-Q)\cap Q=\varnothing$，故同样有 $P\cup Q\in\mathscr{E}$.

对于任一集 $P\subset X\times Y$，定义 $\Omega(P)$ 为所有使得 $P-Q\in\mathfrak{M}$，$Q-P\in\mathfrak{M}$ 和 $P\cup Q\in\mathfrak{M}$ 的那些 $Q\subset X\times Y$ 的类. 下面的性质是明显的.

(*a*) $Q\in\Omega(P)$当且仅当$P\in\Omega(Q)$.

(*b*) 因为$\mathfrak{M}$是单调类,每一个$\Omega(P)$也是单调类.

固定$P\in\mathscr{E}$，上述关于$\mathscr{E}$的评注指出：对于所有的$Q\in\mathscr{E}$，$Q\in\Omega(P)$. 因此$\mathscr{E}\subset\Omega(P)$. 并且现在(*b*)蕴涵着$\mathfrak{M}\subset\Omega(P)$.

其次，固定$Q\in\mathfrak{M}$. 若$P\in\mathscr{E}$，刚才看到了$Q\in\Omega(P)$. 根据(*a*),$P\in\Omega(Q)$,因此$\mathscr{E}\subset\Omega(Q)$. 若再一次运用(*b*),便得

$$\mathfrak{M}\subset\Omega(Q).$$

总结：若P和$Q\in\mathfrak{M}$,则$P-Q\in\mathfrak{M}$,$P\cup Q\in\mathfrak{M}$.

现在推导$\mathfrak{M}$是$X\times Y$内的σ-代数:

(i) $X\times Y\in\mathscr{E}$,因此$X\times Y\in\mathfrak{M}$,

(ii) 因为$\mathfrak{M}$的任意两个元素之差在$\mathfrak{M}$内，所以若$Q\in\mathfrak{M}$，则$Q^c\in\mathfrak{M}$.

(iii) 若$P_i\in\mathfrak{M}$,$i=1,2,3,\cdots$,$P=\bigcup P_i$,令

$$Q_n=P_1\cup\cdots\cup P_n$$

因为$\mathfrak{M}$对有限并运算封闭,$Q_n\in\mathfrak{M}$,

因为$Q_n\subset Q_{n+1}$且$P=\bigcup Q_n$,$\mathfrak{M}$的单调性指出$P\in\mathfrak{M}$.

于是$\mathfrak{M}$是一个σ-代数,$\mathscr{E}\subset\mathfrak{M}\subset\mathscr{S}\times\mathscr{T}$，并且(根据定义)$\mathscr{S}\times\mathscr{T}$是包含$\mathscr{E}$的最小的$\sigma$-代数,因此,$\mathfrak{M}=\mathscr{S}\times\mathscr{T}$.

7.4 定义 对$X\times Y$上每一个函数f和每一个$x\in X$，我们对应一个用$f_x(y)=f(x,y)$定义的Y上的函数f_x.

类似地,若$y\in Y$,f^y是用$f^y(x)=f(x,y)$定义的X上的函数.

因为我们现在与三个σ-代数,$\mathscr{S}$,$\mathscr{T}$，和$\mathscr{S}\times\mathscr{T}$打交道,为了清楚起见,今后提及"可测"一词时，将指明是就三个σ-代数中的那一个而言的.

7.5 定理 设f是一个$X\times Y$上的$(\mathscr{S}\times\mathscr{T})$-可测函数，则

(*a*) 对于每一个$x\in X$,f_x是一个$\mathscr{T}$-可测函数.

(*b*) 对于每一个$y\in X$,f^y是一个$\mathscr{S}$-可测函数.

证明 对任一开集 V，令

$$Q=\{(x,y):\ f(x,y)\in V\}$$

则 $Q\in\mathscr{S}\times\mathscr{T}$，且

$$Q_x=\{y:\ f_x(y)\in V\}$$

定理 7.2 指出 $Q_x\in\mathscr{T}$. 这就证明了 (a)；(b)的证明是类似的.

乘积测度

7.6 定理 设$(X,\mathscr{S},\mu)$和$(Y,\mathscr{T},\lambda)$都是 σ-有限的测度空间. 假定 $Q\in\mathscr{S}\times\mathscr{T}$. 若对每一个 $x\in X$ 和 $y\in Y$,

(1) $$\varphi(x)=\lambda(Q_x),\qquad \psi(y)=\mu(Q^y)$$

则 φ 是$\mathscr{S}$-可测的，ψ 是$\mathscr{T}$-可测的，且

(2) $$\int_X\varphi d\mu=\int_Y\psi d\lambda$$

注 关于测度空间的假定，更明确地说，就是 μ 和 λ 分别是$\mathscr{S}$和$\mathscr{T}$上的正测度，X是可数个不相交的具有 $\mu(X_n)<\infty$ 的集 X_n 的并，Y 是可数个不相交的具有$\lambda(Y_n)<\infty$的集 Y_n 的并.

定理 7.2 表明定义(1)是有意义的. 因为

(3) $$\lambda(Q_x)=\int_Y\chi_Q(x,y)d\lambda(y)\quad(x\in X)$$

对于 $\mu(Q^y)$具有类似的说法，因此结论(2)可以写成下列形式:

(4) $$\int_X d\mu(x)\int_Y\chi_Q(x,y)d\lambda(y)$$

$$=\int_Y d\lambda(y)\int_X\chi_Q(x,y)d\mu(x)$$

证明 设Ω是所有使得定理结论成立的那些$Q\in\mathscr{S}\times\mathscr{T}$ 的类. 我们要证明Ω具有下列四个性质:

(a) 每一个可测矩形属于Ω.

(b) 若 $Q_1\subset Q_2\subset Q_3\subset\cdots$，每个 $Q_i\in\Omega$，且 $Q=\bigcup Q_i$，则 $Q\in\Omega$.

(c) 若$\{Q_i\}$是不相交的Ω的元素的可数集族，且$Q=\bigcup Q_i$，则$Q\in\Omega$.

(d) 设$\mu(A)<\infty,\lambda(B)<\infty$，且

$$A\times B\supset Q_1\supset Q_2\supset Q_3\supset\cdots$$

若对$i=1,2,3,\cdots,Q_i\in\Omega,Q=\bigcap Q_i$，则$Q\in\Omega$.

若$Q=A\times B$，这里$A\in\mathscr{S},B\in\mathscr{T}$，则

(5) $$\lambda(Q_x)=\lambda(B)\chi_A(x)\quad\text{和}\quad\mu(Q^y)=\mu(A)\chi_B(y)$$

因此(2)中每一个积分都等于$\mu(A)\lambda(B)$. 这就给出(a).

为证明(b)，象(1)使φ和ψ与Q对应一样，设φ_i和ψ_i与Q_i相对应. λ和μ的可数可加性指出

(6) $$\varphi_i(x)\to\varphi(x),\quad\psi_i(y)\to\psi(y)\qquad(i\to\infty)$$

在每一点上是单调递增地收敛的. 因为φ_i和ψ_i假定是满足定理结论的，从单调收敛定理即得出(b).

对于不相交集的有限并，(c)是显然的，因为不相交集的并的特征函数是它们的特征函数的和. (c) 的一般情况现在从 (b) 得出.

除了用控制收敛定理代替单调收敛定理以外，(d) 的证明与(b)的证明相象. 因为$\mu(A)<\infty$和$\lambda(B)<\infty$，这样做是合理的.

现在定义

(7) $$Q_{mn}=Q\cap(X_n\times Y_n)\quad(m,n=1,2,3,\cdots)$$

并且设$\mathfrak{M}$是所有使得$Q_{mn}\in\Omega$对m和n所有选择都能成立的那些集$Q\in\mathscr{S}\times\mathscr{T}$的类，则(b)和(d)表明$\mathfrak{M}$是一个单调类；(a)和(c)指出$\mathscr{E}\subset\mathfrak{M}$，并且因为$\mathfrak{M}\subset\mathscr{S}\times\mathscr{T}$，定理 7.3 蕴涵着$\mathfrak{M}=\mathscr{S}\times\mathscr{T}$.

于是，对每一个$Q\in\mathscr{S}\times\mathscr{T}$和对$m,n$的所有选择，$Q_{mn}\in\Omega$. 因为$Q$是集$Q_{mn}$的并，并且这些集是不相交的，从(c)得出$Q\in\Omega$. 这就完成了证明.

7.7 定义 如果$(X,\mathscr{S},\mu)$和$(Y,\mathscr{T},\lambda)$是象定理 7.6 中的

一样，且 $Q\in\mathscr{S}\times\mathscr{T}$. 我们定义

$$(1)\qquad (\mu\times\lambda)(Q)=\int_X \lambda(Q_x)d\mu(x)=\int_Y \mu(Q^y)d\lambda(y)$$

(1)中积分的等式是定理 7.6 的内容. 我们称 $\mu\times\lambda$ 是测度 μ 和 λ 的乘积. 从定理 1.27 直接推得 $\mu\times\lambda$ 实际上是一个测度(即 $\mu\times\lambda$ 是在 $\mathscr{S}\times\mathscr{T}$ 上可数可加的).

也可以看出 $\mu\times\lambda$ 是 σ-有限的.

Fubini 定理

7.8 定理 设 $(X,\mathscr{S},\mu)$ 和 $(Y,\mathscr{T},\lambda)$ 是 σ-有限的测度空间，设 f 是 $X\times Y$ 上的 $(\mathscr{S}\times\mathscr{T})$-可测函数.

(*a*) 若 $0\leqslant f\leqslant\infty$，且若

$$(1)\qquad \varphi(x)=\int_Y f_x d\lambda,\quad \psi(y)=\int_X f^y d\mu\quad (x\in X,\ y\in Y),$$

则 φ 是 $\mathscr{S}$-可测的，ψ 是 $\mathscr{T}$-可测的，且

$$(2)\qquad \int_X \varphi d\mu=\int_{X\times Y} f d(\mu\times\lambda)=\int_Y \psi d\lambda$$

(*b*) 若 f 是复的，且若

$$(3)\qquad \varphi^*(x)=\int_Y |f|_x d\lambda \text{ 和 } \int_X \varphi^* d\mu<\infty$$

则 $f\in L^1(\mu\times\lambda)$.

(*c*) 若 $f\in L^1(\mu\times\lambda)$，则 $f_x\in L^1(\lambda)$，对几乎所有的 $x\in X$ 成立，$f^y\in L^1(\mu)$ 对几乎所有的 $y\in Y$ 成立. 由(1)式 a. e. 定义的函数 φ 和 ψ 分别在 $L^1(\mu)$ 和 $L^1(\lambda)$ 内，并且(2)成立.

注 (2) 的第一个积分和最后一个积分也能写成更为常见的形式.

$$(4)\qquad \int_X d\mu(x)\int_Y f(x,y)d\lambda(y)=\int_Y d\lambda(y)\int_X f(x,y)d\mu(x)$$

这些就是所谓的 f 的“累次积分”，(2)式中间的积分经常被称为重

积分.

(*b*)和(*c*)结合起来给出了下列有用的结果：若 f 是 $(\mathscr{S}\times\mathscr{T})$-可测，且若

$$\int_X d\mu(x)\int_Y |f(x,y)|\,d\lambda(y)<\infty \tag{5}$$

则(4)的两个累次积分是有限的并且相等.

换句话说，对于 $(\mathscr{S}\times\mathscr{T})$-可测函数 f，当 $f\geqslant 0$ 时，“积分的次序可以交换.”当 $|f|$ 的累次积分之一是有限时，“积分的次序也可以交换.”

证明 首先考虑(*a*). 根据定理 7.5，φ 和 ψ 的定义是有意义的. 假定 $Q\in\mathscr{S}\times\mathscr{T}$ 且 $f=\chi_Q$. 根据定义 7.7，(2) 恰好是定理 7.6 的结论. 因此(*a*)对所有非负简单 $(\mathscr{S}\times\mathscr{T})$-可测函数 s 成立. 在一般情况下，存在这种函数序列 s_n，使得 $0\leqslant s_1\leqslant s_2\leqslant\cdots$ 且在 $X\times Y$ 的每一点 $s_n(x,y)\to f(x,y)$. 若象 φ 与 f 对应一样，φ_n 与 s_n 相对应，则有

$$\int_X \varphi_n\,d\mu=\int_{X\times Y} s_n\,d(\mu\times\lambda)\qquad(n=1,2,3,\cdots) \tag{6}$$

单调收敛定理应用于 $(Y,\mathscr{T},\lambda)$ 上时指出，对每一个 $x\in X$，当 $n\to\infty$ 时 $\varphi_n(x)$ 递增到 $\varphi(x)$. 因此，再一次应用单调收敛定理于(6)的两个积分，便得到(2)的第一个等式. 通过交换 x 和 y 的地位，得出(2)的后半部分. 这就完成了(*a*)的证明.

若应用(*a*)于 $|f|$，可看出(*b*)是正确的.

显然，只要对实的 $L^1(\mu\times\lambda)$ 证明(*c*)成立就足够了；复的情况跟着就能得出. 若 f 是实的，将(*a*)应用于 f^+ 和 f^-. 象(1)中 φ 对应于 f 一样，设 φ_1 和 φ_2 对应于 f^+ 和 f^-. 因为 $f\in L^1(\mu\times\lambda)$ 并且 $f^+\leqslant|f|$，又因为 (*a*) 对 f^+ 成立，看出 $\varphi_1\in L^1(\mu)$. 类似地，$\varphi_2\in L^1(\mu)$. 因为

(7) $$f_x=(f^+)_x-(f^-)_x$$

所以对于使得 $\varphi_1(x)<\infty$ 和 $\varphi_2(x)<\infty$ 的每一个 x，有 $f_x\in L^1(\lambda)$；因为 φ_1 和 φ_2 均在 $L^1(\mu)$ 内，故对于几乎所有的 x，都有 $\varphi_1(x)<\infty$ 和 $\varphi_2(x)<\infty$，并且对任一这样的 x，有 $\varphi(x)=\varphi_1(x)-\varphi_2(x)$. 因此 $\varphi\in L^1(\mu)$. 现在用 φ_1 和 f^+ 及用 φ_2 和 f^- 代替 φ 和 f，(2)式都成立；若将所得的两个方程相减，我们可以得到(c)的一半. 用 f^x 和 ψ 代替 f_x 和 φ，同样的方法可证明另一半.

7.9 反例 下面的三个例子将指出，定理 7.6 和 7.8 的各种假定都是不能删掉的.

(a) 设 $X=Y=[0,1]$，$\mu=\lambda=$ 在 $[0,1]$ 上的Lebesgue测度. 选取 $\{\delta_n\}$ 使 $0=\delta_1<\delta_2<\delta_3<\cdots$，$\delta_n\to 1$，且设 g_n 是支集位于 (δ_n,δ_{n+1}) 内的实连续函数，使得 $\int_0^1 g_n(t)dt=1$，对 $n=1,2,3,\cdots$ 成立.

定义

$$f(x,y)=\sum_{n=1}^{\infty}[g_n(x)-g_{n+1}(x)]g_n(y)$$

注意在每一点 (x,y)，式中至多有一项异于 0. 于是在 f 的定义中没有收敛性的问题. 一个简单的计算指出

$$\int_0^1 dx\int_0^1 f(x,y)dy=1\neq 0=\int_0^1 dy\int_0^1 f(x,y)dx$$

因此虽然两个累次积分存在，Fubini 定理的结论却不成立. 注意在这个例子中，除去在 $(1,1)$ 这点以外，f 是连续的. 但是

$$\int_0^1 dx\int_0^1 |f(x,y)|dy=\infty$$

(b) 设 $X=Y=[0,1]$，$\mu=[0,1]$ 上的 Lebesgue 测度，$\lambda=Y$ 上的计数测度. 若 $x=y$，令 $f(x,y)=1$；若 $x\neq y$ 令 $f(x,y)=0$，则

$$\int_X f(x,y)d\mu(x)=0,\qquad \int_Y f(x,y)d\lambda(y)=1$$

对所有[0,1]内的 x 和 y 成立. 因此

$$\int_Y d\lambda(y)\int_X f(x,y)d\mu(x) = 0 \neq 1$$

$$= \int_X d\mu(x)\int_Y f(x,y)d\lambda(y)$$

这次是由于 λ 不是 σ-有限而导致失败.

请注意,若$\mathscr{S}$是所有[0, 1]的 Lebesgue 可测集的类而$\mathscr{T}$是由[0,1]的所有子集组成，则我们的函数 f 是$(\mathscr{S}\times\mathscr{T})$-可测的. 要看出这点,注意 $f=\chi_D$，这里D是单位正方形的对角线. 给定 n, 令

$$I_j=\left[\frac{j-1}{n},\frac{j}{n}\right]$$

并且令

$$Q_n=(I_1\times I_1)\cup(I_2\times I_2)\cup\cdots\cup(I_n\times I_n)$$

则 Q_n 是可测矩形的有限并,且 $D=\cap Q_n$.

(c) 在例(a)和例(b)中,Fubini 定理不成立是由于或者函数或者空间"太大"造成的. 现在转而讨论f关于 σ-代数$\mathscr{S}\times\mathscr{T}$是可测的这一要求所起的作用.

为了更加精确地提出问题，假定 $\mu(X)=\lambda(Y)=1$，$0\leqslant f\leqslant 1$ (因此"大"这一点当然就避免了)；假定对于所有的 x 和 y，f_x 是 $\mathscr{T}$-可测而 f^y 是$\mathscr{S}$-可测的,并且假定 φ 是$\mathscr{S}$-可测和 ψ 是$\mathscr{T}$-可测的. 这里 φ 和 ψ 象 7.8(1)一样地定义. 则 $0\leqslant\varphi\leqslant 1$,和 $0\leqslant\psi\leqslant 1$, 并且两个累次积分都是有限的. (注意在定义累次积分时不需要涉及乘积测度.)能得出 f 的两个累次积分相等吗?

这个(可能令人惊愕的)回答是,否.

在下面(由 Sierpinski 给出)的例子中,我们取

$$(X,\mathscr{S},\mu)=(Y,\mathscr{T},\lambda)=[0,1]$$

具有 Lebesgue 测度. 这个构造依赖于连续统假设.存在一个单位

区间[0，1]到良序集W上的一一映象j，使得对每个$x\in[0,1]$，$j(x)$在W内至多有可数多个先行元素．这是该假设的一个推论．我们承认它的合理性．设Q是单位正方形内所有使得$j(x)$在W内前于$j(y)$的那些(x,y)的集．对于每一个$x\in[0,1]$，Q_x包含了$[0,1]$的除去可数多个点外的所有点；对于每一个$y\in[0,1]$，Q^y至多包含$[0,1]$的可数多个点．若$f=\chi_Q$，得出f_x和f^y是Borel可测的，并且得出

$$\varphi(x)=\int_0^1 f(x,y)dy=1,\quad \psi(y)=\int_0^1 f(x,y)dx=0$$

对所有的x和y成立．因此

$$\int_0^1 dx\int_0^1 f(x,y)dy=1\neq 0=\int_0^1 dy\int_0^1 f(x,y)dx$$

乘积测度的完备化

7.10 若$(X,\mathscr{S},\mu)$和$(Y,\mathscr{T},\lambda)$都是完备测度空间，$(X\times Y,\mathscr{S}\times\mathscr{T},\mu\times\lambda)$却不一定是完备的．这种现象并没有什么毛病：假定存在一个具有$\mu(A)=0$的$A\in\mathscr{S}$，$A\neq\varnothing$；并假定存在一个$B\subset Y$，使$B\notin\mathscr{T}$．则$A\times B\subset A\times Y$，$(\mu\times\lambda)(A\times Y)=0$，但$A\times B\notin\mathscr{S}\times\mathscr{T}$．（后一个断言由定理7.2得出．）

例如，若$\mu=\lambda=m_1$（R^1上的Lebesgue测度），设A由任一点组成，并且设B是R^1的任一不可测集．于是$m_1\times m_1$不是一个完备测度；特别$m_1\times m_1$不是m_2．因为后者根据它的构造是完备的．然而，m_2是$m_1\times m_1$的完备化．这个结果推广到任意维数：

7.11 定理 设m_k表示R^k上的Lebesgue测度．若$k=r+s$，$r\geqslant 1$，$s\geqslant 1$，则m_k是$m_r\times m_s$乘积测度的完备化．

证明 设$\mathscr{B}_k$和$\mathfrak{M}_k$分别是R^k的所有Borel集的σ-代数和R^k的所有Lebesgue可测集的σ-代数．首先证明

(1) $$\mathscr{B}_k\subset\mathfrak{M}_r\times\mathfrak{M}_s\subset\mathfrak{M}_k$$

每一个 k-胞腔属于 $\mathfrak{M}_r \times \mathfrak{M}_s$. 用 k-胞腔所生成的 σ-代数是 $\mathscr{B}_k$. 因此 $\mathscr{B}_k \subset \mathfrak{M}_r \times \mathfrak{M}_s$. 其次，假定 $E \in \mathfrak{M}_r$ 和 $F \in \mathfrak{M}_s$. 根据定理 2.20(b)，容易看出 $E \times R^s$ 和 $R^r \times F$ 同时属于 $\mathfrak{M}_k$. 对它们的交 $E \times F$ 同样是正确的. 于是得出

$$\mathfrak{M}_r \times \mathfrak{M}_s \subset \mathfrak{M}_k.$$

选取 $Q \in \mathfrak{M}_r \times \mathfrak{M}_s$，则 $Q \in \mathfrak{M}_k$. 所以存在 P_1 和 $P_2 \in \mathscr{B}_k$，使得 $P_1 \subset Q \subset P_2$ 且 $m_k(P_2 - P_1) = 0$. m_k 和 $m_r \times m_s$ 二者都是 R^k 平移不变的 Borel 测度. 它们对每一个 k-胞腔指定同一值. 根据定理 2.20(d)，因此在 $\mathscr{B}_k$ 上它们是一致的. 特别，

$$\begin{aligned}(m_r \times m_s)(Q - P_1) &\leqslant (m_r \times m_s)(P_2 - P_1) \\ &= m_k(P_2 - P_1) = 0.\end{aligned}$$

并且因此

$$(m_r \times m_s)(Q) = (m_r \times m_s)(P_1) = m_k(P_1) = m_k(Q)$$

所以在 $\mathfrak{M}_r \times \mathfrak{M}_s$ 上 $m_r \times m_s$ 和 m_k 一致.

现在得出 $\mathfrak{M}_k$ 是 $\mathfrak{M}_r \times \mathfrak{M}_s$ 的 $(m_r \times m_s)$-完备化，而这就是定理所断言的.

我们用 Fubini 定理的另一种说法来结束这一节. 这个结论就定理 7.11 看来是特别感兴趣的部分.

7.12 定理 设 $(X, \mathscr{S}, \mu)$ 和 $(Y, \mathscr{T}, \lambda)$ 是完备的 σ-有限测度空间. 设 $(\mathscr{S} \times \mathscr{T})^*$ 是 $\mathscr{S} \times \mathscr{T}$ 关于测度 $\mu \times \lambda$ 的完备化. 设 f 是 $X \times Y$ 上的 $(\mathscr{S} \times \mathscr{T})^*$-可测函数，则定理 7.8 的结论成立，仅有的不同之处如下：

f_x 的 $\mathscr{T}$-可测性只能对几乎所有 $x \in X$ 断定是正确的. 因此根据 7.8(1)，$\varphi(x)$ 只是 a.e. $[\mu]$ 定义；类似结论对 f^y 和 ψ 也成立.

这个证明依赖于下列两个引理：

引理 1 假定 ν 是 σ-代数 $\mathfrak{M}$ 上的正测度，$\mathfrak{M}^*$ 是 $\mathfrak{M}$ 关于 ν 的完备化，并且 f 是一个 $\mathfrak{M}^*$-可测函数. 则存在一个 $\mathfrak{M}$-可测函数 g，

使得 $f=g$ a. e. $[\mu]$.

(当 ν 是 R^k 的 Lebesgue 测度并且$\mathfrak{M}$是所有的 R^k 的 Borel 集的类时,就出现这个引理的一个有趣的特殊情况.)

引理 2 设 h 是 $X\times Y$ 上 $(\mathscr{S}\times\mathscr{T})^*$-可测函数,使得 $h=0$ a. e. $[\mu\times\lambda]$. 则对于几乎所有的 $x\in X$, $h(x,y)=0$ 对几乎所有的 $y\in Y$ 成立. 特别,对几乎所有的 $x\in X$, h_x 是$\mathscr{T}$-可测的. 对于 f^y 类似的结论也成立.

若我们假定两个引理成立,定理的证明立即可以得到:若 f 如同定理所述的一样,引理 1 (对于 $\nu=\mu\times\lambda$)指出,$f=g+h$,这里 $h=0$ a. e. $[\mu\times\lambda]$,并且 g 是$(\mathscr{S}\times\mathscr{T})$-可测的. 定理 7.8 应用于 g. 引理 2 指出 $f_x=g_x$ a. e. $[\lambda]$ 对几乎所有的 x 成立,且 $f^y=g^y$ a. e. $[\mu]$ 对几乎所有的 y 成立. 因此 f 的两个累次积分和重积分与 g 的累次积分和重积分相同,从而定理得证.

引理 1 的证明 假定 f 是 $\mathfrak{M}^*$-可测且 $f\geqslant 0$. 存在 $\mathfrak{M}^*$-可测的简单函数 $0=s_0\leqslant s_1\leqslant s_2\leqslant\cdots$ 使得对每一个 $x\in X$, 当 $n\to\infty$ 时 $s_n(x)\to f(x)$. 因此 $f=\Sigma(s_{n+1}-s_n)$. 因为 $s_{n+1}-s_n$ 是特征函数的有限的线性组合,所以存在常数 $c_i>0$ 和集 $E_i\in\mathfrak{M}^*$,使得

$$f(x)=\sum_{i=1}^{n}c_i\chi_{E_i}(x)\qquad(x\in X)$$

$\mathfrak{M}^*$的定义指出(见定理 1.36):存在集 $A_i\in\mathfrak{M}$, $B_i\in\mathfrak{M}$,使得 $A_i\subset E_i\subset B_i$ 且 $\nu(B_i-A_i)=0$. 定义

$$g(x)=\sum_{i=1}^{\infty}c_i\chi_{A_i}(x)\qquad(x\in X)$$

则函数 g 是$\mathfrak{M}$-可测的. 且可能除去 $x\in\bigcup(E_i-A_i)\subset\bigcup(B_i-A_i)$ 的那些点外, $g(x)=f(x)$. 因为 $\nu(B_i-A_i)=0$ 对每一个 i 成立,得出 $g=f$ a. e. $[\nu]$. 从这个结果可以得出一般情况 (f 是实的或复的).

引理 2 的证明 设 P 是 $X\times Y$ 的所有使 $h(x,y)\neq 0$ 的点的集. 则 $P\in(\mathscr{S}\times\mathscr{T})^*$ 且 $(\mu\times\lambda)(P)=0$. 因此存在一个 $Q\in\mathscr{S}\times\mathscr{T}$, 使得 $P\subset Q$ 且 $(\mu\times\lambda)Q=0$. 根据定理 7.6 有

$$\int_X \lambda(Q_x)d\mu(x)=0 \tag{1}$$

设 N 是使 $\lambda(Q_x)>0$ 的所有 $x\in X$ 的集. 从(1)得出 $\mu(N)=0$. 对每一个 $x\notin N$, $\lambda(Q_x)=0$. 因为 $P_x\subset Q_x$, 且 $(Y,\mathscr{T},\lambda)$ 是完备的测度空间, 若 $x\notin N$. 则 P_x 的每一个子集属于 $\mathscr{T}$. 若 $y\notin P_x$, 则 $h_x(y)=0$. 于是我们看出: 对每一个 $x\notin N$, h_x 是 $\mathscr{T}$-可测的, 且 $h_x(y)=0$ a. e. $[\lambda]$.

卷积

7.13 经常遇到能用证明某个集确实很大的方法证明该集是不空的. 这个"大"字自然可以涉及各种性质. 其中之一(比较原始的一个)是势. 存在超越数的著名的证明提供了一个例子: 仅存在可数多个代数数但是存在不可数个实数. 因此, 超越实数的集是非空的. Baire 定理的应用基于"大"的拓扑概念: 稠密的 G_δ 集是完备度量空间的"大"子集. "大"的第三种形式是测度论的形式: 人们可以尝试用证明测度空间中某集具有正测度或用证明它的余集具有零测度的方法更加好些, 来证明该集是不空的. Fubini 定理在这种形式的论证中经常出现.

例如, 设 f 和 $g\in L^1(R^1)$, 暂时假定 $f\geqslant 0$ 和 $g\geqslant 0$, 并考虑积分

$$h(x)=\int_{-\infty}^{\infty} f(x-t)g(t)dt \qquad (-\infty<x<\infty) \tag{1}$$

对任一固定的 x, (1)的被积函数是值域在 $[0,\ \infty]$ 内的可测函数, 以致 $h(x)$ 一定可以用(1)很好地确定, 并且 $0\leqslant h(x)\leqslant\infty$.

但是否存在一个 x 使 $h(x)<\infty$ 呢? 注意到对每一个固定的

x,(1)中的被积函数是 L^1 的两个元素的乘积，并且这样的乘积不总是在 L^1 内的．[例如当 $0<x<1$ 时，$f(x)=g(x)=1/\sqrt{x}$，当 x 为其他值时为 0.] Fubini 定理将给出一个肯定的回答．事实上，它将指出 $h\in L^1(R^1)$，因此，$h(x)<\infty$ a. e.

7.14 定理 假定 $f\in L^1(R^1)$，$g\in L^1(R^1)$．则

$$\int_{-\infty}^{+\infty}|f(x-y)g(y)|dy<\infty \tag{1}$$

对几乎所有的 x 成立．对于这些 x，定义

$$h(x)=\int_{-\infty}^{\infty}f(x-y)g(y)dy \tag{2}$$

则 $h\in L^1(R^1)$，并且

$$\|h\|_1\leqslant\|f\|_1\|g\|_1 \tag{3}$$

这里

$$\|f\|_1=\int_{-\infty}^{\infty}|f(x)|dx \tag{4}$$

我们称 h 为 f 和 g 的卷积，并记以 $h=f*g$.

证明 存在Borel 函数 f_0 和 g_0，使得 $f_0=f$ a. e. 和 $g_0=g$ a. e. 若用 f_0 代替 f，用 g_0 代替 g，对每一个 x，积分(1)和(2)不变．因此从一开始就可以假定，f 和 g 都是 Borel 函数．

为了应用 Fubini 定理，首先证明，由

$$F(x,y)=f(x-y)g(y) \tag{5}$$

定义的函数 F 是 R^2 上的 Borel 函数．

由

$$\varphi(x,y)=x-y,\qquad \psi(x,y)=y \tag{6}$$

定义 φ: $R^2\to R^1$ 和 ψ: $R^2\to R^1$．则 $f(x-y)=(f\circ\varphi)(x,y)$ 和 $g(y)=(g\circ\psi)(x,y)$．因为 φ、ψ 都是 Borel 函数，定理 1.12(d)指出，$f\circ\varphi$ 和 $g\circ\psi$ 都是 R^2 上的 Borel 函数．因此，它们的乘积也是 Borel 函数．

其次，我们考察一下：

$$(7)\qquad \int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}|F(x,y)|dx=\int_{-\infty}^{\infty}|g(y)|dy\int_{-\infty}^{\infty}|f(x-y)|dx$$
$$=\|f\|_1\|g\|_1$$

因为根据 Lebesgue 测度的平移不变性，对每一个 $y\in R^1$，

$$(8)\qquad \int_{-\infty}^{\infty}|f(x-y)|dx=\|f\|_1$$

于是 $F\in L^1(R^2)$. 并且 Fubini 定理蕴涵着积分(2) 对几乎所有的 $x\in R^1$存在，并且 $h\in L^1(R^1)$. 最后，根据(7)，

$$\|h\|_1=\int_{-\infty}^{\infty}|h(x)|dx\leqslant\int_{-\infty}^{\infty}dx\int_{-\infty}^{\infty}|F(x,y)|dy$$
$$=\int_{-\infty}^{\infty}dy\int_{-\infty}^{\infty}|F(x,y)|dx=\|f\|_1\|g\|_1$$

这就给出了(3)，并且完成了证明.

卷积将在第九章起着重要的作用.

习　题

1　找出一个集 X 的单调类 $\mathfrak{M}$ 的例子，使得 $\varnothing\in\mathfrak{M}$，$X\in\mathfrak{M}$，但 $\mathfrak{M}$ 不是一个 σ-代数.

2　假定 f 是 R^1 上的 Lebesgue 可测的非负实函数，而 $A(f)$ 是 f 的纵坐标集. 这就是所有 $(x,y)\in R^2$ 使 $0<y<f(x)$ 的点集.

(a) $A(f)$ 在二维的意义下 Lebesgue 可测，对吗？

(b) 若(a)的回答是肯定的，f 在 R^1 上的积分等于 $A(f)$ 的测度吗？

(c) f 的图象是 R^2 的可测子集吗？

(d) 若(c)的回答是肯定的，这个图象的测度等于零吗？

3　找出一个在开单位正方形内的正连续函数 f 的例子，它(关于 Lebesgue 测度)的积分是有限的，但对某个 $x\in(0,1)$(在定理7.8的记号下的) $\varphi(x)$ 是无限的.

4　假定 $1\leqslant p\leqslant\infty$，$f\in L^1(R^1)$，$g\in L^p(R^1)$.

(a) 仿效定理 7.14 的证明去证明定义为 $(f*g)(x)$ 的积分对几乎所有

的 x 存在, $f*g\in L^p(R^1)$, 并证明

$$\|f*g\|_p\leqslant\|f\|_1\|g\|_p$$

(*b*) 指出当 $p=1$ 或 $p=\infty$ 时, 在(*a*)的不等式中等号能够成立. 并且找出使等号成立的条件.

(*c*) 假定 $1<p<\infty$, 并且在(*a*)中等号成立, 由此证明或是 $f=0$ a. e. 或是 $g=0$ a. e.

(*d*) 假定 $1\leqslant p\leqslant\infty$, $\varepsilon>0$, 证明存在 $f\in L^1(R^1)$ 和 $g\in L^p(R^1)$ 使得

$$\|f*g\|_p>(1-\varepsilon)\|f\|_1\|g\|_p$$

5 设 M 是所有的 R^1 上的复 Borel 测度的 Banach 空间. M 的范数是 $\|\mu\|=|\mu|(R^1)$. 对每一 Borel 集 $E\subset R^1$, 对应一个集

$$E_2=\{(x,y):x+y\in E\}\subset R^2$$

若 μ 和 $\lambda\in M$, 定义它们的卷积 $\mu*\lambda$ 为对每一个 Borel 集 $E\subset R^1$, 由

$$(\mu*\lambda)(E)=(\mu\times\lambda)(E_2)$$

所给定的集函数; $\mu\times\lambda$ 同定义 7.7 一样.

(*a*) 证明 $\mu*\lambda\in M$, 并且 $\|\mu*\lambda\|\leqslant\|\mu\|\|\lambda\|$.

(*b*) 证明 $\mu*\lambda$ 是唯一的 $\nu\in M$, 使得

$$\int f\,d\nu=\int\int f(x+y)\,d\mu(x)\,d\lambda(y)$$

对每一个 $f\in C_0(R^1)$ 成立. (所有积分展布在 R^1 上).

(*c*) 证明 M 中的卷积是交换的, 结合的, 并且关于加法是分配的.

(*d*) 证明公式

$$(\mu*\lambda)(E)=\int\mu(E-t)\,d\lambda(t)$$

对每一个 μ 和 $\lambda\in M$ 和对每一个 Borel 集 E 成立. 这里

$$E-t=\{x-t:x\in E\}$$

(*e*) 若 μ 集中在可数集上, 就定义 μ 是离散的; 若对每一个 $x\in R^1$, $\mu(\{x\})=0$, 就定义 μ 是连续的; 设 m 是 R^1 上的 Lebesgue 测度 (注意, $m\notin M$). 证明若 μ 和 λ 都是离散的, 则 $\mu*\lambda$ 是离散的; 若 μ 是连续的且 $\lambda\in M$, 则 $\mu*\lambda$ 是连续的, 并且若 $\mu\ll m$, 则 $\mu*\lambda\ll m$.

(*f*) 假定 $d\mu=f\,dm$, $d\lambda=g\,dm$, $f\in L^1(R^1)$ 和 $g\in L^1(R^1)$, 证明 $d(\mu*\lambda)=(f*g)\,dm$.

(*g*) 性质(*a*)和(*c*)表明: Banach 空间 M 就是人们所称为的交换 Bana-

ch 代数. 证明(e)和(f)蕴涵着M内所有的离散测度的集是 M 的一个子代数；连续测度构成M的一个理想；并且关于m绝对连续的测度构成M的一个理想，它(作为一个代数)同构于 $L^1(R^1)$.

(h) 证明M具有单位元，即证明存在一个 $\delta\in M$，使得 $\delta*\mu=\mu$ 对所有的 $\mu\in M$ 成立.

(i) 在这些讨论中，仅用到 R^1 的两个性质：R^1 是一个交换(加法)群，存在一个 R^1 上的平移不变 Borel 测度 m，这个测度不恒等于 0，并且它在 R^1 的所有紧子集上是有限的. 证明若用 R^k 代替 R^1 或用 T(单位圆周)或用 T^k(k 维环面，即 k 个 T 的笛卡儿乘积)代替 R^1，当适当地阐明定义之后，同样的结果也都成立.

6 (在 R^k 内的极坐标). 设 S_{k-1} 是 R^k 内的单位球面，即所有到原点的距离是 1 的那些 $u\in R^k$ 的集. 证明每一个 $x\in R^k$，除去 $x=0$，具有唯一的表达形式 $x=ru$，这里 r 是一个正实数且 $u\in S_{k-1}$. 于是 $R^k-\{0\}$ 可以看作 $(0,\infty)\times S_{k-1}$ 的笛卡儿乘积.

设 m_k 是 R^k 上的 Lebesgue 测度，在 S_{k-1} 上定义一个测度 σ_{k-1} 如下：若 $A\subset S_{k-1}$ 且 A 是 Borel 集，设 $\tilde{A}$ 是所有点 ru 的集，这里 $0<r<1$ 和 $u\in A$，并且定义

$$\sigma_{k-1}(A)=k\cdot m_k(\tilde{A})$$

证明公式

$$\int_{R^k} f\,dm_k=\int_0^\infty r^{k-1}dr\int_{S_{k-1}} f(ru)\,d\sigma_{k-1}(u)$$

对每一个 R^k 上非负 Borel 函数 f 成立. 验证这个结果在 $k=2$ 时和 $k=3$ 时与熟知的结果相一致.

提示：若 $0<r_1<r_2$，且若 A 是 S_{k-1} 的开子集，设 E 是所有具有 $r_1<r<r_2$，$u\in A$ 的 ru 的集. 验证公式对 E 的特征函数成立. 从而过渡到 R^k 中 Borel 集的特征函数.

7 假定 $(X,\mathscr{S},\mu)$ 和 $(Y,\mathscr{T},\lambda)$ 是 σ-有限测度空间，并且假定 ψ 是 $\mathscr{S}\times\mathscr{T}$ 上的测度，使得

$$\psi(A\times B)=\mu(A)\lambda(B)$$

当 $A\in\mathscr{S}$ 和 $B\in\mathscr{T}$ 时成立. 证明 $\psi(E)=(\mu\times\lambda)(E)$ 对每一个 $E\in\mathscr{S}\times\mathscr{T}$ 成立.

8 (a) 假定 f 是 R^2 上的实函数，使得每一个截口 f_x 是 Borel 可测的和每一个截口 f^y 是连续的. 证明 f 在 R^2 上是 Borel 可测的. 注意这和

例 7.9(c)之间的对立.

(b) 假定 g 是 R^k 上的实函数，它分别对 k 个变量的每一个是连续的. 更确切地说，对 $x_2, \cdots, x_k$ 的每一个选择，映射 $x_1 \to g(x_1, x_2, \cdots, x_k)$ 是连续的，等等. 证明 g 是一个 Borel 函数.

提示：若 $(i-1)/n = a_{i-1} \leqslant x \leqslant a_i = i/n$，令

$$f_n(x, y) = \frac{a_i - x}{a_i - a_{i-1}} f(a_{i-1}, y) = \frac{x - a_{i-1}}{a_i - a_{i-1}} f(a_i, y)$$

9 假定 E 是 R^1 中的稠密集，而 f 是 R^2 上的实函数，使得 (a) 对每一个 $x \in E$，f_x 是 Lebesgue 可测的，(b) 对几乎所有的 $y \in R^1$，f^y 是连续的. 证明 f 在 R^2 上是 Lebesgue 可测的.

10 假定 f 是 R^2 上的实函数，对每一个 x，f_x 是 Lebesgue 可测的，对每一个 y，f^y 是连续的. 假定 g：$R^1 \to R^1$ 是 Lebesgue 可测的，且令 $h(y) = f(g(y), y)$. 证明 h 在 R^1 上是 Lebesgue 可测的.

提示：象习题 8 一样定义 f_n，令 $h_n(y) = f_n(g(y), y)$，证明 h_n 是可测的并证明 $h_n(y) \to h(y)$.

11 设 $\mathscr{B}_k$ 是 R^k 的所有 Borel 集的 σ-代数. 证明 $\mathscr{B}_{m+n} = \mathscr{B}_m \times \mathscr{B}_n$. 这是与定理 7.14 有关的结果.

12 利用 Fubini 定理和关系式

$$\frac{1}{x} = \int_0^\infty e^{-xt} dt \qquad (x > 0)$$

去证明

$$\lim_{A \to \infty} \int_0^A \frac{\sin x}{x} dx = \frac{\pi}{2}$$

第八章　微　　分

测度的导数

为了引出定义 8.3，我们从关于某些点函数的导数的一个定理开始.

8.1　定理　假设 μ 是 R^1 上的复 Borel 测度和

$$f(x)=\mu((-\infty,x)) \qquad (x\in R^1) \tag{1}$$

如果 $x\in R^1$ 且 A 是一个复数，则下面两个命题的每一个都蕴涵着另一个：

(a) f 在 x 可微且 $f'(x)=A$.

(b) 对每一个 $\varepsilon>0$，存在一个 $\delta>0$，使得

$$\left|\frac{\mu(I)}{m(I)}-A\right|<\varepsilon \tag{2}$$

对每一个长度小于 δ 的包含 x 的开区间 I 成立.

这里 m 表示 R^1 上的 Lebesgue 测度.

证明　限制在某一个包含 x 的线段上，用 $\mu-Am$ 代替 μ（因此新的测度是有限的）. 这就表明可以假定 $A=0$ 而不失去一般性.

若 $f'(x)=0$ 和 $\varepsilon>0$，则存在一个 $\delta>0$，当 $|t-x|<\delta$ 时，

$$|f(t)-f(x)|\leqslant\varepsilon|t-x|$$

假设 $x\in I$，$I=(s,\ t)$，且 $t-s<\delta$. 选取 s_n 使 $x>s_1>s_2>\cdots$，$s_n\to s$. 则

$$\begin{aligned}|\mu([s_n,t))|&=|f(t)-f(s_n)|\\&\leqslant|f(t)-f(x)|+|f(x)-f(s_n)|\end{aligned}$$

$$\leqslant \varepsilon(t-x)+\varepsilon(x-s_n)$$
$$=\varepsilon(t-s_n)<\varepsilon m(I)$$

并且因为 $I=\bigcup([s_n,t))$，得出 $|\mu(I)|\leqslant \varepsilon m(I)$. 于是(a)蕴涵着(b).

其次，假设(b)对 $A=0$ 成立. 选取 $\varepsilon>0$，且象在(b)内一样地选取 $\delta>0$. 如果 $s<x<t$ 和 $t-s<\delta$，选取 s_n 使 $t-\delta<s_1<s_2<\cdots$，并且 $\lim s_n=s$. 则对所有充分大的 n

$$|\mu((s_n,t))|<\varepsilon(t-s_n)$$

因为 $[s,t)=\bigcap(s_n,t)$ 和

$$f(t)-f(s)=\mu([s,t))$$

由此得出

(3) $$|f(t)-f(s)|\leqslant \varepsilon|t-s| \qquad (s<x<t<s+\delta)$$

若(b)成立，则 $\mu(\{x\})=0$，这里 $\{x\}$ 是仅由 x 所组成的集. 因此 f 在点 x 连续. 所以在(3)内，不是 s 就是 t 能被 x 代替，我们便得出 $f'(x)=0$.

8.2 定义 定理 8.1 启发我们，μ 在 x 的导数可以定义为当区间 I 收缩至点 x 时，商 $\dfrac{\mu(I)}{m(I)}$ 的极限. 当然，也能考虑商 $\mu(E)/m(E)$ 的极限，这里 E 跑遍某一个另外的集族. 下面将不仅在 R^1 上，而是在任一欧氏空间 R^k 上做这些工作.

假设 $x\in R^k$. 对 R^k 内的 Borel 集序列 $\{E_i\}$，若存在一个具有下列性质的数 $\alpha>0$：每一个 E_i 在一个以 x 为中心半径 $r_i>0$ 的开球 $B(x,r_i)$ 内，使得

(1) $$m(E_i)\geqslant \alpha m(B(x,r_i)) \qquad (i=1,2,3,\cdots)$$

并且当 $i\to\infty$ 时，$r_i\to 0$，就称 $\{E_i\}$ 良好地收缩于 x. (这里以及今后，m 表示 R^k 上的 Lebesgue 测度).

注意，这既不要求 $x\in E_i$，甚至也不要求 x 在 E_i 的闭包内. 条

件(1)是要求每一个 E_i 必须占有 x 的某个球形邻域的相当大的部分的定量叙述. 例如，直径趋于零的最长的棱比最短的棱至多大1000倍的 k-胞腔的序列套是良好地收缩的. (在 R^2 内)棱长分别为 $\frac{1}{i}$ 和 $\left(\frac{1}{i}\right)^2$ 的矩形序列套则不是良好地收缩的.

8.3 定义 若 μ 是 R^k 上的复 Borel 测度，若 $x\in R^k$，A 是一个复数，并且若

$$\lim_{i\to\infty}\frac{\mu(E_i)}{m(E_i)}=A \tag{1}$$

对每一个良好地收缩于 x 的序列 $\{E_i\}$ 成立，则称 A 为 μ 在 x 的导数，并且记以

$$(D\mu)(x)=A \tag{2}$$

一个更加完全的叙述是，$(D\mu)(x)$ 是 μ 在 x 关于 Lebesgue 测度 m 的导数.

这章的主要结果是定理 8.6，下面两个引理将在它的证明中用到.

8.4 引理 若 Ω 是 R^k 内的开球族，其并是 W，并且若 $t<m(W)$，则 Ω 具有有限的不相交的子族 $(B_1, B_2, \cdots, B_n)$ 使得

$$\sum_{i=1}^{n}m(B_i)>3^{-k}t$$

证明 选取紧集 $K\subset W$，使 $m(K)>t$. (这用到 m 的正则性.)根据紧性，K 可以被 Ω 的有限多个元素所覆盖，比如说 S_1，$S_2, \cdots, S_p$. 我们可以把它们排列起来，使得它们的半径 $r(S_j)$ 对 $r=1, 2, \cdots, p-1$ 满足 $r(S_j)\geqslant r(S_{j+1})$.

令 $B_1=S_1$. 把所有与 B_1 相交且 $j>1$ 的那些 S_j 删去. 设 $B_2=S_{i_2}$ 是余下的 S_j 中的第一个(如果存在的话). 把所有的与 B_2 相交且 $j>i_2$ 的那些 S_j 删去. 设 $B_3=S_{i_3}$ 是余下的 S_j 中的第一

个．只要可能，就进行下去．

这就给出了不相交的集族$\{B_1, \cdots, B_n\}$．

每一个删去的S_j是与某个B_i同心的其半径3倍于B_i之半径的球β_i的子集．于是$K \subset \beta_1 \cup \beta_2 \cup \cdots \cup \beta_n$．由此得出

$$t < m(K) \leqslant \sum_{i=1}^{n} m(\beta_i) = 3^k \sum_{i=1}^{n} m(B_i)$$

8.5 引理 假设μ是R^k上的正的Borel测度，它在紧集上是有限的．设A是Borel集，且$\mu(A)=0$．则存在一个集$A' \subset A$，使得

(a) $m(A-A')=0$;

(b) $(D\mu)(x)=0$，对每个$x \in A'$成立．

证明 定义A'是所有满足

$$\lim_{r \to 0} \frac{\mu(B(x,r))}{m(B(x,r))} = 0 \tag{1}$$

的那些$x \in A$的集．

[若(1)的极限存在，通常称为μ在x的对称导数．回忆一下，$B(x,r)$是中心在x，半径为r的开球．]

对于$j=1,2,3,\cdots$，设P_j是所有满足

$$\limsup_{r \to 0} \frac{\mu(B(x,r))}{m(B(x,r))} > \frac{1}{j} \tag{2}$$

的那些$x \in A$的集．我们将要证明

$$m(P_j)=0 \qquad (j=1,2,3,\cdots) \tag{3}$$

因为$A-A' = \bigcup P_j$，从(3)就可得出(a)．

固定j，固定$\varepsilon > 0$．因为$\mu(A)=0$，存在一个开集$V \supset A$，且有$\mu(V) < \varepsilon$．每一个$x \in P_j$是一个开球$B_x \subset V$的中心，使得

$$m(B_x) < j\mu(B_x) \tag{4}$$

设W是这些B_x的并．选取$t < m(W)$．引理8.4指出$\{B_x : x \in P_j\}$

有不相交的子集族$\{B_{x_1},\cdots,B_{x_n}\}$，使得

$$t<3^k\sum_{i=1}^{n}m(B_{x_i})<3^k j\sum_{i=1}^{n}\mu(B_{x_i})$$
$$=3^k j\mu(B_{x_1}\cup B_{x_2}\cup\cdots\cup B_{x_n})<3^k j\mu(V)<3^k je$$

于是$m(W)\leqslant 3^k je$. 因为$P_j\subset W$，而e是任意的，看出P_j在Lebesgue测度任意小的开集内. 因此m的完备性蕴涵着P_j是可测的，且$m(P_j)=0$. 这就证明了(3). 因此证明了(a).

为了证明(b)，固定$x\in A'$，并设$\{E_i\}$是一个良好地收缩于x的序列. 根据定义8.2，存在开球$B(x,r_i)\supset E_i$，使得$r_i\to 0$，并且使得

$$m(E_i)\geqslant\alpha m(B(x,r_i)) \tag{5}$$

对所有的i和某个$\alpha>0$成立. 因此

$$\frac{\mu(E_i)}{m(E_i)}\leqslant\frac{1}{\alpha}\cdot\frac{\mu(B(x,r_i))}{m(B(x,r_i))} \tag{6}$$

因为(1)对每一个$x\in A'$成立，所以(6)证明了$(D\mu)(x)=0$.

8.6 定理 若μ是R^k上的复Borel测度. 则

(a) $(D\mu)(x)$a. e. $[m]$存在，

(b) $D\mu\in L^1(R^k)$，并且

(c) 对每一个Borel集E，

$$\mu(E)=\mu_s(E)+\int_E(D\mu)(x)dx$$

这里$\mu_s\perp m$，以及$(D\mu_s)(x)=0$ a. e. $[m]$.

注意(c)是μ关于m的Lebesgue分解. 这就导致下列的系.

系 (i) $\mu\perp m$，当且仅当$(D\mu)(x)=0$ a. e. $[m]$.

(ii) $\mu\ll m$，当且仅当

$$\mu(E)=\int_E(D\mu)(x)dx$$

对每一个Borel集E成立. 在这种情况下，(作为商的极限所计算

的)导数 $D\mu$ a. e. $[m]$和 Radon-Nikodym 导数 $d\mu/dm$ 一致.

证明 由于 Lebesgue分解定理,分别对 $\mu\perp m$ 和对 $\mu\ll m$ 证明定理成立就够了. 同时仅考虑实的 μ 也就足够了.

若 $\mu\perp m$,并且 $\mu=\mu^+-\mu^-$ 是 μ 的 Jordan 分解, 则 $\mu^+\perp m$, 所以存在一个 Borel 集 A, 具有 $\mu^+(A)=0$ 和 $m(A^c)=0$. 根据引理 8.5, $(D\mu^+)(x)=0$ a. e. $[m]$. 同样的讨论也证明$(D\mu^-)(x)=0$ a. e. $[m]$. 因此, $(D\mu)(x)=0$ a. e. $[m]$.

其次,假定 $\mu\ll m$ 和 μ 是实的.

根据 Radon-Nikodym 定理,存在一个实 Borel 函数

$$f\in L^1(R^k)$$

使得

$$\mu(E)=\int_E f(x)dx \tag{1}$$

对每一个 Borel 集 E 成立. 因此,从我们现在将要证明的等式

$$(D\mu)(x)=f(x) \quad (\text{a. e. } [m]) \tag{2}$$

可得出定理的结论.

将每个有理数 r 对应于集

$$A_r=\{x: f(x)<r\},\ B_r=\{x: f(x)\geqslant r\} \tag{3}$$

并用

$$\lambda_r(E)=\int_{E\cap B_r}(f(x)-r)dx \tag{4}$$

定义正的 Borel 测度 λ_r. 因为 $\lambda_r(A_r)=0$, 引理 8.5 指出存在一个集 $A_r'\subset A_r$, 使得 $m(A_r-A_r')=0$, 并且, 若 $x\in A_r'$

$$(D\lambda_r)(x)=0 \tag{5}$$

令 $Y=\bigcup_r(A_r-A_r')$. 则 $m(Y)=0$. 挑选 Y 以外的某个 x, 设 $\{E_i\}$良好地收缩于 x, 并选择 $r>f(x)$. 则 $x\in A_r$. 因为

$$\mu(E)-rm(E)=\int_E(f(y)-r)dy\leqslant\lambda_r(E)$$

我们有

$$\frac{\mu(E_i)}{m(E_i)}\leqslant\frac{\lambda_r(E_i)}{m(E_i)}+r \qquad (i=1,2,3,\cdots) \tag{6}$$

因为 $x\in Y$，有 $x\in A'_r$. 因此(5)和(6)给出

$$\limsup_{i\to\infty}\frac{\mu(E_i)}{m(E_i)}\leqslant r \tag{7}$$

这对大于 $f(x)$的每一个有理数 r，(7)式都是正确的. 因此

$$\limsup_{i\to\infty}\frac{\mu(E_i)}{m(E_i)}\leqslant f(x) \tag{8}$$

现在我们已经证明了：几乎对每一个 $x\in R^k$ 和每一个良好地收缩于 x 的序列$\{E_i\}$，(8)式都能满足.

如果我们用$-\mu$代替 μ，因之用$-f$代替 f，便得出几乎对每一个$x\in R^k$ 和每一个良好地收缩于 x 的序列$\{E_i\}$满足

$$\liminf_{i\to\infty}\frac{\mu(E_i)}{m(E_i)}\geqslant f(x) \tag{9}$$

因为(8)和(9)给出(2)，故定理证明完毕.

8.7 评注 通过考察定理 8.6 的证明将会显示出，当 μ 仅仅定义在某个开集 $V\subset R^k$ 内的 Borel 集上时，定理 8.6 显然也是成立的.（因此，这个结论当然仅在 V 内成立.）或者依照实际情况，我们对所有的 $E\subset V^c$ 简单地定义 $\mu(E)=0$，并且应用定理.

若 $f\in L^1(R^k)$和

$$\mu(E)=\int_E f(x)dx \tag{1}$$

称 μ 为 f 的不定积分是合理的. 用这样的术语，定理 8.6 断定一个不定积分的导数 a. e. $[m]$等于它的被积函数，并且断定每一个 $\mu\ll m$ 是其导数的不定积分.

不定积分的可微性可以解释为一类 Lebesgue 可积函数的“平均连续性”. 因为若(1)成立，$(D\mu)(x_0)=f(x_0)$的断言无非是说，

当E良好地收缩于x时，

$$\frac{1}{m(E)}\int_E f(x)dx\to f(x_0) \tag{2}$$

或者说

$$\frac{1}{m(E)}\int_E \{f(x)-f(x_0)\}dx\to 0 \tag{3}$$

(2)的左边是f在靠近x_0的很小的集E上的平均；(3)的左边是$f-f(x_0)$在E上的平均.

现在(3)能归结为消除在几乎所有点x_0的很小的邻域内由$f-f(x_0)$的符号的改变所引起的影响；或者一个更强的结论可能是正确的，即，对几乎所有的x_0，$|f-f(x_0)|$的平均在x_0的很小的邻域内实际上变得很小. 这个更强的结论事实上是正确的：

8.8 定理 假设$f\in L^1(R^k)$. 定义f的"Lebesgue 集"L_f为所有$x_0\in R^k$的集，使得

$$\lim_{i\to\infty}\frac{1}{m(E_i)}\int_{E_i}|f(x)-f(x_0)|dx=0 \tag{1}$$

对每一个良好地收缩于x_0的序列$\{E_i\}$成立. 则几乎所有的$x_0\in R^k$属于L_f.

证明 设S是复平面内的可数稠密集. 对$r\in S$，定义

$$\mu_r(E)=\int_E|f(x)-r|dx \tag{2}$$

定理 8.6 指出

$$(D\mu_r)(x_0)=|f(x_0)-r| \tag{3}$$

对几乎所有的x_0成立. 设Y_r是例外集. 令$Y=\cup Y_r$，则$m(Y)=0$.

若$x_0\notin Y$，$\{E_i\}$良好地收缩于x_0，且$e>0$. 则存在$r\in S$使$|f(x_0)-r|<e$；因此，$|f(x)-f(x_0)|<|f(x)-r|+e$，并且

(4) $$\frac{1}{m(E_i)}\int_{E_i}|f(x)-f(x_0)|dx<\frac{\mu_r(E_i)}{m(E_i)}+e$$

因为 $x_0\notin Y$，(3)成立，所以对充分大的 i，(4)的左边小于 $2e$. 这就对每一个 $x_0\notin Y$ 证明了(1)式，从而完成了定理的证明.

把我们的注意限制于良好地收缩序列的特殊形式有时是有好处的. 我们现在就来考虑这样一种情况.

8.9 定义 设 μ 是 R^k 上的实 Barel 测度. 对于 $x\in R^k$ 和 $n=1,2,3,\cdots$，设 $\Delta_n(x)$ 是商 $\mu(C)/m(C)$ 的上确界，这里 C 取遍所有使得 $x\in C$，其棱平行坐标轴，并且棱长小于 $\frac{1}{n}$ 的 R^k 内的开立方体. 定义

$$(\mathscr{D}^Q\mu)(x)=\lim_{n\to\infty}\Delta_n(x)$$

因为 $\Delta_n\geqslant\Delta_{n+1}$，对每一个 $x\in R^k$，这个极限作为 $[-\infty,\infty]$ 内的一个数是存在的.

定义 $(\mathscr{D}_Q\mu)(x)$，只要在上一段中用“下确界”代替“上确界”即可.

数 $(\mathscr{D}^Q\mu)(x)$ 和 $(\mathscr{D}_Q\mu)(x)$ 可以称为 μ 在 x 点的上“立方体导数”和下“立方体导数”. 注意，当 $k=1$ 时，立方体 C 恰好是在第 8.1 节所出现的区间 I.

容易看出，$\mathscr{D}^Q\mu$ 和 $\mathscr{D}_Q\mu$ 是 Borel 函数：若 $\Delta_n(x)>\alpha$，则对某一个 C（棱长 $<1/n$）使得 $x\in C$，$\mu(C)>\alpha m(C)$ 成立；显然对在这个 C 内的所有的 y，$\Delta_n(y)>\alpha$；于是 Δ_n 是下半连续的，所以 $\mathscr{D}^Q\mu$ 是一个 Borel 函数. 同样的论证方法适用于 $\mathscr{D}_Q\mu$.

8.10 定理 假设 μ 是 R^1 上的实 Borel 测度，$\mu\geqslant 0$ 并且 $\mu\perp m$，则

$$(\mathscr{D}_Q\mu)(x)=\infty \qquad \text{a. e. } [\mu]$$

这个定理补充了断定 $(\mathscr{D}\mu)(x)=0$ a. e. $[m]$ 的定理 8.6.

证明 固定 $e>0$. 因为 $\mu\perp m$, 故 μ 集中在一个具有 $m(V)<e$ 的开集 V 上. 对 $n=1,2,3,\cdots$, 定义 E_n 为所有使得 $(\mathscr{D}_Q\mu)(x)<n$ 的那些 $x\in V$ 的集. 设 K 是 E_n 的紧子集. 假如能证明 $\mu(K)=0$, 则 μ 的正则性(定理 2.18)蕴涵着 $\mu(E_n)=0$ 对每一个 n 成立. 这就给出所要的结果.

每一个 $x\in K$ 在一个区间 $I_x\subset V$ 内, 使得 $\mu(I_x)<n\cdot m(I_x)$ 成立. 由于紧性, K 被有限多个这样的线段 I_{x_i} 所覆盖. 若 R^1 的某个点在三个区间内, 这些区间之一必在另外两个区间的并中, 并且可以去掉它而不改变其并. 用这样的方法, 我们能删去多余的区间, 并且能达到没有一点在两个以上的区间 I_{x_i} 内. 这时

$$\mu(K)\leqslant\mu\left(\bigcup_i I_{x_i}\right)\leqslant\sum_i\mu(I_{x_i})<n\sum_i m(I_{x_i})$$

$$\leqslant 2n\cdot m\left(\bigcup_i I_{x_i}\right)\leqslant 2n\cdot m(V)<2ne$$

因为 e 是任意的, 所以 $\mu(K)=0$. 证明完毕.

注意, 在上面的证明中用到直线的一个很特殊的性质. 习题 3 指出, 若 $k>1$, 定理 8.10 事实上是不成立的. 然而, 下面弱一点的结果是正确的, 它的系在定理 8.26 的证明中将用到.

8.11 定理 假设 μ 是 R^k 上的实 Borel 测度, $\mu\geqslant 0$, 并且 $\mu\perp m$. 则

$$(\mathscr{D}^Q\mu)(x)=\infty \qquad \text{a. e. } [\mu]$$

系 若 $\mu\geqslant 0$, 且 $(\mathscr{D}^Q\mu)(x)<\infty$ 对每个 $x\in R^k$ 成立, 则 $\mu\ll m$.

证明 因为 $\mu(R^k)<\infty$, 容易看出, 在某一 $x\in R^k$ 上 $(\mathscr{D}^Q\mu)(x)<\infty$ 当且仅当存在一个整数 $j(x)$, 使得 $\mu(C)<j(x)m(C)$ 对每一个包含 x 的开立方体 C (其棱平行于坐标轴)成立.

对于 $j=1,2,3,\cdots$, 设 E_j 是所有使得 $\mu(C)<jm(C)$ 对每一个包含 x 的 C 成立的那些 $x\in R^k$ 的集. 注意每一个 E_j 是一个闭

集.

选取 $\varepsilon>0$. 因为 $\mu\perp m$, μ 是集中在某个 $m(V)<\varepsilon$ 的开集 V 上. 由于是开集,象第 2.19 节一样, V 是可数多个不相交的单元的并.

固定 j. 设 $Q_1, Q_2, Q_3, \cdots$ 是上面那些与 $V\cap E_j$ 相交的单元. 因为每一个 Q_n 是开立方体 C 的递降序列的交, E_j 的定义指出 $\mu(Q_n)\leqslant jm(Q_n)$ 对每一个 n 成立. 因此

$$\mu(E_j)=\mu(V\cap E_j)\leqslant\mu\left(\bigcup_n Q_n\right)=\sum_n\mu(Q_n)$$

$$\leqslant j\sum_n m(Q_n)=jm\left(\bigcup_n Q_n\right)\leqslant jm(V)<j\varepsilon$$

因为 ε 是任意的, $\mu(E_j)=0$. 根据此证明的第一段,当且仅当 $x\in\bigcup E_j$ 时, $(\mathscr{D}^q\mu)(x)<\infty$. 但 $\mu(\bigcup E_j)=0$. 这就完成了定理的证明.

有界变差函数

8.12 定义 我们将每一个 R^1 上的复函数 f 都对应于它的全变差函数 T_f, 用

$$(1)\qquad T_f(x)=\sup\sum_{j=1}^{N}|f(x_j)-f(x_{j-1})|\qquad(-\infty<x<\infty)$$

定义,这里上确界是对所有的 N 和对所有使得

$$(2)\qquad -\infty<x_0<x_1<\cdots<x_N=x$$

的 $\{x_j\}$ 的可能选择而取的.

一般地,

$$(3)\qquad 0\leqslant T_f(x)\leqslant T_f(y)\leqslant\infty\qquad(x<y)$$

若 T_f 是有界函数,则(3)蕴涵着

$$(4)\qquad V(f)=\lim_{x\to\infty}T_f(x)$$

存在并且是有限的. 在这种情况下,我们称 f 是有界变差的,并称 $V(f)$ 为 f 的全变差;所有有界变差函数 f 的类记为 BV.

若 $-\infty<x\leqslant\infty$,如果对应于每一个 $e>0$, 存在一个实数 $\alpha<x$,使得

(5) $\qquad\qquad \alpha<t<x$ 蕴涵着 $|f(t)-f(x^-)|<e$

则 f 称为在 x 点具有左极限,记以 $f(x^-)$,若 $f(x^-)=f(x)$,则 f 称为在 x 点左连续.

右极限和从右的连续性在 $[-\infty,\infty)$ 上类似地定义.

我们称函数 $f\in BV$ 为正规化的,若当 $x\to-\infty$ 时, $f(x)\to0$,并且 f 在 R^1 的每一点上是左连续的. 这些函数的类记为 NBV.

上面仅仅考虑了定义在 R^1 全体上的函数. 实际上我们也能同样地考虑定义在任一线段或 R^1 上的区间上的函数. 无论是上面的定义或是下面的定理,在任何主要方面都不会有影响.

8.13 定理

(*a*) 若 $f\in BV$ 且 $x<y$. 则

$$|f(y)-f(x)|\leqslant T_f(y)-T_f(x)$$

(*b*) 若 $f\in BV$,则 $f(x^-)$ 在 $(-\infty,\infty]$ 的每一点存在, $f(x^+)$ 在 $[-\infty,\infty)$ 的每一点存在, 使 f 不连续的那些点的集至多是可数的,并且存在唯一的一个常数 c 和唯一的一个函数 $g\in NBV$, 使得

$$f(x)=c+g(x)$$

在 f 的所有的连续点上成立. 同时 $V(g)\leqslant V(f)$.

(*c*) 若 $f\in NBV$,则 $T_f\in NBV$.

证明

(*a*) 若 $x<y$ 和 $e>0$,则存在点 $x_0<x_1<\cdots<x_n=x$,使得

(1) $\qquad\qquad \Sigma|f(x_i)-f(x_{i-1})|>T_f(x)-e$

因此

$$T_f(y)\geqslant|f(y)-f(x)|+\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$$

$$>|f(y)-f(x)|+T_f(x)-e$$

这就证明了(a).

(b) 从(a)得出，若$\{x_i\}$是一个序列，对这样的$\{x_i\}$，$\{T_f(x_i)\}$是一个 Cauchy 序列. 则$\{f(x_i)\}$也是一个 Cauchy 序列. 因为单调函数(具体地说 T_f)在所有的点上具有左极限和右极限，并且因为它们至多有可数个不连续点，所以同样的结论对 f 也成立. 因此我们能定义

(2) $$c=\lim_{t\to-\infty}f(t),\quad g(x)=f(x^-)-c\quad(x\in R^1)$$

显然，当$x\to-\infty$时，$g(x)\to0$. 若$x\in R^1$和$e>0$，则存在一个$\alpha<x$，使得$|f(t)-f(x^-)|<e$对所有$t\in(\alpha,x)$成立. 因为$f(t^-)$是当$\alpha<s<t$时所有函数值$f(s)$的集的极限点，这就得出，当$\alpha<t<x$时，$|f(t^-)-f(x^-)|\leqslant e$成立. 于是 g 是左连续的.

若$x_0<x_1<\cdots<x_n$和$\delta>0$，则

(3) $$\sum_{i=1}^{n}|g(x_i)-g(x_{i-1})|=\lim_{\delta\to0}\sum_{i=1}^{n}|f(x_i-\delta)-f(x_{i-1}-\delta)|$$

因为(3)式右边的和没有一个超过$V(f)$，有$V(g)\leqslant V(f)$. 特别$g\in BV$.

除唯一性外，这就证明了(b). 但是，若两个左连续函数在R^1的稠密子集上一致，则它们是恒等的. 现在就容易推出 g 的唯一性了.

(c) 若$f\in NBV$. 固定$x\in R^1$，$e>0$，选择点

$$x_0<x_1<\cdots<x_n=x$$

使(1)成立. 若$t_0<\cdots<t_N=x_0$，则

(4) $$T_f(x)\geqslant\sum_{j=1}^{N}|f(t_j)-f(t_{j-1})|+\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|$$

根据(1),(4)的第一个和小于 e,因此 $T_f(x_0)\leqslant e$,这就是说当 $t\to-\infty$时,$T_f(t)\to 0$.

最后,选取 t 使 $x_{n-1}<t<x_n$. 则

$$(5)\quad \sum_{i=1}^{n-1}|f(x_i)-f(x_{i-1})|+|f(t)-f(x_{n-1})|\leqslant T_f(t)\leqslant T_f(x^-)\leqslant T_f(x)$$

如果我们令 $t\to x_n=x$,因为 $f(x)=f(x^-)$,(5)的左边趋于(1)的左边,这就给出了 $T_f(x)-e<T_f(x^-)$. 与(5)比较,现在证明了

$$T_f(x^-)=T_f(x),$$

于是证明完毕.

下面的定理阐明了函数类 NBV 的重要性. 请注意观察 f 和 μ 之间如何把 f 的全变差和 μ 的全变差联系起来的,并且看到 Lebesgue 测度的存在性在(*b*)内是如何用于构造 μ 的.

8.14 定理

(*a*) 若 μ 是 R^1 上的复 Borel 测度,并且,若

$$(1)\qquad f(x)=\mu((-\infty,x))\qquad (x\in R^1)$$

则 $f\in NBV$.

(*b*) 反之,对每一个 $f\in NBV$,对应唯一的一个复 Borel 测度 μ,使得(1)成立. 对于这个 μ

$$(2)\qquad T_f(x)=|\mu|((-\infty,x))\qquad (x\in R^1)$$

(*c*) 若(1)成立,则 f 在 $\mu(\{x\})=0$ 的这些点 x 上确实是连续的.

证明 如果 f 是由(1)定义,并且 $x_1<x_2<\cdots<x_n\to x$. 则由于

$$(3)\qquad (-\infty,x)=\bigcup_{n=1}^{\infty}(-\infty,x_n)$$

$f(x_n)\to f(x)$. 于是 f 是左连续的. 若 $x_n\to-\infty$,则 $\bigcap(-\infty,x_n)$

$=\varnothing$，根据定理 1.19(e)，所以当 $x\to-\infty$ 时 $f(x)\to 0$。若 $x_0<x_1<\cdots<x_n=x$，则

$$\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|=\sum_{i=1}^{n}|\mu([x_{i-1},x_i))|$$
$$\leqslant|\mu|((-\infty,x))$$

所以

$$T_f(x)\leqslant|\mu|((-\infty,x)) \tag{4}$$

这就证明了(a)。

在(b)的证明中，首先假设 $f\in NBV$，且 f 是不减的，$f\neq 0$。对每一 $x\in R^1$，对应一个集 $\Phi[x]$ 如下：如果 f 在 x 点连续，$\Phi[x]$ 是点 $f(x)$；如果 $f(x^+)>f(x)$，则 $\Phi[x]$ 是区间 $[f(x),\ f(x^+)]$。若 $E\subset R^1$，设 $\Phi(E)$ 是所有对 $x\in E$ 的集 $\Phi[x]$ 的并。我们断言，定义

$$\mu(E)=m(\Phi(E)) \tag{5}$$

给出一个满足(1)的测度；这里 m 是 R^1 上的 Lebesgue 测度。

令 $J=\Phi[R^1]$，则 J 是一个 1-胞腔(即一个包括端点或不包括端点的有界区间)。至多存在可数多个点 $y_i\in J$，使得 $f^{-1}(y)$ 由多于一个点组成。对于这些 y_i，$f^{-1}(y_i)$ 是一个 1-胞腔。

设 Σ 是所有使得 $\Phi[E]$ 是 Borel 集的那些 $E\subset R^1$ 的类。若 E 是一个 1-胞腔，$\Phi[E]$ 也是 1-胞腔。因此 $E\in\Sigma$。对任一 $E\subset R^1$，$\Phi[E^c]$ 是 $J-\Phi[E]$ 加上一个至多可数的集($\{y_i\}$的子集)的并；于是 $E\in\Sigma$ 蕴涵着 $E^c\in\Sigma$。其次，

$$\Phi[E_1\cup E_2\cup\cdots]=\Phi[E_1]\cup\Phi[E_2]\cup\cdots$$

这就证明了 Σ 是包含所有区间的 σ-代数，因此包含了所有的 Borel 集，所以对所有的 Borel 集 E，$m(\Phi[E])$ 是确定的。此外，μ 是可数可加的，因为，如果 $\{E_i\}$ 是不相交的 R^1 的 Borel 集的类，则除去对于可数集 $\{y_i\}$ 外，$\{\Phi[E_i]\}$ 是不相交的。因为对每一个可数集 E，$m(E)=0$，所以这并不影响 μ 的可数可加性。

于是(5)定义一个 Borel 测度. 因为 $\Phi[(-\infty, x)]$是端点为 0 和 $f(x)$的 1-胞腔,(5)式表明(1)成立.

现在转向(b)的一般情况. 若 $f\in NBV$,则 $f=u+iv$,u 和 v 是实函数,$u\in NBV$. 根据定理 8.13(c),$T_u\in NBV$. 令

$$u_1=\frac{1}{2}(T_u+u),\qquad u_2=\frac{1}{2}(T_u-u) \tag{6}$$

从定理 8.13(a)容易得出,u_1 和 $u_2\in NBV$,并且它们是不减的. 前面的构造 u_1 和 u_2 对应于测度 μ_1 和 μ_2. 与 $u=u_1-u_2$ 相对应的将是 $\mu_1-\mu_2$. 若我们类似地处理 v,并且综合这两个结果,我们获得在(1)成立的意义下与 f 对应的测度 μ.

若两个正则测度(注意定理 2.18)在所有的形如 $(-\infty, x)$的区间上是一致的,则它们在所有的形如$[\alpha,\beta)$的 1-胞腔上是一致的. 因而在所有的开集上一致,因此在所有的 Borel 集上一致. 这就证明了(b)的断言的唯一性.

最后,设 λ 是在同样方式下与 T_f 相对应的测度. 若 $\alpha<\beta$,则

$$\mu([\alpha,\beta))=f(\beta)-f(\alpha),\quad \lambda([\alpha,\beta))=T_f(\beta)-T_f(\alpha) \tag{7}$$

因此,若 $E=[\alpha,\beta)$,不等式

$$|\mu(E)|\leqslant\lambda(E) \tag{8}$$

成立. 因为在 R^1 内的每一个开集是可数个不相交的这种 1-胞腔的并,(8)对每一个开集成立. 因此(8)对每一个 Borel 集成立. μ 的全变差$|\mu|$的定义现在蕴涵着$|\mu|\leqslant\lambda$. 特别

$$|\mu|((-\infty,x))\leqslant\lambda((-\infty,x))=T_f(x) \tag{9}$$

现在从(4)和(9)就得出(2).

(c) 的证明留作练习.

点函数的微分法

8.15 绝对连续函数 一个 R^1 上的复函数 f,若对每一个

$\varepsilon>0$，存在一个对应的 $\delta>0$，当$(\alpha_1,\beta_1),\cdots,(\alpha_N,\beta_N)$是不相交的区间时，使得

(1) $$\sum_{i=1}^{N}(\beta_i-\alpha_i)<\delta \text{ 蕴涵着 } \sum_{i=1}^{N}|f(\beta_i)-f(\alpha_i)|<\varepsilon$$

则称 f 为绝对连续.

请注意每一个绝对连续的函数是一致连续的（取 $N=1$）. 任一绝对连续函数限制在有界区间上是一有界变差函数. 然而，若 $f(x)=\sin x$ 或者 $f(x)=x+|x|$，则 $f(x)$是绝对连续的，但 $f\notin BV$.

术语"绝对连续"的两个含义具有如下关系:

8.16 定理 假设 $f\in NBV$. μ 是如在定理8.14一样与 f 相对应，则 $\mu\ll m$ 当且仅当 f 是绝对连续的.

（这里 m 表示 R^1 上的 Lebesgue 测度.）

证明 假设 f 是绝对连续的. 设 E 是使得 $m(E)=0$ 的 Borel 集. 选取 $\varepsilon>0$，并按照 8.15 节那样选择 $\delta>0$. μ 的正则性表明存在开集 $W_1\supset W_2\supset\cdots\supset E$，使得 $m(W_1)<\delta$ 和当 $n\to\infty$时，$\mu(W_n)\to\mu(E)$. 因为 W_n 是不相交的区间 $I_j=(\alpha_j,\beta_j)$的并，并且

$$\Sigma(\beta_j-\alpha_j)<\delta,$$

得出

$$|\mu(W_n)|\leqslant\sum_j|\mu(I_j)|=\sum_j|f(\beta_j)-f(\alpha_j)|\leqslant\varepsilon$$

因此，$\mu(E)=0$. 这就证明了 $\mu\ll m$.

从定理 6.11 可得出逆命题.

我们现在能够把前面集函数微分法的结果翻译成点函数语言了. 定理 8.17 到 8.19 是由 Lebesgue 所提出的经典的结果. 定理 8.18 和 8.21 推广了微积分的基本定理.

8.17 定理 如果 $g\in L^1(R^1)$，且

(1) $$f(x)=\int_{-\infty}^{x}g(t)dt \qquad (-\infty<x<\infty)$$

则 $f\in NBV$，f 是绝对连续的，并且

$$f'(x)=g(x) \qquad \text{a. e.}[m] \tag{2}$$

证明 对每一个 Borel 集 E 定义

$$\mu(E)=\int_E g(t)dt \tag{3}$$

则 $f(x)=\mu((-\infty,x))$ 和 $\mu\ll m$. 根据定理 8.14，$f\in NBV$；根据 8.16，f 是绝对连续的；定理 8.1 和 8.6 蕴涵着

$$f'(x)=(D\mu)(x)=g(x) \qquad \text{a. e.}[m] \tag{4}$$

8.18 定理 若 $f\in NBV$，则 f a. e. $[m]$ 可微，$f'\in L^1(R^1)$，并且存在一个函数 $f_s\in NBV$，具有 $f_s'(x)=0$ a. e. $[m]$，使得

$$f(x)=f_s(x)+\int_{-\infty}^{x} f'(t)dt \qquad (-\infty<x<\infty) \tag{1}$$

$f_s=0$ 当且仅当 f 是绝对连续；若 f 是不减的，f_s 也是不减的.

我们称 f_s 为 f 的奇异部分. 存在一个不恒为常数的连续的奇异函数，这一点也许是出乎意料的，在 8.20(b)这一节将给出一些例子. 象应用于测度一样，“奇异”这个字具有它的在这种现象上的起源.

证明 根据定理 8.14，存在一个 R^1 上的复测度 μ，使得

$$\mu((-\infty,x))=f(x).$$

根据定理 8.6，

$$\mu(E)=\mu_s(E)+\int_E (D\mu)(t)dt \tag{2}$$

令

$$f_s(x)=\mu_s((-\infty,x)) \qquad (-\infty<x<\infty) \tag{3}$$

定理 8.6 和 8.1 表明 $f_s'(x)=0$ a. e. $[m]$ 和

$$f'(x)=(D\mu)(x) \qquad \text{a. e.}[m]$$

因此，若我们取 $E=(-\infty,x)$，就从(2)得出(1).

根据定理 8.16，当且仅当 $\mu\ll m$，即当且仅当 $\mu_s=0$ 时，f 是绝对连续的.

最后，若 f 是不减的，则 $\mu \geqslant 0$，因此 $\mu_s \geqslant 0$，从而 f_s 是不减的.

8.19 定理 如果 $f \in BV$，则 f 是 a.e.[m]可微的，且

$$f' \in L^1(R^1).$$

证明 根据定理 8.13，存在一个 $g \in NBV$，使得

$$f(x) = g(x) + c$$

在所有的 f 的连续点上成立. 将定理 8.18 应用于 g.

因此下面引理(令 $h = f - g$ 时)蕴涵着本定理:

引理 如果 $h \in BV$，并且除开一个至多是可数的集之外，$h(x) = 0$. 则 $h'(x) = 0$ a.e.

为了证明引理，设 $S = \{x_i\}$ 是至多可数的集. 在这些点上，$h(x_i) = c_i \neq 0$. 因为 $h \in BV$，容易看出 $\Sigma |c_i| < \infty$. 固定 k，设 A_k 是所有 $x \notin S$ 的集，在这些 x 上，

$$\left| \frac{h(y) - h(x)}{y - x} \right| > \frac{1}{k} \tag{1}$$

对无穷多个 y 成立. 于是 $x \in A_k$ 当且仅当 $|x - x_i| < k|c_i|$ 对无穷多个 i 成立. 若 J_i 是中心在 x_i 长度为 $2k|c_i|$ 的区间，于是得出

$$\Sigma m(J_i) = 2k\Sigma |c_i| < \infty \tag{2}$$

根据定理 1.41，因此 $m(A_k) = 0$，对 $k = 1, 2, 3, \cdots$ 成立.

但如果 $x \notin S \cup A_1 \cup A_2 \cup A_3 \cup \cdots$，则 $h'(x) = 0$. 这就完成了证明.

习题 5 和 6 与这个引理有关.

8.20 例 上面的定理指出: 方程

$$f(x) - f(a) = \int_a^x f'(t)dt \tag{1}$$

(在这个方程的右边是一个 Lebesgue 积分) 当且仅当 f 是在 $[a, b]$ 上是绝对连续时(1)对某个区间 $[a, b]$ 内的所有的 x 成立. 人们可能会问，f' 的存在性是否蕴涵着 f 的绝对连续性? 问题的这一

提法是不够严格的．我们将给出两个例子说明(1)为什么会不成立，然后给出一个定理，它从另一些充分条件来推出(1)式．

(*a*) 当 $x\neq 0$ 时，令 $f(x)=x^2\sin(x^{-2})$，$f(0)=0$．则 f 在每一点是可微的，但

$$\int_0^1 |f'(t)|\,dt=\infty \tag{2}$$

所以 $f'\notin L^1$．同时在 $[0,1]$ 上，$f\notin BV$．

若我们把(1)中的积分（用[0，1]代替$[a,b]$）看作在$[e,1]$上的积分当 $e\to 0$ 时的极限，则对这个 f，(1)式还是成立的．

在这一类取极限过程不能使用的地方可能会出现更加复杂的情况．有一种由 Denjoy 和 Perron(参看[18]，[28]) 作出的积分过程，它是这样设计的，使得 f 在每一点都可微时，(1)式成立．这些过程却不具备由 f 的可积性能推出$|f|$的可积性这个性质，所以在分析中并不起多大作用．

(*b*) 假设 f 在$[a,b]$上连续，f 在$[a,b]$的几乎每一点是可微的，并且在$[a,b]$上 $f'\in L^1$．这些假定蕴涵着(1)成立吗？

回答：否．

选取$\{\delta_n\}$使得 $1=\delta_0>\delta_1>\delta_2>\cdots$，$\delta_n\to 0$．令 $E_0=[0,1]$．假定 $n\geqslant 0$ 并构造 E_n，使 E_n 是 2^n 个不相交的每一个长为 $2^{-n}\delta_n$ 的闭区间的并．在这 2^n 个区间的每一个的中心去掉一个线段，使余下的 2^{n+1} 个区间的每一个具有长度 $2^{-n-1}\delta_{n+1}$．（这是可能的，因为 $\delta_{n+1}<\delta_n$．）并设 E_{n+1} 是这 2^{n+1} 个区间的并．则 $E_1\supset E_2\supset\cdots$，$m(E_n)=\delta_n$，并且，若

$$E=\bigcap_{n=1}^{\infty} E_n \tag{3}$$

则 E 是紧的，且 $m(E)=0$．（事实上，E 是完全集．）令

(4) $$g_n=\delta_n^{-1}\chi_{E_n} \text{ 和 } f_n(x)=\int_0^x g_n(t)dt \quad (n=0,1,2,\cdots)$$

则 $f_n(0)=0, f_n(1)=1$，且每一个 f_n 是单调函数，它在 E_n 的余集的每一个线段上是常数．若 I 是其并为 E_n 的 2^n 个区间之一，则

(5) $$\int_I g_n(t)dt=\int_I g_{n+1}(t)dt=2^{-n}$$

从(5)得出

(6) $$f_{n+1}(x)=f_n(x) \qquad (x\notin E_n)$$

和

(7) $$|f_n(x)-f_{n+1}(x)|\leqslant\int_I |g_n-g_{n+1}|<2^{-n+1} \quad (x\in E_n)$$

因此$\{f_n\}$一致收敛于一个连续单调函数 f，使 $f(0)=0, f(1)=1$，且对所有的 $x\notin E, f'(x)=0$．因为 $m(E)=0$，有 $f'=0$ a. e.

于是(1)式不成立．附带地，现在我们已经构造了一个在定理 8.18 的叙述之后所定义的连续奇异函数的例子．

若 $\delta_n=(2/3)^n$，集 E 就是 Cantor 三分集．

8.21 定理 假设 f 是$[a,b]$上的实函数，它在$[a,b]$的每一点是可微的．假定在$[a,b]$上 $f'\in L^1$，则

(1) $$f(x)-f(a)=\int_a^x f'(t)dt \quad (a\leqslant x\leqslant b)$$

注意可微性在$[a,b]$的每一点都假定是具备的．

证明 显然对 $x=b$ 证明结论成立就足够了．固定 $e>0$．定理 2.24 保证存在$[a,b]$上的一个下半连续函数 g，使得 $g>f'$，且

(2) $$\int_a^b g(t)dt<\int_a^b f'(t)dt+e$$

确实，定理 2.24 仅给出 $g\geqslant f'$，但因为 $m([a,b])<\infty$，我们能在不影响(2)的前提下加一个小的常数于 g 上．对于任一 $\eta>0$，定义

(3) $$F_\eta(x)=\int_a^x g(t)dt-f(x)+f(a)+\eta(x-a)$$

$$(a\leqslant x\leqslant b)$$

暂时保持 η 固定. 因为 g 是下半连续的，且 $g(x)>f'(x)$，对每一个 $x\in[a,b]$，存在对应的 $\delta_x>0$，使得

(4) $$g(t)>f'(x) \quad 和 \quad \frac{f(t)-f(x)}{t-x}<f'(x)+\eta$$

对所有的 $t\in(x,x+\delta_x)$ 成立. 对于任一这样的 t，因此有

$$\begin{aligned}F_\eta(t)-F_\eta(x)&=\int_x^t g(s)ds-[f(t)-f(x)]+\eta(t-x)\\&>(t-x)f'(x)-(t-x)[f'(x)+\eta]+\eta(t-x)\\&=0\end{aligned}$$

因为 $F_\eta(a)=0$ 而 F_η 是连续的，所以存在一个满足 $F_\eta(x)=0$ 的最大的点 $x\in[a,b]$. 若 $x<b$，上面的计算蕴涵着 $F_\eta(t)>0$ 对 $t\in(x,b]$ 成立. 在任何情况下，$F_\eta(b)\geqslant 0$. 因为这个结论对每一个 $\eta>0$ 成立，所以(2)和(3)现在给出

(5) $$f(b)-f(a)\leqslant\int_a^b g(t)dt<\int_a^b f'(t)dt+e$$

因为 e 是任意的，得出

(6) $$f(b)-f(a)\leqslant\int_a^b f'(t)dt$$

若 f 满足定理的假设，则 $-f$ 也满足定理的假设；因此用 $-f$ 代替 f，(6)式成立，并且这两个不等式一起给出(1)式.

可微变换

8.22 定义 对任一 $x=(\xi_1,\cdots,\xi_k)\in R^k$，对应于一个函数

(1) $$\|x\|=\max(|\xi_1|,\cdots,|\xi_k|)$$

对处理立方体来说，采用这个范数比采用通常的欧几里得范数

(2) $$\|x\|_2=(\xi_1^2+\cdots+\xi_k^2)^{1/2}$$

要好一些. 因为$\|x\|\leqslant\|x\|_2\leqslant\sqrt{k}\,\|x\|$, 用这两个范数所引进的度量给出同一个 R^k 上的拓扑. 特别, 当 $n\to\infty$ 时, $\|x_n\|\to0$ 当且仅当 $\|x_n\|_2\to0$.

假设 V 是 R^k 内的一个开集, T 是 V 到 R^k 内的映射, $x\in V$, 且 A 是 R^k 上的一个线性算子(即, 像定义 2.1 一样, 是 R^k 到 R^k 内的一个线性映射). 若对每一个 $\varepsilon>0$, 存在一个 $\delta>0$, 使得不等式

(3) $$\|T(x+h)-T(x)-Ah\|\leqslant\varepsilon\|h\|$$

对于所有使 $|h|<\delta$ 的 $h\in R^k$ 成立, 就称 T 在 x 点是可微的且定义

(4) $$T'(x)=A$$

线性算子 $T'(x)$称为 T 在 x 的导数. 微分这个术语对 $T'(x)$ 也经常采用. 因此, 人们往往不说 T 在 x 点是可微的, 而说 T 在 x 点具有微分或说 T 的微分在 x 点存在.

在(3)式中用范数(2)代替范数(1)时, 无论是 T 的可微性或是 $T'(x)$的值均不受影响.

若 T 在 V 的每一点上可微, 就称 T 在 V 内可微. 在那种情况下, 对每一个 $x\in V$ 都有一个对应的线性算子 $T'(x)$; 对于固定的 x 和很小的 h, $T(x+h)-T(x)$被 h 的线性函数 $T'(x)h$ 在(3)的意义下逼近.

因为每一个实数 α 能解释为 R^1 上的(映 t 到 αt 的)线性算子, 当 $k=1$ 时, $T'(x)$的上述定义和通常的定义一致.

对 R^k 上的每一个线性算子 A 对应于一个数

(5) $$\Delta(A)=m(A(Q))$$

这里 m 是 R^k 上 Lebesgue 测度, Q 是单位立方体: $x\in Q$ 当且仅当 $0\leqslant\xi_i<1$ 对 $1\leqslant i\leqslant k$ 成立.

8.23 评注 如果 A 是 R^k 上的一个线性算子, 则

(1) $$A'(x)=A\qquad(x\in R^k)$$

若对所有的 Borel 集 E, 我们定义

(2) $$\mu(E)=m(A(E))$$

则 μ 是平移不变的, 因为

$$\mu(E+x)=m(A(E+x))=m(A(E)+Ax)=m(A(E))$$
$$=\mu(E)$$

从定理 2.20(*d*)得出

(3) $$\mu(E)=\Delta(A)m(E)$$

并且因此

(4) $$(D\mu)(x)=\Delta(A'(x)) \qquad (x\in R^k)$$

因为每一个可微变换能用一个常数加上一个线性变换来局部地逼近, 我们可以猜想在适当的条件下, (4)可以扩展到可微变换上去. 事情确是这样, 我们首先叙述一个没有涉及任何诱导测度 μ 的结果.

8.24 定理 假设 T 是开集 $V\subset R^k$ 到 R^k 内的连续开映射, 假设 T 在某一点 $x\in V$ 可微, 则对每一个 $\varepsilon>0$, 存在一个 $\delta>0$, 使得

(1) $$\left|\frac{m(T(C))}{m(C)}-\Delta(T'(x))\right|<\varepsilon$$

对每一个棱平行于坐标轴、棱长小于 δ 的、包含 x 的开立方体 C 成立.

证明 不失一般性, 假设 $x=0$ 和 $T(x)=0$. 令 $A=T'(0)$.

将用到关于有限维向量空间上线性算子的下列基本事实 (至于证明, 参看任何一本线性代数的书): R^k 上的线性算子 A 是一一的当且仅当 A 的值域是 R^k 的全部. 在一一的情况下, A 的逆 A^{-1} 是 R^k 上的线性算子, 并且称 A 为非奇异的或者可逆的.

为方便起见, 这个证明分成两部份.

情况 1 A 是非奇异的. 考虑由

(2) $$S(x)=A^{-1}T(x) \qquad (x\in V)$$

所定义的映射 S. 显然, S 是连续的、开的且 $S'(0)=A^{-1}T'(0)=I$, I 是恒等算子. 我们将证明

(3) $$\left|\frac{m(S(C))}{m(C)}-1\right|<\varepsilon$$

对所有的包含 0 的充分小的 C 成立.

因为 $T(x)=AS(x)$, 根据 8.23(3), 有

(4) $$m(T(C))=m(A(S(C)))=\Delta(A)m(S(C))$$

因此(3)将给出所要的结果.

固定 $\eta>0$, 使 $\eta<\frac{1}{4}$, 并且

(5) $$1-\varepsilon<(1-2\eta)^k<(1+2\eta)^k<1+\varepsilon$$

因为 $S(0)=0$ 和 $S'(0)=I$, 故存在一个 $\delta>0$, 使得 $\|x\|<\delta$ 时,

(6) $$\|S(x)-x\|\leqslant\eta\|x\|$$

设 C 是一个包含 0 的开立方体, 它的棱平行于坐标轴且棱长 $\lambda<\delta$. 设 C_1 和 C_2 是同心于 C 的开立方体, 它们的棱长分别为

(7) $$\lambda_1=(1-2\eta)\lambda,\quad \lambda_2=(1+2\eta)\lambda$$

注意(6)式蕴涵着

(8) $$\|S(x)-x\|\leqslant\eta\lambda \qquad (x\in\bar{C})$$

这里 $\bar{C}$ 是 C 的闭包.

若 $x\in C$, 则(8)指出 $S(x)\in C_2$. 于是 $S(C)\subset C_2$.

我们下一个目标是证明 $C_1\subset S(C)$. 令

(9) $$E_1=C_1\cap S(C),\quad E_2=C_1-S(C)$$

根据(8), S 决不会把 C 的边界点映到 C_1 内. 因此在(9)内能用 $\bar{C}$ 代替 C 而不致影响 E_1 和 E_2 的定义. 由 C_1 是开的和 $S(\bar{C})$ 是紧的, 看出 E_2 是开的. 因为 $\eta<\frac{1}{4}$, 故(8)指出 S 映 C 的中心到 C_1 内, 所以 E_1 是非空的. 最后, S 假设为开映射, 所以 $S(C)$ 是开的.

因此 E_1 是开的. 这样, C_1 就是两个不相交的开集 E_1 和 E_2 的并, 且 $E_1\neq\varnothing$. 但 C_1 是一个连通集(R^k 内的每一个凸集是连通的), 所以 C_1 不是两个不相交的非空开集的并. 于是得出 $E_2=\varnothing$, 因此 $E_1=C_1$, 且这就给出 $C_1\subset S(C)$.

现在已经证明了 $C_1\subset S(C)\subset C_2$. 因此

$$(10)\qquad (1-2\eta)^k\leqslant\frac{m(S(C))}{m(C)}\leqslant(1+2\eta)^k$$

并且从(5)得出(3).

这就完成了情况 1 的证明.

情况 2　A 是奇异的. 在这种情况下, A 映 R^k 到低维的子空间内, 即映 R^k 到一个零测集内. 特别

$$(11)\qquad \Delta(A)=m(A(Q))=0$$

如果给定 $e>0$, 则存在一个 $\eta>0$, 使得当 E_η 是所有与 $A(Q)$ 的距离小于 η 的点的集时, $m(E_\eta)<e$. 因为 $A=T'(0)$, 故存在一个 $\delta>0$, 使得当 $\|x\|<\delta$ 时,

$$(12)\qquad \|T(x)-Ax\|\leqslant\eta\|x\|$$

设 C 象在情况 1 一样是开立方体, 它的棱长 $\lambda<\delta$. 则对所有 $x\in C$, $\|T(x)-Ax\|<\eta\lambda$. 这意味着: $T(C)$ 包含在由与 $A(C)$ 距离小于 $\eta\lambda$ 的那些点所组成的集 E 内. η 的选择表明 $m(E)<e\lambda^k$. 因此, 当 $\lambda<\delta$ 时就有

$$(13)\qquad \frac{m(T(C))}{m(C)}\leqslant\frac{m(E)}{m(C)}=\frac{m(E)}{\lambda^k}<e$$

并且因为 $\Delta(A)=0$, 我们就已经证明了在情况 2 下, (1)式也成立.

现在定理的证明完了.

8.25　评注

(a) 若从假设中删去"开"映射这个条件, 上面的定理还是正确的, 但证明要困难些. 我们用到 T 是开的这一性质仅仅是为了

证明下列结论：如果 C 和 C_1 是同心的立方体，C_1 在 C 内部，C 和 C_1 的对应的面之间的距离小于 e，并且如果 C 的点被 T 移动得决不超过 e 这么远，则 $T(C)\supset C_1$，对连续的 T 这是正确的，但是其证明则要依靠一些比我们所希望读者具备的更深的关于 R^k 的拓扑知识.

下面是所需要的命题的不同叙述方法，并给出了其证明的梗概.

假设 $C=\{x:\|x\|<1\}$，$\|x_0\|<1-e$，$f:\bar{C}\to R^k$ 是连续的，并且，当 $\|x\|=1$ 时，$\|f(x)-x\|<e$. 则 $x_0\in f(C)$.

为证明这个结论，假设 $x_0\notin f(C)$. $x\neq 0$ 时，定义 $u(x)=x/\|x\|$. 在 $\bar{C}$ 上用

$$F(x)=\begin{cases} f(2x) & \text{若}\|x\|<\frac{1}{2} \\ (2\|x\|-1)u(x)+2(1-\|x\|)f(u(x)) & \text{若}\frac{1}{2}\leqslant\|x\|\leqslant 1 \end{cases}$$

定义 F. 则 F 是连续的. 若 $\frac{1}{2}\leqslant\|x\|\leqslant 1$，则 $F(x)$ 在 $u(x)$ 和 $f(u(x))$ 之间的线段上，因此 $\|F(x)\|>1-e$. 由此得出 $x_0\notin F(\bar{C})$. 同时，若 $\|x\|=1$，则 $F(x)=x$.

设 h 是 R^k 到 R^k 上的同胚，使得若 $\|x\|=1$，则 $h(x)=x$，并且 $h(x_0)=0$. 定义

$$g(x)=-u(h(F(x))) \qquad (x\in\bar{C})$$

因为 $x_0\notin F(\bar{C})$，$g(x)$ 是一个 $\bar{C}$ 到它的边界内的连续映射. 若 $\|x\|=1$，则 $g(x)=-x$. 于是对每一个 $x\in\bar{C}$，$g(x)\neq x$. 这就与 Brouwer 不动点定理矛盾.

(*b*) 同时也应该指出，若假设 T 是连续可微的，即若我们假设 $x\to T'(x)$ 是 V 到 R^k 上的全体线性算子的空间内的连续映射，则反函数定理([26]，定理 9.17)保证：当 $T'(x)$ 是非奇异时，T 是

开的. 所以,若我们假设 T' 是连续的，就能从假设中删去"开的"这个条件.

在下面的应用中，假定了 $T'(x)$的存在性，但没有对 T' 加上连续性的假定，所加在 T 上的拓扑学假定避开了可测性的困难.

8.26 定理 假设T是开集$V\subset R^k$到有界开集$W\subset R^k$上的可微映射,同时设T是一一的,且T的逆映射是连续的. 则

(*a*) 对每一个 Borel 集 $E\subset V$,$T(E)$是一个 Borel 集.

(*b*) 若E是 Lebesgue 可测的,则 $T(E)$ 也是 Lebesgue 可测的.

(*c*) 若 $\mu(E)=m(T(E))$,则 μ 是一个 V 上的正有界 Borel 测度,且 $\mu\ll m$.

(*d*) 对每一个 Lebesgue 可测集 $E\subset V$,

$$m(T(E))=\int_E \Delta(T'(x))dx$$

(*e*) 更一般地,若 $f\in L^1(W)$,则有

$$\int_W f(y)dy=\int_V f(T(x))\Delta(T'(x))dx$$

证明 因为T的逆映射是连续的,若E是开的，则$T(E)$是开的；所有使得$T(E)$是 Borel 集的那些$E\subset V$的集族是V内的σ-代数;这就蕴涵着(*a*).

若如(*c*)中一样定义 μ,因为T是一一的，则从m的可数可加性得出μ的可数可加性. 因为 $m(W)<\infty$，故μ是有界的. 根据定理 8.24,对每一个 x,$(\mathscr{D}^0\mu)(x)<\infty$. 因此，定理 8.11 的系指出 $\mu\ll m$. 这就证明了(*c*).

结合定理8.24和18.6,我们看出$(D\mu)(x)=\Delta(T'(x))$a. e. [m],并且对所有的 Borel 集E获得(*d*)的方程，但若E是 Borel 集且$m(E)=0$，则 $m(T(E))=0$；于是，在这种情况下对所有的具有

$m(A)=0$ 的 Lebesgue 可测集 $A\subset V$，得出 $T(A)$ 是 Lebesgue 可测的，并且 $m(T(A))=0$. 这就完成了(d)的证明，同时建立了(b).

剩下证明(e). 若 A 是 W 中的 Borel 集，且 $E=T^{-1}(A)$. 则 E 是 Borel 集，$\chi_E(x)=\chi_A(T(x))$，所以(d)蕴涵着

$$\int_W \chi_A(y)dy=m(A)=m(T(E))=\int_V \chi_A(T(x))\Delta(T'(x))dx$$

于是，若 $f=\chi_A$，则(e)成立. 因此若 f 是任一简单的 Borel 函数，则(e)也成立，并且可得出一般的情况.

8.27 Jacobi 行列式 变量代换公式 8.26(e)通常用下列形式叙述.

(1) $$\int_{T(V)} f(y)dy=\int_V f(T(x))|J_T(x)|dx$$

这里 $J_T(x)$ 是 T 在 x 点的 Jacobi 行列式. 根据定义，这是线性算子 $T'(x)$ 的行列式. 若我们证明了

(2) $$\Delta(T'(x))=|J_T(x)|$$

成立，则这两个公式的等价性就会确立起来，而且这显然是下列结果的一个推论.

8.28 定理 若 A 是 R^k 上的线性算子，并且 $\Delta(A)$ 是与 A 对应的一个纯量因子，使得

(1) $$m(A(E))=\Delta(A)m(E)$$

对每一个可测集 E 成立. 则

(2) $$\Delta(A)=|\det A|$$

我们选择用几何语言定义的量 $\Delta(A)$ 来叙述并证明定理 8.24 和 8.26，是为了突出这些定理的几何的和测度论的面貌. $\Delta(A)$ 也以 A(与 A 相联系的矩阵)的行列式的绝对值形式出现，这乃是一种代数性质的附加知识，这对计算显然是很重要的. 然而，过早地引入这个行列式并不会简化我们的工作.

证明 设$\{e_1,\cdots,e_k\}$是R^k的标准基：当$i=j$时，e_j的第i个坐标是1；当$i\neq j$时为0. 若A是R^k上的线性算子，且

$$Ae_j=\sum_{i=1}^{k}\alpha_{ij}e_i \qquad (1\leqslant j\leqslant k) \tag{3}$$

则根据定义，$\det A$是矩阵$[A]$的行列式，它的第i行第j列的元素是α_{ij}.

如果$A=A_1A_2$，则(1)蕴涵着$\Delta(A)=\Delta(A_1)\Delta(A_2)$. 因此，行列式的乘法定理指出，若对$A_1$和$A_2$，(2)式成立，则对$A$，(2)式也成立. 因为每一个$R^k$上的线性算子$A$是有限多个下列三种形式的算子的乘积. 对这些形式的每一个建立(2)就足够了.

(I) $\{Ae_1,\cdots,Ae_k\}$是$\{e_1,\cdots,e_k\}$的置换.

(II) $Ae_1=\alpha e_1$，对$i=2,\cdots,k$，$Ae_i=e_i$.

(III) $Ae_1=e_1+e_2$，对$i=2,\cdots,k$，$Ae_i=e_i$.

若A是(I)型的，则$[A]$恰好每一行每一列有一个1，在其余的地方为0. 所以$\det A=\pm 1$. 若Q是单位立方体，则$A(Q)=Q$，因此$\Delta(A)=1=|\det A|$.

若A是(II)型的，则显然$\Delta(A)=|\alpha|=|\det A|$.

若A是(III)型的，则$\det A=1$，且$A(Q)$是所有的点$\Sigma\xi_ie_i$的集，这些点的坐标满足

$$\xi_1\leqslant\xi_2<\xi_1+1，\ 若\ i\neq 2，则\ 0\leqslant\xi_i<1 \tag{4}$$

若S_1是在$A(Q)$内具有$\xi_2<1$的那些点的集，并且若S_2是$A(Q)$的其余部份，则

$$S_1\cup(S_2-e_2)=Q \tag{5}$$

这里S_2-e_2是S_2的平移. 因此

$$\Delta(A)=m(S_1\cup S_2)=m(S_1)+m(S_2-e_2)=m(Q)=1$$

这就完成了证明.

习　题

1　假定$\{\mu_n\}$是R^k上的正的 Borel 测度序列，且

$$\mu(E)=\sum_{n=1}^{\infty}\mu_n(E)$$

假定$\mu(R^k)<\infty$，证明μ是一个 Borel 测度．μ_n的 Lebesgue 分解和μ的 Lebesgue 分解之间的关系怎样？证明

$$(D\mu)(x)=\sum_{n=1}^{\infty}(D\mu_n)(x)\quad \text{a. e.}$$

2　假设每一个f_n是R^1上的正的不减的函数，且

$$f(x)=\sum_{n=1}^{\infty}f_n(x)<\infty$$

对所有的x成立．证明

$$f'(x)=\sum_{n=1}^{\infty}f_n'(x)\quad \text{a. e.}$$

3　设J是R^2内直线$y=x$上的紧区间．设m_1是J上的一维 Lebesgue 测度，对每一个 Borel 集$E\subset R^2$定义$\mu(E)=m_1(E\cap J)$．证明：虽然$\mu\perp m$，但$(D_Q\mu)(x)=0$对每一个$x\in R^2$成立．（与定理 8.10 比较．）

4　构造一个R^1上的连续单调函数，使得虽然$f'(x)=0$ a. e.，但f在任一线段上都不是常数．

5　若f象定理 8.19 的引理内的函数一样，能否存在一个不可数集E，使得f在E的任一点上不可微？

6　假设$\{c_i\}$是正数序列，使得$\Sigma|c_i|=\infty$．证明存在线段$J_i=(x_i-c_i, x_i+c_i)$，使得若$i\neq j$，则$x_i\neq x_j$，$\{x_i\}$在R^1内是稠密的，且每一个$x\in R^1$都在无穷多个J_i内．令$f(x_i)=c_i(i=1, 2, 3, \cdots)$，对所有其余的$x$，令$f(x)=0$．证明$f$是处处不可微的．

7　假设E是R^1(或R^k)内无孤立点的紧集(一个完全集)．证明E是连续正的 Borel 测度μ的支集(见第二章习题 11 的定义)．若$m(E)=0$，这就给出奇异测度的例子．

8　假设$E\subset[a, b]$，$m(E)=0$．构造一个$[a, b]$上的绝对连续单调函数

f，使得对每一个 $x\in E$，$f'(x)=\infty$。提示：$E\subset\bigcap V_n$，$m(V_n)<2^{-n}$，V_n 是开集。考虑 χ_{V_n} 的和。

9　证明两个 $[a,b]$ 上的两个绝对连续函数的乘积是绝对连续的。用这个结果推导出分部积分的定理。

10　若 f 是 $[0,1]$ 上的实函数，且

$$\gamma(t)=t+if(t)$$

根据定义，f 的图像的长度是 γ 在 $[0,1]$ 上的全变差。证明当且仅当 $f\in BV$ 时这个长度是有限的。假定 $f(0)=0$，f 是连续和不减的，且 f_s 是 f 的奇异部份（见定理 8.18）。证明 f 的图像的长度是

$$f_s(1)+\int_0^1\sqrt{1+[f'(t)]^2}\,dt$$

若 $f\in BV$，但 f 不一定是单调的，这个公式将怎样改变？在例 8.20(*b*) 中所构造的函数的图像长度是多少？

11　若 E 是 R^1 内的 Lebesgue 可测集，商

$$\frac{m(E\cap(x-\delta,x+\delta))}{2\delta}$$

的上极限和下极限叫做 E 在 x 点的上密度和下密度，记为 $\overline{D}_E(x)$ 和 $\underline{D}_E(x)$。若它们相等，其共同值 $D_E(x)$ 是 E 在 x 的密度（有时叫做度量密度）。若 $D_E(x)=1$，则 x 是 E 的密集点。

证明几乎所有的 $x\in E$，$D_E(x)=1$，而几乎所有的 $x\notin E$，$D_E(x)=0$。

构造一个集 E，使得 $\underline{D}_E(0)\neq\overline{D}_E(0)$。$\underline{D}_E(0)=0$ 和 $\overline{D}_E(0)=1$ 的情况能出现吗？

12　若 $A\subset R^1$ 和 $B\subset R^1$。令 $A+B=\{a+b:a\in A,\ b\in B\}$。设 $m(A)>0$，$m(B)>0$。证明 $A+B$ 包含一个线段。

提示：或者用 A 和 B 内密集点的存在性，或者假设 $m(A)<\infty$，$m(B)<\infty$，并证明 χ_A 和 χ_B 的卷积是一个连续函数，它在 R^1 上的积分不为 0。

设 C 是 Cantor 三分集。证明：虽然 $m(C)=0$，但 $C+C$ 却是一个区间。

将这个结果推广到 R^k 内的集上。

13　证明存在一个闭集 $E\subset R^2$，具有 $m_2(E)>0$，它不包含测度为正数的可测矩形（见定义 7.1）。提示：选取 $K\subset R^1$，它是紧的，完全不连通的，具有 $m_1(K)>0$。设 $E=\{(x,y):x+y\in K\}$。若 $A\subset R^1$，$B\subset R^1$，且 $A\times B\subset E$，则 $A+B\subset K$。用习题 12。

14 假设G是一个R^1的子群(关于加法)，$G\neq R^1$，并且G是Lebesgue可测的. 证明$m(G)=0$(与习题12比较).

15 证明至多存在一个算子A，满足定义8.22对$T'(x)$的要求

16 证明R^2上的Lebesgue测度是旋转不变的.

17 构造一个R^1上的单调函数，它在每一点的导数存在，但不是R^1上的连续函数.

18 对于$f\in L^\infty(R^1)$. 定义$f_t(x)=f(x-t)$，并假设

$$\lim_{t\to 0}\|f_t-f\|_\infty=0$$

这个范数是本性上确界. 证明在这些条件下，存在一个R^1上的一致连续函数g，使得$g(x)=f(x)$a. e.

提示：若$|t|<1/(2n)$，令$h_n(t)=n$，否则$h_n(t)=0$. 令g_n是卷积$f*h_n$. 证明$g_n(x)\to f(x)$a. e. 和

$$|g_n(x+t)-g_n(x)|\leqslant\|f_t-f\|_\infty$$

利用序列$\{g_n\}$的同等连续性([26]，定理7.23).

19 若$f(x+t)=f(x)$对所有的$x\in R^1$成立，则称t为R^1上的函数f的一个周期. 假设f是实的Lebesgue可测函数，具有周期s和t，使得s/t是无理数. 证明存在一个常数λ使得$f(x)=\lambda$a. e. 但f不一定要是常数. 提示：f的周期构成一个稠密集. 对实数α，注意集$E_\alpha=\{x:f(x)>\alpha\}$的密集点.

20 假设f是在由不等式$a\leqslant x\leqslant b$和$A\leqslant y\leqslant B$决定的矩形上的实函数. 找出那些关于f的条件(使它们尽可能地弱)，在这些条件下使下列结论有意义并且是正确的(D_1f表示f关于第一个变量x的偏导数). 若

$$g(x)=\int_A^B f(x,y)\,dy$$

则

$$\begin{aligned}g(x)&=\int_A^B dy\left[f(a,y)+\int_a^x (D_1f)(t,y)\,dt\right]\\&=\int_a^x dt\int_A^B (D_1f)(t,y)\,dy+\int_A^B f(a,y)\,dy\end{aligned}$$

以致

$$g'(x)=\int_A^B (D_1f)(x,y)\,dy$$

同时看到在什么条件下，能通过直接考虑商$[g(t)-g(x)]/(t-x)$导出

上面最后的公式.

21 假设 f 是 $[a, b]$ 上的连续复函数，具有全变差 V. 证明对每一个 $W<V$，对应一个 $\delta>0$，具有下列性质：若 $a=x_0<x_1<\cdots<x_n=b$，且 $|x_i-x_{i-1}|<\delta$ 对 $i=1, 2, \cdots, n$ 成立，则

$$\sum_{i=1}^{n}|f(x_i)-f(x_{i-1})|>W$$

22 假设 f 是一个$[a, b]$上的连续实函数. 对每一个实数 y，设$M(y)$是使得 $f(x)=y$ 的那些$[a, b]$内的点 x 的个数. ($M(y)$是有限的或无限的. M 可以称为 f 的重数函数). 证明M 是 Borel 函数，并且$\int M(y)\,dy$ 是 f 在 $[a, b]$上的全变差.

提示：对图像是有限个直线区间的并的函数，这个结果是显然的. 通过适当地选取这种函数序列来逼近 f (它们的重数函数应该递增地趋于 f 的重数函数)，运用习题 21.

23 每一个 R^1 上的左连续复函数是 Borel 函数吗？若 V 是开集，关于 $f^{-1}(V)$ 你能说些什么？

24 (籍助于 Hausdorff 极大性定理)证明存在一个 R^1 上的实的不连续函数 f，使得

(1) $$f(x+y)=f(x)+f(y)$$

对所有的 x 和 $y\in R^1$ 成立.

证明若(1)成立和 f 是 Lebesgue 可测的，则 f 连续.

证明若(1)成立和 f 的图像在平面上不是稠密的，则 f 是连续的.

找出所有的满足(1)的连续函数，把这个结果与第九章的习题 17 加以比较.

第九章　Fourier 变式

形式上的性质

9.1　定义　本章我们将从以前的记号出发，并用字母 m 记 R^1 上的 Lebesgue 测度被 $\sqrt{2\pi}$ 除的结果，而不是 R^1 上的 Lebesgue 测度. 这个约定简化了一些结果的外貌，例如反演定理和 Plancherel 定理. 因此，我们将用记号

$$(1)\qquad \int_{-\infty}^{\infty} f(x)dm(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)dx$$

其中 dx 是通常的 Lebesgue 测度，并定义

$$(2)\qquad \|f\|_p=\left\{\int_{-\infty}^{\infty}|f(x)|^p dm(x)\right\}^{1/p}\qquad (1\leqslant p<\infty)$$

$$(3)\qquad (f*g)(x)=\int_{-\infty}^{\infty} f(x-y)g(y)dm(y)\qquad (x\in R^1)$$

和

$$(4)\qquad \hat{f}(t)=\int_{-\infty}^{\infty} f(x)e^{-ixt}dm(x)\qquad (t\in R^1)$$

整个这章，我们将用 L^p 代替 $L^p(R^1)$，而 C_0 将记 R^1 上所有在无穷远点为零的连续函数的空间.

如果 $f\in L^1$，则积分(4)对每个实数 t 都是完全确定的. 函数 $\hat{f}$ 称为 f 的 Fourier 变式. 注意，术语“Fourier 变式”也用于把 f 映为 $\hat{f}$ 的映射.

在定理 9.2 中列出的形式上的性质是与 m 的平移不变性密切相关的，也与这样一个事实，即对每个实数 α，映射 $x\longrightarrow e^{i\alpha x}$ 是加法群 R^1 的特征标密切相关的. 根据定义，如果对一切实数 s 和 t，

$|\varphi(t)|=1$ 并且

$$\varphi(s+t)=\varphi(s)\varphi(t) \tag{5}$$

则函数 φ 是 R^1 的特征标. 换句话说，φ 是加法群 R^1 到绝对值等于 1 的复数乘法群内的一个同态映射. 以后将会看到(定理 9.23 的证明中)，R^1 的每个连续的特征标可由一个指数函数给出.

9.2 定理 设 $f\in L^1$，α 和 λ 是实数.

(a) 如果 $g(x)=f(x)e^{i\alpha x}$，则 $\hat{g}(t)=\hat{f}(t-\alpha)$.

(b) 如果 $g(x)=f(x-\alpha)$，则 $\hat{g}(t)=\hat{f}(t)e^{-i\alpha t}$.

(c) 如果 $g\in L^1$，$h=f*g$，则 $\hat{h}(t)=f(t)\hat{g}(t)$.

因此，Fourier 变式用特征标来乘化为平移，反之亦然，并把卷积化为点态相乘的积.

(d) 如果 $g(x)=\overline{f(-x)}$，则 $\hat{g}(t)=\overline{\hat{f}(t)}$.

(e) 如果 $g(x)=f(x/\lambda)$ 且 $\lambda>0$，则 $\hat{g}(t)=\lambda\hat{f}(\lambda t)$.

(f) 如果 $g(x)=-ixf(x)$ 且 $g\in L^1$，则 $\hat{f}$ 是可微的，且 $\hat{f}'(t)=\hat{g}(t)$.

证明 (a)，(b)，(d)和(e)由直接代入公式 9.1(4)予以证明. (c)的证明是 Fubini 定理的一个应用(关于要求可测性的证明参看定理 7.14)：

$$\begin{aligned}\hat{h}(t)&=\int_{-\infty}^{\infty}e^{-itx}dm(x)\int_{-\infty}^{\infty}f(x-y)g(y)dm(y)\\&=\int_{-\infty}^{\infty}g(y)e^{-ity}dm(y)\int_{-\infty}^{\infty}f(x-y)e^{-it(x-y)}dm(x)\\&=\int_{-\infty}^{\infty}g(y)e^{-ity}dm(y)\int_{-\infty}^{\infty}f(x)e^{-itx}dm(x)\\&=\hat{g}(t)\hat{f}(t)\end{aligned}$$

注意如何应用 m 的平移不变性.

为证明(f)，注意

(1) $$\frac{\hat{f}(s)-\hat{f}(t)}{s-t}=\int_{-\infty}^{\infty}f(x)e^{-ixt}\varphi(x,s-t)dm(x)\quad(s\neq t)$$

这里 $\varphi(x,u)=(e^{-ixu}-1)/u$. 由于对所有实数 $u\neq0$, $|\varphi(x,u)|\leqslant|x|$, 且当 $u\to0$ 时 $\varphi(x,u)\to-ix$, 把控制收敛定理用于(1)式, 如果 s 趋向于 t, 我们得到

(2) $$\hat{f}'(t)=-i\int_{-\infty}^{\infty}xf(x)e^{-ixt}dm(x)$$

9.3 评注

(*a*) 由于控制收敛定理仅涉及函数的可数序列, 在上述证明中, 求助于控制收敛定理似乎是不合逻辑的. 然而, 对每个收敛于 t 的序列$\{s_n\}$, 它却能帮助我们得到,

$$\lim_{n\to\infty}\frac{\hat{f}(s_n)-\hat{f}(t)}{s_n-t}=-i\int_{-\infty}^{\infty}xf(x)e^{-ixt}dm(t)$$

而这恰好是说,

$$\lim_{s\to t}\frac{\hat{f}(s)-\hat{f}(t)}{s-t}=-i\int_{-\infty}^{\infty}xf(x)e^{-ixt}dm(t)$$

我们还会遇到类似情况, 当应用收敛定理于那些情况时将不再作更多的声明.

(*b*) 定理 9.2(*b*)表明

$$[f(x+\alpha)-f(x)]/\alpha$$

的 Fourier 变式是

$$\hat{f}(t)\frac{e^{i\alpha t}-1}{\alpha}$$

这一点暗示定理 9.2(*f*) 的类似情形在一定的条件下也应该成立, 即是说, f' 的 Fourier 变式是 $it\hat{f}(t)$. 如果 $f\in L^1$, $f^1\in L^1$, 而且 f 是 f'的不定积分的话, 则此结果容易用分部积分得到. 我们把它以及与它有关的一些结果留作习题. Fourier 变式把微分法化为用 ti 相乘这一事实, 使 Fourier 变式在微分方程的研究中成为一种有用的工具.

反演定理

9.4 刚才我们看到，关于函数的某些运算同关于它们的Fourier变式的运算是很好地对应着的. 如果有一种方法可以从变式又得回函数，即是说，有一种反演公式，那么，当然将会提高这种对应的有用性和兴趣.

我们通过与Fourier级数相类比，来看这样一个公式会像什么样子. 如果

$$c_n=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)e^{-inx}dx \qquad (n\in\mathbf{Z}) \tag{1}$$

则反演公式是

$$f(x)=\sum_{-\infty}^{\infty}c_ne^{inx} \tag{2}$$

我们知道，如果 $f\in L^2(T)$，在 L^2-收敛的意义下，(2)式成立. 我们也知道，纵然 f 是连续的，但在点态收敛的意义下，(2)式也不一定成立. 现在设 $f\in L^1(T)$，$\{c_n\}$ 用(1)给出，并设

$$\sum_{-\infty}^{\infty}|c_n|<\infty \tag{3}$$

令

$$g(x)=\sum_{-\infty}^{\infty}c_ne^{inx} \tag{4}$$

由于(3)，(4)中的级数一致收敛(因此，g 是连续的)，并容易算出 g 的Fourier系数：

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}g(x)e^{-ikx}dx&=\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\sum_{n=-\infty}^{\infty}c_ne^{inx}\right\}e^{-ikx}dx\\&=\sum_{n=-\infty}^{\infty}c_n\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{i(n-k)x}dx\end{aligned} \tag{5}$$

$$= c_k$$

因此，f 和 g 有相同的 Fourier 系数. 由此得出 $f=g$ a. e.，于是 f 的 Fourier 级数收敛于 $f(x)$a. e.

在 Fourier 变式的上下文里，类似的假设是 $f\in L^1$ 和 $\hat{f}\in L^1$，而且我们可以希望，像

$$f(x)=\int_{-\infty}^{\infty}\hat{f}(t)e^{itx}dm(t) \tag{6}$$

这样的公式是正确的. 确实，如果 $\hat{f}\in L^1$，(6)式的右边含义是明确的；记它为 $g(x)$；但是，如果要像(5)那样来论证，我们就会遇到实际上是毫无意义的积分

$$\int_{-\infty}^{\infty}e^{i(t-s)x}dx \tag{7}$$

因此，纵然在很强的假设，即 $\hat{f}\in L^1$ 之下，(6)式(它是正确的)的证明也还得经历一个更曲折的路程.

[应该提一下，如果把在$(-\infty,\infty)$上的积分理解为在$(-A,A)$上的积分当 $A\to\infty$ 时的极限的话，甚至当 $\hat{f}\notin L^1$ 时，(6)式也可以成立. (类似于一个不是绝对收敛的级数可以收敛一样.)然而，我们不打算探讨这个问题.]

9.5 定理 对 R^1 上的任意函数 f 和每个 $y\in R^1$，令 f_y 是 f 的平移，定义为

$$f_y(x)=f(x-y)\qquad(x\in R^1) \tag{1}$$

如果 $1\leqslant p<\infty$ 且 $f\in L^p$，则映射

$$y\longrightarrow f_y \tag{2}$$

是一个 R^1 到 $L^p(R^1)$ 内的一致连续映射.

证明 固定 $\varepsilon>0$. 由于 $f\in L^p$，存在连续函数 g，它的支集属于有界区间$[-A,A]$，使得

$$\|f-g\|_p<\varepsilon$$

(定理 3.14). g 的一致连续性表明存在 $\delta\in(0,A)$，使得$|s-t|<\delta$

时，有

$$|g(s)-g(t)|<(3A)^{-1/p}\varepsilon$$

如果$|s-t|<\delta$，便得到

$$\int_{-\infty}^{\infty}|g(x-s)-g(x-t)|^p dx<(3A)^{-1}\varepsilon^p(2A+\delta)<\varepsilon^p$$

于是$\|g_s-g_t\|<\varepsilon$.

注意L^p-范数(关于Lebesgue测度)是平移不变的：$\|f\|_p=\|f_s\|_p$. 因此，当$|s-t|<\delta$时

$$\begin{aligned}\|f_s-f_t\|_p&\leqslant\|f_s-g_s\|_p+\|g_s-g_t\|_p+\|g_t-f_t\|_p\\&=\|(f-g)_s\|_p+\|g_s-g_t\|_p+\|(g-f)_t\|_p<3\varepsilon\end{aligned}$$

这就完成了证明.

9.6 定理 如果$f\in L^1$，则$\hat{f}\in C_0$，且

(1) $$\|\hat{f}\|_\infty\leqslant\|f\|_1$$

证明 由9.1(4)，不等式(1)是显然的. 如果$t_n\to t$，则

(2) $$|\hat{f}(t_n)-\hat{f}(t)|\leqslant\int_{-\infty}^{\infty}|f(x)|\,|e^{-it_nx}-e^{-itx}|dm(x)$$

此被积函数是以$2|f(x)|$为界的，并当$n\to\infty$时，对每个x都趋向于0. 根据控制收敛定理，于是$\hat{f}(t_n)\to\hat{f}(t)$. 因此，$\hat{f}$是连续的.

由于$e^{\pi i}=-1$，9.1(4)给出

(3) $$\begin{aligned}\hat{f}(t)&=-\int_{-\infty}^{\infty}f(x)e^{-it(x+\pi/t)}dm(x)\\&=-\int_{-\infty}^{\infty}f(x-\pi/t)e^{-itx}dm(x)\end{aligned}$$

因此

(4) $$2\hat{f}(t)=\int_{-\infty}^{\infty}\left\{f(x)-f\left(x-\frac{\pi}{t}\right)\right\}e^{-itx}dm(x)$$

于是

(5) $$2|\hat{f}(t)|\leqslant\|f-f_{\pi/t}\|_1$$

根据定理9.5，当$t\to\pm\infty$时趋向于0.

9.7 一对辅助函数 在反演定理的证明中，知道一个正的函数 H，使得它有正的 Fourier 变式，而且这个变式的积分又易于计算，这将是便利的. 在许多可能性当中，我们选择了与半平面内的调和函数的关系是有趣的那一种. （参看第十一章的习题 21. ）

令

(1) $$H(t)=e^{-|t|}$$

并定义

(2) $$h_\lambda(x)=\int_{-\infty}^{\infty}H(\lambda t)e^{itx}dm(t) \qquad (\lambda>0)$$

经过简单计算，得出

(3) $$h_\lambda(x)=\sqrt{\frac{2}{\pi}}\frac{\lambda}{\lambda^2+x^2}$$

因此

(4) $$\int_{-\infty}^{\infty}h_\lambda(x)dm(x)=1$$

也要注意到，$0<H(t)\leqslant 1$ 并且当 $\lambda\to 0$ 时 $H(\lambda t)\to 1$.

9.8 命题 如果 $f\in L^1$，则

$$(f*h_\lambda)(x)=\int_{-\infty}^{\infty}H(\lambda t)\hat{f}(t)e^{ixt}dm(t)$$

证明 这是 Fubini 定理的一个简单应用.

$$\begin{aligned}(f*h_\lambda)(x)&=\int_{-\infty}^{\infty}f(x-y)dm(y)\int_{-\infty}^{\infty}H(\lambda t)e^{ity}dm(t)\\&=\int_{-\infty}^{\infty}H(\lambda t)dm(t)\int_{-\infty}^{\infty}f(x-y)e^{ity}dm(y)\\&=\int_{-\infty}^{\infty}H(\lambda t)dm(t)\int_{-\infty}^{\infty}f(y)e^{it(x-y)}dm(y)\\&=\int_{-\infty}^{\infty}H(\lambda t)\hat{f}(t)e^{itx}dm(t)\end{aligned}$$

9.9 定理 如果 $g\in L^\infty$，g 在点 x 连续，则

(1) $$\lim_{\lambda\to 0}(g*h_\lambda)(x)=g(x)$$

证明 由于 9.7(4), 有

$$\begin{aligned}(g*h_\lambda)(x)-g(x)&=\int_{-\infty}^{\infty}[g(x-y)-g(x)]h_\lambda(y)dm(y)\\&=\int_{-\infty}^{\infty}[g(x-y)-g(x)]\lambda^{-1}h_1\left(\frac{y}{\lambda}\right)dm(y)\\&=\int_{-\infty}^{\infty}[g(x-\lambda s)-g(x)]h_1(s)dm(s)\end{aligned}$$

最后那个被积函数被 $2\|g\|_\infty h_1(s)$ 所控制, 并且当 $\lambda\to 0$, 对每个 s 按点收敛于 0. 因此, 由控制收敛定理便得到(1).

9.10 定理 如果 $1\leqslant p<\infty$ 和 $f\in L^p$, 则

(1) $$\lim_{\lambda\to 0}\|f*h_\lambda-f\|_p=0$$

$p=1$ 和 $p=2$ 的情形我们是有兴趣的, 但是一般情形也不难证明.

证明 由于 $h_\lambda\in L^q$, q 是共轭于 p 的指数, 对每个 x, $(f*h_\lambda)(x)$ 的意义是明确的. (事实上, $f*h_\lambda$ 是连续的; 参看习题 8.)因为 9.7(4), 我们有

(2) $$(f*h_\lambda)(x)-f(x)=\int_{-\infty}^{\infty}[f(x-y)-f(x)]h_\lambda(y)dm(y)$$

而定理 3.3 给出

(3) $$\begin{aligned}&|(f*h_\lambda)(x)-f(x)|^p\\&\qquad\leqslant\int_{-\infty}^{\infty}|f(x-y)-f(x)|^p h_\lambda(y)dm(y)\end{aligned}$$

将(3)式对 x 积分, 并应用 Fubini 定理:

(4) $$\|f*h_\lambda-f\|_p^p\leqslant\int_{-\infty}^{\infty}\|f_y-f\|_p^p h_\lambda(y)dm(y)$$

如果 $g(y)=\|f_y-f\|_p^p$, 根据定理 9.5, g 是有界和连续的, 且 $g(0)=0$. 根据定理 9.9, 因此当 $\lambda\to 0$ 时, (4)的右边趋向于 0.

9.11 反演定理 如果 $f\in L^1$ 和 $\hat{f}\in L^1$，且

$$(1)\qquad g(x)=\int_{-\infty}^{\infty}\hat{f}(t)e^{ixt}dm(t)\qquad (x\in R^1)$$

则 $g\in C_0$ 和 $f(x)=g(x)$ a. e.

证明 由命题 9.8，

$$(2)\qquad (f*h_\lambda)(x)=\int_{-\infty}^{\infty}H(\lambda t)\hat{f}(t)e^{ixt}dm(x)$$

(2)式右边的被积函数以 $|\hat{f}(t)|$ 为界，由于当 $\lambda\to 0$ 时，$H(\lambda t)\to 1$，根据控制收敛定理，(2)的右边对每个 $x\in R^1$ 收敛于 $g(x)$.

如果综合定理 9.10 和 3.12，我们看出，存在序列 $\{\lambda_n\}$ 使得 $\lambda_n\to 0$，且

$$(3)\qquad \lim_{n\to\infty}(f*h_{\lambda_n})(x)=f(x)\ \text{a. e.}$$

因此，$f(x)=g(x)$ a. e.. 由定理 9.6 推出 $g\in C_0$.

9.12 唯一性定理 如果 $f\in L^1$ 并且对一切，$t\in R^1$，$\hat{f}(t)=0$，则 $f(x)=0$ a. e. .

证明 由 $\hat{f}=0$ 得到 $\hat{f}\in L^1$，由反演定理得出结果.

Plancherel 定理

由于 R^1 的勒贝格测度是无限的，L^2 不是 L^1 的子集. 因此，根据公式 9.1(4)，Fourier 变式的定义对每个 $f\in L^2$ 不是直接可以应用的. 但如果 $f\in L^1\cap L^2$，就能应用定义并得出 $\hat{f}\in L^2$. 事实上，$\|\hat{f}\|_2=\|f\|_2$! 把由 $L^1\cap L^2$ 到 L^2 内的这个等距映射扩充为 L^2 到 L^2 上的等距，且此扩充对每个 $f\in L^2$ 都定义了 Fourier 变式(有时称为 Plancherel 变式). 事实上，所得出的 L^2 理论比 L^1 的情形更具有对称性. 在 L^2 中，f 和 $\hat{f}$ 确实处于相同的地位.

9.13 定理 对每个 $f\in L^2$ 都能对应一个函数 $\hat{f}\in L^2$，使得下述性质成立：

(a) 如果 $f\in L^1\cap L^2$, 则 $\hat{f}$ 是 f 的前面定义过的Fourier变式.

(b) 对每个 $f\in L^2$, $\|\hat{f}\|_2=\|f\|_2$.

(c) 映射 $f\to\hat{f}$ 是一个 L^2 到 L^2 上的 Hilbert 空间的同构.

(d) 在 f 和 $\hat{f}$ 之间存在下述对称关系: 如果

$$\varphi_A(t)=\int_{-A}^{A}f(x)e^{-ixt}dm(x)\text{和}\ \psi_A(x)=\int_{-A}^{A}\hat{f}(t)e^{-ixt}dm(t)$$

则当 $A\to\infty$ 时, $\|\varphi_A-\hat{f}\|_2\to 0$ 和 $\|\psi_A-f\|_2\to 0$.

注: 由于 $L^1\cap L^2$ 在 L^2 中是稠密的, 性质(a)和(b)唯一决定了映射 $f\to\hat{f}$. 称性质(d)为 L^2 反演定理.

证明 我们的首要目标是建立关系

$$\|\hat{f}\|_2=\|f\|_2\qquad(f\in L^1\cap L^2)\tag{1}$$

固定 $f\in L^1\cap L^2$, 令 $\tilde{f}(x)=\overline{f(-x)}$, 并规定 $g=f*\tilde{f}$. 则

$$\begin{aligned}g(x)&=\int_{-\infty}^{\infty}f(x-y)\overline{f(-y)}dm(y)\\&=\int_{-\infty}^{\infty}f(x+y)\overline{f(y)}dm(y)\end{aligned}\tag{2}$$

或者

$$g(x)=(f_{-x},f)\tag{3}$$

这里的内积是在 Hilbert 空间 L^2 内取的, 像在定理 9.5 中一样, f_{-x} 表示 f 的平移. 根据该定理, $x\to f_{-x}$ 是 R^1 到 L^2 内的一个连续映射, 因此, 由内积的连续性 (定理 4.6)得出 g 是一个连续函数. Schwarz 不等式表明

$$|g(x)|\leqslant\|f_{-x}\|_2\|f\|_2=\|f\|_2^2\tag{4}$$

于是 g 是有界的. 且由于 $f\in L^1$ 和 $\tilde{f}\in L$, 也就有 $g\in L^1$.

由于 $g\in L^1$, 可以应用命题 9.8:

$$(g*h_\lambda)(0)=\int_{-\infty}^{\infty}H(\lambda t)\hat{g}(t)dm(t)\tag{5}$$

由于 g 是连续和有界的, 定理 9.9 表明

(6) $$\lim_{\lambda\to 0}(g*h_\lambda)(0)=g(0)=\|f\|_2^2$$

定理 9.2(d)表明 $\hat{g}=|\hat{f}|^2\geqslant 0$，由于当 $\lambda\to 0$ 时 $H(\lambda t)$递增到 1，单调收敛定理给出

(7) $$\lim_{\lambda\to 0}\int_{-\infty}^{\infty}H(\lambda t)\hat{g}(t)dm(t)=\int_{-\infty}^{\infty}|\hat{f}(t)|^2dm(t)$$

于是(5)，(6)和(7)表明 $\hat{f}\in L^2$ 和(1)成立.

这曾经是证明的难点.

对任何 $A>0$，令 f_A 是 f 和区间$[-A,A]$的特征函数的积. 如果 $f\in L^2$，则当 $A\to\infty$ 时，显然有$\|f_A-f\|_2\to 0$. 又 $f_A\in L^1\cap L^2$，并且当 φ_A 如(d)中所定义时，则 $\varphi_A=\hat{f}_A$. 由于$\{f_A\}$是 L^2 内的 Cauchy 序列，关系(1)表明$\{\varphi_A\}$是 L^2 内的 Cauchy 序列；且由于 L^2 是完备的，当 $A\to\infty$ 时，$\{\varphi_A\}$收敛于 L^2 的一个元素，我们称为 $\hat{f}$. 注意，如果 $f\in L^1\cap L^2$，则 $\varphi_A(t)$点态收敛于原先定义了的 f 的 Fourier 变式，而这个点态极限几乎处处与 L^2-极限一致(定理 3.12).

映射 $f\to\hat{f}$ 的定义域已由 $L^1\cap L^2$ 扩充到 L^2. 此外，由(1)得到$\|\varphi_A\|_2=\|f_A\|_2$，于是

(8) $$\|\hat{f}\|_2=\lim_{A\to\infty}\|\varphi_A\|_2=\lim_{A\to\infty}\|f_A\|_2=\|f\|_2$$

这样，我们证明了(a)，(b)，和(d)的头一半.

我们对 $\hat{f}$ 引进临时性的记号 Φf. 于是 Φf 是函数 φ_A 的 L^2-极限. 如果 $f\in L^2$，则 Φ 是 L^2 到 L^2 内的一个等距. 对 $g\in L^2$，类似地把 Ψg 定义为函数

(9) $$\int_{-A}^{A}g(t)e^{ixt}dm(t)$$

的 L^2-极限. 由于$(\Psi g)(x)=\Phi g(-x)$，因此，Ψ 是一个 L^2 到 L^2 内的等距. 如果 f 和 $\hat{f}$ 两者均属于 $L^1\cap L^2$，则由定理 9.11 得到

(10) $$\Psi\Phi f=f$$

根据命题 9.8，如果 $f\in L^1\cap L^2$，且 $\lambda>0$，$f*h_\lambda$ 便满足这些要求. 因此

(11) $$\Psi\Phi(f*h_\lambda)=f*h_\lambda$$

当 $\lambda\to 0$ 时，$\|f*h_\lambda-f\|_2\to 0$(定理 9.10)，由于 $\Psi\Phi$ 是一个等距，对每个 $f\in L^1\cap L^2$ 得到(10). 因为 $L^1\cap L^2$ 在 L^2 内是稠密的，对每个 $f\in L^2$，(10)式成立. 这就证明了(*d*)的后半部分.

但由于算子 Φ 和 Ψ 所处的地位显然可以互相交换，因而对每个 $g\in L^2$，关系式 $\Phi\Psi g=g$ 也都是正确的. 这就是说，如果 $f=\Psi g$，则 $g=\hat{f}$. 因此，映射 $f\to\hat{f}$ 把 L^2 映到 L^2 上.

最后，恒等式

(12) $$4f\bar{g}=|f+g|^2-|f-g|^2+i|f+ig|^2-i|f-ig|^2$$

表明 L^2 到 L^2 上的每个等距映射也保持内积. 换句话说，Parseval 公式

(1) $$\int_{-\infty}^{\infty}f(x)\overline{g(x)}dm(x)=\int_{-\infty}^{\infty}\hat{f}(t)\overline{\hat{g}(t)}dm(t)$$

对 $f\in L^2$ 和 $g\in L^2$ 是成立的.

这就得出(*c*)，证完.

9.14 定理 如果 $f\in L^2$ 和 $\hat{f}\in L^1$，则

$$f(x)=\int_{-\infty}^{\infty}\hat{f}(t)e^{ixt}dm(t)\ \text{ a. e.}$$

证明 这是定理 9.13(*d*)的一个系.

9.15 评注 如果 $f\in L^1$，对每个 t，公式 9.1(4)明确地定义 $\hat{f}(t)$. 如果 $f\in L^2$，Plancherel 定理唯一地确定 $\hat{f}$ 作为 Hilbert 空间 L^2 的元素，但作为点函数的 $\hat{f}(t)$ 仅仅是几乎处处被确定. 这是 L^1 和 L^2 中 Fourier 变式理论之间的重要不同之处. 作为点函数，$\hat{f}(t)$的不确定性将给我们正要讨论的问题带来一些困难.

9.16 L^2 的平移不变子空间 L^2 的子空间 M 称为平移不变的，如果对每个实数 α，$f\in M$ 蕴涵着 $f_\alpha\in M$，而 $f_\alpha=f(x-\alpha)$. 在我们研究 Fourier 变式的时候，平移早已起着重要的作用. 现在我们提出一个问题，它的解决将提供一个如何应用 Plancherel 定理的实例. (在第十九章中将出现其它的应用.)此问题是:

描述 L^2 的闭平移不变子空间.

令 M 是 L^2 的闭平移不变子空间，$\hat{M}$ 是 M 在 Fourier 变式下的像. 则 $\hat{M}$ 是闭的(因为 Fourier 变式是 L^2-等距.)如果 f_α 是 f 的一个平移，f_α 的 Fourier 变式是 $\hat{f}e_\alpha$ 这里 $e_\alpha(t)=e^{-i\alpha t}$; 在定理 9.2 中，对 $f\in L^1$，我们证明过这一点; 如同在定理 9.13(d)中所看到的一样，此结果扩充到了 L^2. 由此得到，对一切 $\alpha\in R^1$，$\hat{M}$ 用 e_α 来乘是不变的.

设 E 是 R^1 中任一个可测集. 如果 $\hat{M}$ 是一切 $\varphi\in L^2$ 的集，而 φ 在 E 上几乎处处为零，则 $\hat{M}$ 确实是 L^2 的一个子空间，它用所有的 e_α 来乘是不变的(注意，$|e_\alpha|=1$，因此，如果 $\varphi\in L^2$，则 $\varphi e_\alpha\in L^2$)，而 $\hat{M}$ 还是闭集. 证明: 当且仅当 φ 与每个 $\psi\in L^2$ 正交，而 ψ 在 E 的余集上几乎处处为零时，$\varphi\in\hat{M}$.

设 M 为在 Fourier 变式的作用下 $\hat{M}$ 的逆像，则 M 是一个具有所要性质的空间.

现在，我们可以猜到，每一个我们所要求的空间 M 都可以从集 $E\subset R^1$ 通过这种方式得出. 要证明这一点. 我们就得证明，对每个闭的平移不变的 $M\subset L^2$ 对应一个集 $E\subset R^1$，使得当且仅当在 E 上几乎处处 $\hat{f}(t)=0$ 时 $f\in M$. 从 M 构造 E 的明显方法是对每个 $f\in M$ 联系着一个由使 $\hat{f}(t)=0$ 的所有点组成集 E_f，并定义 E 为这些集 E_f 的交. 但这个明显的方法带来了严重的困难: 每个 E_f 只能确定到相差一个测度为零的集. 如果 $\{A_i\}$ 是可数集族，每个 A_i 确定到相差一个测度为零的集则 $\bigcap A_i$ 也确定到相差一个测度

为零的集. 但是存在有不可数个 $f\in M$，这样在 $\cap E_f$ 上我们就会失去一切控制.

如果考虑我们的函数是作为 Hilbert 空间 L^2 的元素，而本来就不是作为点函数的话，这些困难就完全不会产生.

现在我们将证明这个猜想. 令 $\hat{M}$ 是闭平移不变子空间 $M\subset L^2$ 在 Fourier 变式作用下的象. 设 P 是 L^2 到 $\hat{M}$ 上的正交射影(定理 4.11)：对每个 $f\in L^2$ 对应唯一的 $Pf\in\hat{M}$，使得 $f-Pf$ 正交于 $\hat{M}$，因此

(1) $$f-Pf\perp Pg \qquad (f\text{ 和 }g\in L^2)$$

并由于 $\hat{M}$ 对用 e_α 相乘是不变的，故有

(2) $$f-Pf\perp(Pg)e_\alpha \qquad (f,g\in L^2,\alpha\in R^1)$$

如果我们回忆起在 L^2 里内积是如何定义的，我们就会看到(2)式等价于

(3) $$\int_{-\infty}^{\infty}(f-Pf)\cdot\overline{Pg\cdot}\ e_{-\alpha}dm=0 \qquad (f,g\in L^2,\alpha\in R^1)$$

这就说明

(4) $$(f-Pg)\cdot\overline{Pg}$$

的 Fourier 变式是 0. 函数(4)是两个 L^2-函数的乘积，因此属于 L^1. 现在，Fourier 变式的唯一性定理表明函数(4)几乎处处为零. 如果用 Pg 代替 $\overline{Pg}$ 也是真实的. 因此

(5) $$f\cdot Pg=(Pf)\cdot(Pg) \qquad (f,g\in L^2)$$

交换 f 和 g 的位置，由(5)得到

(6) $$f\cdot Pg=g\cdot Pf \qquad (f,g\in L^2)$$

现设 g 是 L^2 内的一个固定的正函数；例如，令 $g(t)=e^{-|t|}$. 定义

(7) $$\varphi(t)=\frac{(Pg)(t)}{g(t)}$$

$(Pg)(t)$仅可能几乎处处被确定；选取任一个在(7)中决定的 φ. 于是(6)变为

(8) $$Pf=\varphi\cdot f \qquad (f\in L^2)$$

如果 $f\in\hat{M}$, 则 $Pf=f$. 这就是说, $P^2=P$ 并由此得到 $\varphi^2=\varphi$, 因为

(9) $$\varphi^2\cdot g=\varphi\cdot Pg=P^2g=Pg=\varphi\cdot g$$

由于 $\varphi^2=\varphi$, 我们几乎处处有 $\varphi=0$ 或 1, 如果我们令 E 是使得 $\varphi(t)=0$ 的一切 t 的集, 因为当且仅当 $f=Pf=\varphi\cdot f$ 时, $f\in\hat{M}$, 故 $\hat{M}$ 确实由那些在 E 上几乎处处为零的 $f\in L^2$ 所组成.

因此, 我们的问题得到了下述的解答.

9.17 定理 每个可测集 $E\subset R^1$ 都对应于一个由所有在 E 上使 $\hat{f}=0$ a. e. 的 $f\in L^2$ 组成的空间 M_E, 则 M_E 是 L^2 的闭平移不变子空间. L^2 的每个闭平移不变子空间是某个 E 的 M_E, 而且 $M_A=M_B$ 当且仅当

$$m((A\cup B)\cup(B-A))=0$$

唯一性的证明是容易的；我们把它留给读者来完成.

当然, 在其它函数空间也可能提出上述问题. 此问题在 L^1 中已较详细地研究过. 已知的结果表明, 这一场合要比 L^2 的场合复杂得难以想像.

Banach 代数 L^1

9.18 定义 设 A 是一个 Banach 空间, 如果 A 中定义了一种乘法, 并满足不等式

(1) $$\|xy\|\leqslant\|x\|\|y\| \qquad (x\text{ 和 }y\in A)$$

结合律 $x(yz)=(xy)z$, 分配律

(2) $$x(y+z)=xy+xz,\ (y+z)x=yx+zx \qquad (x,y,z\in A)$$

和关系

(3) $$(\alpha x)y=x(\alpha y)=\alpha(xy)$$

而 α 是任意纯量，则称 A 为Banach 代数.

9.19 例

(*a*) 令 $A=C(X)$, X 是紧 Hausdorff 空间，赋以上确界范数，并赋以函数的通常的点态乘法: $(fg)(x)=f(x)g(x)$. 这是一个交换的 Banach 代数($fg=gf$)，有单位元(常量函数 1).

(*b*) $C_0(R^1)$是一个交换 Banach 代数，但没有单位元，即不存在元素 u 使得对所有 $f\in C_0(R^1)$有 $uf=f$.

(*c*) R^k(或任一个 Banach 空间)上所有线性算子的集，如果其算子的范数如定义 5.3 一样，并由

$$(A+B)(x)=Ax+Bx,\ ABx=A(Bx)$$

定义算子间的加法和乘法，则此算子集是一个带有单位元的非交换 Banach 代数(除非 $k=1$).

(*d*) 如果用卷积定义乘法，则 L^1 是 Banach 空间；由于

$$\|f*g\|_1\leqslant\|f\|_1\|g\|_1$$

说明满足范数不等式. 可以直接证明结合律(Fubini 定理的一个应用)，但我们也可以处理如下:

因为 $f*g$ 的 Fourier 变式是 $\hat{f}\cdot\hat{g}$，而已知映射 $f\to\hat{f}$ 是一一的，对每个 $t\in R^1$，根据复数的结合律有

$$\hat{f}(t)[\hat{g}(t)\hat{h}(t)]=[\hat{f}(t)\hat{g}(t)]\hat{h}(t)$$

由此得到

$$f*(g*h)=(f*g)*h$$

用同样的方法，立即会得到 $f*g=g*f$. 容易看到定义 9.18 中剩下的要求在 L^1 中也是成立的.

因此 L^1 是一个交换的 Banach 代数. Fourier 变式是一个 L^1 到 C_0 内的代数同构. 因此，没有 $f\in L^1$ 使得 $\hat{f}\equiv1$，所以 L^1 没有单位元.

9.20 复同态 Banach 代数上最重要的复函数是 A 到复数域内的同态. 确切地说是存在保持乘法的线性泛函, 即是说, 此函数 φ 对所有的 $x, y\in A$ 和一切纯量 α, β, 有

$$\varphi(\alpha x+\beta y)=\alpha\varphi(x)+\beta\varphi(y), \varphi(xy)=\varphi(x)\varphi(y)$$

请注意, 在这个定义中并没有提出有界性的假设. 有趣的是这种假设是多余的:

9.21 定理 如果 φ 是 Banach 代数 A 上的复同态, 则作为线性泛函, φ 的范数至多是 1.

证明 为了得出矛盾, 假设对某个 $x_0\in A$, $|\varphi(x_0)|>\|x_0\|$. 令 $\lambda=\varphi(x_0)$, $x=x_0/\lambda$. 则 $\|x\|<1$ 而 $\varphi(x)=1$.

由于 $\|x^n\|\leqslant\|x\|^n$ 和 $\|x\|<1$, 则元素

(1) $$s_n=-x-x^2-\cdots-x^n$$

是 A 的一个 Cauchy 序列. 因 A 作为 Banach 空间是完备的, 故存在 $y\in A$, 使得 $\|y-s_n\|\to 0$, 并容易看出 $x+s_n=xs_{n-1}$, 于是

(2) $$x+y=xy$$

因此, $\varphi(x)+\varphi(y)=\varphi(x)\varphi(y)$, 然而, 当 $\varphi(x)=1$ 时, 这是不可能的.

9.22 L^1 的复同态 设 φ 是 L^1 的复同态, 即 φ 是保持下述关系的线性泛函(根据定理 9.21, 其范数至多是 1).

(1) $$\varphi(f*g)=\varphi(f)\varphi(g)\qquad (f\text{ 和 }g\in L^1)$$

根据定理 6.16, 存在 $\beta\in L^\infty$, 使得

(2) $$\varphi(f)=\int_{-\infty}^{\infty}f(x)\beta(x)dm(x)\qquad (f\in L^1)$$

现在我们利用关系(1), 来看一看对于 β 我们还能说些什么. 一方面,

$$\varphi(f*g)=\int_{-\infty}^{\infty}(f*g)(x)\beta(x)dm(x)$$

$$= \int_{-\infty}^{\infty} \beta(x)dm(x) \int_{-\infty}^{\infty} f(x-y)g(y)dm(y)$$

(3)
$$= \int_{-\infty}^{\infty} g(y)dm(y) \int_{-\infty}^{\infty} f_y(x)\beta(x)dm(x)$$

$$= \int_{-\infty}^{\infty} g(y)\varphi(f_y)dm(y)$$

另一方面，

(4)
$$\varphi(f)\varphi(g) = \varphi(f) \int_{-\infty}^{\infty} g(y)\beta(y)dm(y)$$

设 φ 不恒等于零. 固定 $f \in L^1$ 使得 $\varphi(f) \neq 0$. 由于对每个 $g \in L^1$，(3)的最后一个积分都等于(4)的右边. 因此，定理 6.16的唯一性表明对几乎所有的 y

(5)
$$\varphi(f)\beta(y) = \varphi(f_y)$$

但是，$y \to f_y$ 是 R^1 到 L^1 内的连续映射(定理 9.5)和在 L^1 上 φ 是连续的. 因此(5)的右边是 y 的连续函数，并且可以假设 β 是连续的[如果需要，可以在测度为零的集上改变 $\beta(y)$ 而不影响(2)]. 如果我们用 $x+y$ 代替 y，然后在(5)中用 f_x 代替 f，得到

$$\varphi(f)\beta(x+y) = \varphi(f_{x+y}) = \varphi((f_x)_y)$$
$$= \varphi(f_x)\beta(y) = \varphi(f)\beta(x)\beta(y)$$

于是

(6)
$$\beta(x+y) = \beta(x)\beta(y) \qquad (x \text{ 和 } y \in R^1)$$

由于 β 不恒等于 0，则由(6)得到 $\beta(0) = 1$，而 β 的连续性表明，存在 $\delta > 0$ 使得

(7)
$$\int_0^{\delta} \beta(y)dy = c \neq 0$$

于是

(8)
$$c\beta(x) = \int_0^{\delta} \beta(y)\beta(x)dy = \int_0^{\delta} \beta(y+x)dy$$

$$=\int_x^{x+\delta}\beta(y)dy$$

由于β连续，最后一个积分是x的可微函数；于是(8)表明β是可微的．对于y微分(6)，然后令$y=0$；结果有

(9) $$\beta'(x)=A\beta(x),\qquad A=\beta'(0)$$

因此，$\beta(x)e^{-Ax}$的导数是0，并由于$\beta(0)=1$，得到

(10) $$\beta(x)=e^{Ax}$$

但β在R^1上有界，因此A必定是纯虚数，并得出结论：存在$t\in R^1$，使得

(11) $$\beta(x)=e^{-itx}$$

这样我们就得出 Fourier 变式．

9.23 定理 L^1上的每个复同态φ(除$\varphi=0$外)，都对应有唯一的$t\in R^1$，使得$\varphi(f)=\hat{f}(t)$．

上面已证明了t的存在性．唯一性可由观察得出：如果$t\neq s$，则存在$f\in L^1$使得$\hat{f}(t)\neq\hat{f}(s)$；把$f(x)$当作$e^{-|x|}$的一个适当的平移．

习　题

1　设$f\in L^1$，$f>0$．证明对每个$y\neq 0$，$|\hat{f}(y)|<\hat{f}(0)$．

2　计算一个区间的特征函数的 Fourier 变式．对$n=1, 2, 3, \cdots$，令g_n是$[-n, n]$的特征函数，h是$[-1, 1]$的特征函数，并明确地计算g_n*h．(图象是逐段线性的．)证明g_n*h是函数$f_n\in L^1$的 Fourier 变式；除一个常数因子外

$$f_n(x)=\frac{\sin x\cdot\sin nx}{x^2}$$

证明$\|f_n\|_1\to\infty$，并得到：映射$f\to\hat{f}$把L^1映到C_0的一个真子集内，而这个映射的值域在C_0中是稠密的．

3　求极限

$$\lim_{A\to\infty}\int_{-A}^{A}\frac{\sin\lambda t}{t}e^{itx}dt \qquad (-\infty<x<\infty)$$

其中 λ 是一个正常数.

4 给出一个 $f\in L^2$ 的例子，使得 $f\notin L^1$ 但 $\hat{f}\in L^1$. 这在什么情况下才会发生?

5 如果 $f\in L^1$ 和 $\int|t\hat{f}(t)|dm(t)<\infty$，证明 f 几乎处处同一个可微函数一致，而此函数的导数是

$$i\int_{-\infty}^{\infty}t\hat{f}(t)e^{ixt}dm(t)$$

6 设 $f\in L^1$，f 几乎处处可微，且 $f'\in L^1$. 由此能得出 f' 的 Fourier 变式就是 $ti\hat{f}(t)$吗?

7 令 S 是 R^1 中具有下述性质的所有函数 f 的类: f 是无限次可微的，并且对 m 和 $n=0,1,2,\cdots$，存在数 $A_{mn}(f)<\infty$，使得

$$|x^nD^mf(x)|\leqslant A_{mn}(f) \qquad (x\in R^1)$$

D 是通常的微分算子.

证明 Fourier 变式把 S 映到 S 上.

找出 S 的元素的例子.

8 如果 p 和 q 是共轭指数，$f\in L^p$，$g\in L^q$，而 $h=f*g$，证明 h 是一致连续的. 如果还有 $1<p<\infty$，则 $h\in C_0$; 说明对某些 $f\in L^1$，$g\in L^\infty$ 这是不对的.

9 设 $1\leqslant p<\infty$，$f\in L^p$ 和

$$g(x)=\int_x^{x+1}f(t)dt$$

证明 $g\in C_0$. 如果 $f\in L^\infty$，关于 g 能有些什么结果?

10 令 C^∞ 是 R^1 上一切无限次可微复函数的类，C_c^∞ 是由一切支集是紧的 $g\in C^\infty$ 所组成. 证明 C_c^∞ 不能单独由 0 组成.

令 L^1_{loc} 是一切局部地属于 L^1 的 f 的类; 即是说，$f\in L^1_{loc}$，如果 f 是可测的，且对每个有界区间 I，$\int_I|f|<\infty$.

如果 $f\in L^1_{loc}$ 和 $g\in C_c^\infty$，证明 $f*g\in C^\infty$.

证明存在 C_c^∞ 中的序列 $\{g_n\}$，当 $n\to\infty$时，对每个 $f\in L^1$ 有

$$\|f*g-f\|_1\to 0$$

(与定理 9.10 比较.) 证明 $\{g_n\}$ 也可以这样选取，使得对每个 $f\in L^1_{loc}$ 几乎处

处有$(f*g_n)(x)\to f(x)$；事实上，对适当的$\{g_n\}$，在每一个使f等于它的不定积分的导数的点x都能保证收敛.

证明如果$f\in L^1$，当$\lambda\to 0$时，纵然h_λ没有紧的支集(h_λ定义在9.7节)，还是几乎处处有$(f*h_\lambda)(x)\to f(x)$，而且$f*h_\lambda\in C^\infty$.

11 求关于f和$\hat{f}$共同具有或单独具有的条件，以保证下述形式论证的正确性：如果

$$\varphi(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)e^{-itx}dx$$

和

$$F(x)=\sum_{k=-\infty}^{\infty}f(x+2k\pi)$$

则$F(x)$是周期函数，周期为2π，F的第n个Fourier系数是$\varphi(n)$，因此$F(x)=\sum\varphi(n)e^{inx}$. 特别，

$$\sum_{k=-\infty}^{\infty}f(2k\pi)=\sum_{n=-\infty}^{\infty}\varphi(n)$$

更一般，如果$\alpha>0,\beta>0,\alpha\beta=2\pi$，有

$$(*)\qquad \sum_{k=-\infty}^{\infty}f(k\beta)=\alpha\sum_{n=-\infty}^{\infty}\varphi(\alpha)$$

当$\alpha\to 0$时，等式(*)右边的极限有什么可说的(当然是对"好的"函数)？这与反演定理是一致的吗？

[(*)是著名的Poisson求和公式.]

12 在习题11中取$f(x)=e^{-|x|}$并导出恒等式

$$\frac{e^{2\pi\alpha}+1}{e^{2\pi\alpha}-1}=\frac{1}{\pi}\sum_{n=-\infty}^{\infty}\frac{\alpha}{\alpha^2+n^2}$$

13 如果$0<c<\infty$，定义$f_c(x)=\exp(-cx^2)$

(a) 计算$\hat{f}_c$. 提示：如果$\varphi=\hat{f}_c$，由分部积分得到

$$2c\varphi'(t)+t\varphi(t)=0.$$

(b) 证明存在一个(且仅一个)c，使得$\hat{f}_c=f_c$.

(c) 证明$f_a*f_b=\gamma f_c$；用a和b的显式表示γ和c.

(d) 在习题11中，取$f=f_c$. 其结果是什么样一个恒等式？

14 对$f\in L^1(R^k)$，Fourier变式能用

$$\hat{f}(y)=\int_{R^k} f(x)e^{-i(x\cdot y}dm_k(x) \quad (y\in R^k)$$

来定义. 这里的 $x\cdot y=\sum\xi_i\eta_i$, 如果 $x=(\xi_1,\cdots,\xi_n)$, $y=(\eta_1,\cdots,\eta_n)$. m_k 是 R^k 上的 Lebesgue测度, 为了方便将它除以 $(2\pi)^{k/2}$. 证明反演定理和本文中的 Plancherel 定理, 以及与定理 9.23 类似的命题.

15 如果 $f\in L^1(R^k)$, A 是 R^k 上的线性算子并且 $g(x)=f(Ax)$, $\hat{g}$ 与 $\hat{f}$ 有何联系? 如果 f 是旋转不变的, 即是说, 如果 $f(x)$ 仅依赖于从原点到 x 的欧氏距离, 证明这对 $\hat{f}$ 同样是对的.

16 R^k 上函数 f 的 Laplace 算子是

$$\Delta f=\sum_{j=1}^{k}\frac{\partial^2 f}{\partial x_j^2}$$

这里假设偏导数是存在的. 如果 $g=\Delta f$ 且满足一切必要的可积条件, 则 $\hat{f}$ 和 $\hat{g}$ 之间有什么关系? Laplace 算子同平移是可交换的. 证明它同旋转也是可交换的, 即当 f 有连续二阶导数, A 是 R^k 的一个旋转时, 有

$$\Delta(f\circ A)=\Delta f\circ A$$

(说明在 f 有紧支集的附加条件下证明此点就足够了.)

17 证明 R^1 的每个 Lebesgue 可测特征标是连续的. 对 R^k 同样证明之. (改写定理 9.23 的部分证明.)同第八章习题 24 比较.

第十章　全纯函数的初等性质

复微分

我们现在来研究定义在复平面的子集内的复函数. 为了方便起见，本书以后将采用一些标准符号.

10.1　定义　若 $r>0$，且 a 为一复数，

(1)　　$$D(a;r)=\{z:|z-a|<r\}$$

是中心在 a，半径为 r 的开圆盘. $\bar{D}(a;r)$ 是 $D(a;r)$ 的闭包，且

(2)　　$$D'(a;r)=\{z:0<|z-a|<r\}$$

是中心在 a，半径为 r 的去心圆盘.

拓扑空间 X 内的集 E 称为不连通的，如果 E 是两个非空集 A 及 B 的并，且

(3)　　$$\bar{A}\cap B=\varnothing=A\cap\bar{B}$$

若 A 和 B 如上述，且 V 和 W 分别为 $\bar{A}$ 及 $\bar{B}$ 的余集，便有 $A\subset W$ 和 $B\subset V$，因此

(4)　　$$E\subset V\cup W,E\cap V\neq\varnothing,E\cap W\neq\varnothing,E\cap V\cap W=\varnothing$$

相反，如果开集 V 和 W 存在，使(4)成立，取 $A=E\cap W$，$B=E\cap V$，容易看出 E 是不连通的.

如果 E 是闭的且不连通，则(3)表明 E 是两个不相交的非空闭集的并；因若 $\bar{A}\subset A\cup B$ 且 $\bar{A}\cap B=\varnothing$，则 $\bar{A}=A$.

如果 E 是开的且不连通，则(4)表明 E 是两个不相交的非空开集，即 $E\cap V$ 及 $E\cap W$ 的并.

每一个由单独的一点组成的集明显是连通的. 若 $x\in E$，包含 x 的 E 的连通子集的集族 Φ_x 因此非空. 容易看出 Φ_x 的所有元素的

并是连通的，并且它是E的一个最大连通子集. 这些集称为E的分支. 因此E的任何两个分支是不相交的. 并且E是它的分支的并.

区域，是指复平面的一个非空连通开子集. 因为在这平面内每一个开集Ω都是圆盘的并，又因为所有圆盘是连通的，故Ω的每个分支都是开集. 这样，每个平面开集都是不相交区域的并. 从现在起，字母Ω将表示一个平面开集.

10.2 定义 假设f为定义在Ω内的复函数. 如果$z_0\in\Omega$，且

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0} \tag{1}$$

存在，我们记这极限为$f'(z_0)$，并称它为f在z_0的导数. 如果对每一个$z_0\in\Omega$，$f'(z_0)$都存在，我们就称f在Ω内是全纯的(或解析的). 所有在Ω内全纯的函数类将记为$H(\Omega)$.

很明显，若对每个$\varepsilon>0$，对应有$\delta>0$，使得对所有$z\in D'(z_0;\delta)$

$$\left|\frac{f(z)-f(z_0)}{z-z_0}-f'(z_0)\right|<\varepsilon \tag{2}$$

则$f'(z_0)$存在. 这样，$f'(z_0)$是作为复数的商的极限而得出的一个复数. 注意f是Ω到R^2内的映射，并且定义8.22将这样的映射对应于另一类型的导数，即R^2上的一个线性算子. 在现在的情况下，若(2)满足，这个线性算子就是乘以$f'(z_0)$(把R^2看作复数域)，我们留下它给读者验证.

10.3 评注 如果$f\in H(\Omega)$和$g\in H(\Omega)$，则应用通常的微分法则，也有$f+g\in H(\Omega)$和$fg\in H(\Omega)$，因此$H(\Omega)$是一个环.

更有趣的是全纯函数叠加仍是全纯函数：若$f\in H(\Omega)$，$f(\Omega)\in\Omega_1$，$g\in H(\Omega_1)$，且$h=g\circ f$，则$h\in H(\Omega)$，并且h'可由链式法则计算

$$h'(z_0)=g'(f(z_0))f'(z_0)\qquad(z_0\in\Omega) \tag{1}$$

为证明这一点，固定 $z_0\in\Omega$，并令 $w_0=f(z_0)$，则

$$f(z)-f(z_0)=[f'(z_0)+\varepsilon(z)](z-z_0) \tag{2}$$

$$g(w)-g(w_0)=[g'(w_0)+\eta(w)](w-w_0) \tag{3}$$

这里当 $z\to z_0$ 时，$\varepsilon(z)\to 0$，且当 $w\to w_0$ 时，$\eta(w)\to 0$，令 $w=f(z)$，并把(2)代入(3)：若 $z\neq z_0$，

$$\frac{h(z)-h(z_0)}{z-z_0} \tag{4}$$

$$=[g'(f(z_0))+\eta(f(z))][f'(z_0)+\varepsilon(z)]$$

f的可微性保证 f 在 z_0 连续．因此(1)由(4)得出．

10.4 例 对 $n=0,1,2,\cdots$，z^n在整个平面全纯，并且 z 的每一个多项式也在整个平面全纯．容易直接验证 $1/z$ 在$\{z: z\neq 0\}$全纯．因此，在链式法则中取 $g(w)=1/w$，我们看到，如果 f_1及 f_2均在 $H(\Omega)$中，而且 Ω_0为 Ω 的开子集，f_2在 Ω_0内无零点，则

$$f_1/f_2\in H(\Omega_0).$$

在整个平面全纯的函数(这种函数称为整函数)的另外一个例子是在引言中定义的指数函数．事实上，我们在那里看到在定义 10.2 意义下，exp 是处处可微的，并且对每个复数 z，$\exp'(z)=\exp(z)$．

10.5 幂级数 对幂级数的理论我们仅假定下面这一点是已知的．就是对每一个幂级数

$$\sum_{n=0}^{\infty}c_n(z-a)^n \tag{1}$$

对应有一个数 $R\in[0,\infty]$，使得对每一个 $r<R$，此级数在$\bar{D}(a;r)$内绝对一致收敛，而当 $z\notin\bar{D}(a;R)$时，这个级数发散．这个“收敛半径”R 可由根式判别法给出

$$\frac{1}{R}=\limsup_{n\to\infty}|c_n|^{1/n}$$

如果对每个圆盘 $D(a;r)\subset\Omega$，对应有一个级数(1)，这级数对所有 $z\in D(a;r)$ 均收敛于 $f(z)$，我们称在 Ω 内定义的函数 f 在 Ω 内可表示为幂级数.

10.6 定理 如果 f 在 Ω 内可表示为幂级数，则 $f\in H(\Omega)$，而且 f' 在 Ω 内也可表示为幂级数. 事实上，若对 $z\in D(a;r)$

$$f(z)=\sum_{n=0}^{\infty}c_n(z-a)^n \tag{1}$$

则对这些 z 我们同时有

$$f'(z)=\sum_{n=1}^{\infty}nc_n(z-a)^{n-1} \tag{2}$$

证明 若级数(1)在 $D(a;r)$ 内收敛，则根式判别法表明，级数(2)也在 $D(a;r)$ 内收敛. 不失一般性，取 $a=0$，记级数(2)的和为 $g(z)$. 固定 $w\in D(a;r)$，并选取 ρ，使 $|w|<\rho<r$. 若 $z\neq w$，我们有

$$\frac{f(z)-f(w)}{z-w}-g(w)=\sum_{n=1}^{\infty}c_n\left[\frac{z^n-w^n}{z-w}-nw^{n-1}\right] \tag{3}$$

方括号内的表达式当 $n=1$ 时为 0，当 $n\geqslant 2$ 时为

$$(z-w)\sum_{k=1}^{n-1}kw^{k-1}z^{n-k-1} \tag{4}$$

如果 $|z|<\rho$，和式(4)的绝对值便小于

$$\frac{n(n-1)}{2}\rho^{n-2} \tag{5}$$

于是

$$\left|\frac{f(z)-f(w)}{z-w}-g(w)\right|\leqslant|z-w|\sum_{n=2}^{\infty}n^2|c_n|\rho^{n-2} \tag{6}$$

因为 $\rho<r$，后面的级数收敛，所以(6)式左边当 $z\to w$ 时趋于零.

这说明$f'(w)=g(w)$，并且完成了证明.

系 因为f'同样满足f的假设条件，这定理能应用于f'. 这就得出f具有所有阶的导数. 每个导数在Ω内均可表示为幂级数，并且若(1)成立，则

$$f^{(k)}(z)=\sum_{n=k}^{\infty}n(n-1)\cdots(n-k+1)c_n(z-a)^{n-k} \tag{7}$$

由(1)推出

$$k!c_k=f^{(k)}(a) \qquad (k=0,1,2,\cdots)$$

因此，对每个$a\in\Omega$，有唯一的序列$\{c_n\}$使(1)成立.

我们现在叙述一个构造可表示为幂级数的函数的方法. 它的特殊情形在下一步中是重要的.

10.7 定理 假设μ为可测空间X上复(有限)测度，φ为X上复可测函数，Ω为在平面上与$\varphi(X)$不相交的一个开集，且

$$f(z)=\int_X\frac{d\mu(\xi)}{\varphi(\xi)-z} \qquad (z\in\Omega) \tag{1}$$

则f在Ω内可表示为幂级数.

证明 假设$D(a;r)\subset\Omega$，由于对每个$z\in D(a;r)$及每个$\xi\in X$，

$$\left|\frac{z-a}{\varphi(\xi)-a}\right|\leqslant\left|\frac{z-a}{r}\right|<1 \tag{2}$$

成立、故几何级数

$$\sum_{n=0}^{\infty}\frac{(z-a)^n}{(\varphi(\xi)-a)^{n+1}}=\frac{1}{\varphi(\xi)-z} \tag{3}$$

对每个固定的$z\in D(a;r)$在X上一致收敛. 因此级数(3)可代入(1)，且$f(z)$可通过交换求和及积分的次序来计算. 这样，便得出

$$f(z)=\sum_{n=0}^{\infty}c_n(z-a)^n \qquad (z\in D(a;r)) \tag{4}$$

此处

$$c_n=\int_X \frac{d\mu(\zeta)}{(\varphi(\zeta)-a)^{n+1}} \qquad (n=0,1,2,\cdots) \tag{5}$$

注　级数(4)在 $D(a;r)$ 内的收敛性是这个证明的一个推论. 我们也可以由(5)推出这个结论,因为(5)表明

$$|c_n|\leqslant\frac{|\mu|(X)}{r^{n+1}} \qquad (n=0,1,2,\cdots) \tag{6}$$

沿路径的积分

在这一章中，我们的首要目标是定理 10.6 的逆定理：每个 $f\in H(\Omega)$ 在 Ω 内可表示为幂级数. 最快的途径是通过 Cauchy定理. 它导出一个重要的全纯函数积分表达式. 在这节里，将展示所需要的积分理论. 我们尽可能使它简单，并且仅仅把它看作研究全纯函数性质的一个有用工具.

10.8　定义　如果 X 是一个拓扑空间. X 中的一条曲线是指紧区间 $[\alpha,\beta]\subset R^1$ 到 X 内的连续映射 γ;这里 $\alpha<\beta$,我们称 $[\alpha,\beta]$ 为 γ 的参数区间,且记 γ 的值域为 γ^*. 就是说,γ 是一个映射,而 γ^* 则是所有点 $\gamma(t)$ 的集,其中 $\alpha\leqslant t\leqslant\beta$.

若 γ 的始点 $\gamma(\alpha)$ 和它的终点 $\gamma(\beta)$ 重合,我们称 γ 为闭曲线.

路径是指平面内一条逐段连续可微的曲线. 更明白地说，一条具有参数区间 $[\alpha,\beta]$ 的路径就是 $[\alpha,\beta]$ 上的一个满足下列条件的连续复函数 γ:存在有限个点 s_j，使得 $\alpha=s_0<s_1<\cdots<s_n=\beta$,且 γ 限制在每个区间 $[s_{j-1},s_j]$ 时,在 $[s_{j-1},s_j]$ 上有连续导数,而 γ 在点 $s_1,\cdots,s_{n-1}$ 的左、右导数可以不同.

闭路径是一条闭曲线,它同时也是一条路径.

现假设 γ 是一条路径,f 为 γ^* 上的连续函数,f 沿 γ 的积分定义为在 γ 的参数区间 $[\alpha,\beta]$ 上的积分

(1) $$\int_\gamma f(z)dz=\int_\alpha^\beta f(\gamma(t))\gamma'(t)dt$$

设 φ 为区间$[\alpha_1,\beta_1]$到$[\alpha,\beta]$上的连续可微一一映射，使得 $\varphi(\alpha_1)=\alpha,\varphi(\beta_1)=\beta$，并令$\gamma_1=\gamma\cdot\varphi$. 则$\gamma_1$是具有参数区间$[\alpha_1,\beta_1]$的路径，$f$ 沿 γ_1的积分为

$$\begin{aligned}&\int_{\alpha_1}^{\beta_1} f(\gamma_1(t))\gamma_1'(t)dt\\&=\int_{\alpha_1}^{\beta_1} f(\gamma(\varphi(t)))\gamma'(\varphi(t))\varphi'(t)dt\\&=\int_\alpha^\beta f(\gamma(s))\gamma'(s)ds\end{aligned}$$

于是我们"改变参变量"并不改变这积分：

(2) $$\int_{\gamma_1} f(z)dz=\int_\gamma f(z)dz$$

当(2)对一对路径 γ 及 γ_1 (且对所有 f)成立时，我们认为 γ 和 γ_1 是等价的.

可以用等价的路径代替一条路径，也就是说，可以按我们的愿望选择参数区间，这一点对我们是会方便的. 例如，如果 γ_1的终点重合于 γ_2的起点，我们就可以设置它们的参数区间，使得 γ_1及 γ_2联成一条路径 γ，对每个在 $\gamma^*=\gamma_1^*\cup\gamma_2^*$上的连续函数 f，具有性质

(3) $$\int_\gamma f=\int_{\gamma_1} f+\int_{\gamma_2} f$$

假设$[0,1]$是路径 γ 的参数区间，且 $\gamma_1(t)=\gamma(1-t)$，$0\leqslant t\leqslant 1$. 我们称 γ_1为 γ 的反向路径. 理由是，对任何$\gamma_1^*=\gamma^*$上连续的函数 f，我们有

$$\begin{aligned}&\int_0^1 f(\gamma_1(t))\gamma_1'(t)dt\\&=-\int_0^1 f(\gamma(1-t))\gamma'(1-t)dt\end{aligned}$$

$$=-\int_0^1 f(\gamma(s))\gamma'(s)ds$$

于是

(4) $$\int_{\gamma_1} f=-\int_{\gamma} f \ .$$

由(1)我们得到不等式

(5) $$\left|\int_{\gamma} f(z)dz\right| \leqslant \|f\|_{\infty}\int_{\alpha}^{\beta} |\gamma'(t)|dt$$

这里$\|f\|_{\infty}$是$|f|$在γ^*上的最大值，并且(5)式后面的积分(由定义)是γ的长度.

10.9 特殊情形

(*a*) 如果a是一个复数且$r>0$，则由

(1) $$\gamma(t)=a+re^{it} \qquad (0\leqslant t\leqslant 2\pi)$$

定义的路径称为中心在a，半径为r的正向圆周. 我们有

(2) $$\int_{\gamma} f(z)dz=ir\int_0^{2\pi} f(a+re^{i\theta})e^{i\theta}d\theta$$

且γ的长为$2\pi r$，如所期望的一样.

(*b*)如果a和b是复数，由

(3) $$\gamma(t)=a+(b-a)t \qquad (0\leqslant t\leqslant 1)$$

给出的路径γ是有向区间$[a,b]$，它的长为$|b-a|$，且

(4) $$\int_{[a,b]} f(z)dz=(b-a)\int_0^1 f[a+(b-a)t]dt$$

如果

(5) $$\gamma_1(t)=\frac{a(\beta-t)+b(t-\alpha)}{\beta-\alpha} \qquad (\alpha\leqslant t\leqslant\beta)$$

便得到一条等价的路径. 我们仍记为$[a,b]$. $[a,b]$的反向路径是$[b,a]$.

(*c*) 设$\{a,b,c\}$是复数的三元序组，设

$$\triangle=\triangle(a,b,c)$$

是顶点在 a, b, c 的三角形(△为包含 a, b 及 c 的最小凸集)，且对任意在△边界上连续的函数 f，定义

$$\int_{\partial\triangle} f=\int_{[a,b]} f+\int_{[b,c]} f+\int_{[c,a]} f \tag{6}$$

我们可以认为(6)是它左边式子的定义，或者我们也可以把 $\partial\triangle$ 看作为如定义 10.8 所描述的，联结 $[a,b]$、$[b,c]$到$[c,a]$而得的路径. 这时容易证明公式(6)是正确的.

如果把 $\{a,b,c\}$ 轮换，由(6)我们看出对(6)式的左边并无影响. 若 $\{a,b,c\}$ 被 $\{a,c,b\}$ 代替，则(6)式左边改变符号.

我们现在给出一个定理，它在函数论中起着非常重要的作用.

10.10 定理 设 γ 是一条闭路径，设 Ω 为 γ^* 的余集(相对于平面)，并定义

$$\mathrm{Ind}_\gamma(z)=\frac{1}{2\pi i}\int_\gamma \frac{d\xi}{\xi-z} \qquad (z\in\Omega) \tag{1}$$

则 Ind_γ 是 Ω 上的一个整数值函数. 它在 Ω 的每一分支内是常数，并且在 Ω 的无界分支内为零.

我们称 $\mathrm{Ind}_\gamma(z)$ 为 z 关于 γ 的指数. 注意 γ^* 是紧的，因此 γ^* 在一有界圆盘 D 内. D 的余集 D^c 是连通的. 这样，D^c 就在 Ω 的某一分支内. 这表明 Ω 确实有一个无界分支.

证明 令 $[\alpha,\beta]$ 为 γ 的参数区间，固定 $z\in\Omega$，则

$$\mathrm{Ind}_\gamma(z)=\frac{1}{2\pi i}\int_\alpha^\beta \frac{\gamma'(s)}{\gamma(s)-z}ds \tag{2}$$

由于当且仅当 $e^w=1$ 时，$w/2\pi i$ 才是整数，定理的第一个论断，就是 $\mathrm{Ind}_\gamma(z)$ 是整数，等价于断言 $\varphi(\beta)=1$. 这里

$$\varphi(t)=\exp\left\{\int_\alpha^t \frac{\gamma'(s)}{\gamma(s)-z}ds\right\} \qquad (\alpha\leqslant t\leqslant\beta) \tag{3}$$

(3)的微分表明，除了可能在使 γ 不可微的有限集 S 外，都有

$$\frac{\varphi'(t)}{\varphi(t)}=\frac{\gamma'(t)}{\gamma(t)-z} \tag{4}$$

因此$\varphi/(\gamma-z)$是$[\alpha,\beta]$上的连续函数，它在$[\alpha,\beta]-S$内的导数为零．因为S有限，$\varphi/(\gamma-z)$在$[\alpha,\beta]$上是常数．又因为$\varphi(\alpha)=1$，我们得出

$$\varphi(t)=\frac{\gamma(t)-z}{\gamma(\alpha)-z} \qquad (\alpha\leqslant t\leqslant\beta) \tag{5}$$

我们现在利用γ为闭路径，也就是$\gamma(\beta)=\gamma(\alpha)$的假设．(5)表示$\varphi(\beta)=1$，而这正如我们上面已看到的，蕴涵着$\mathrm{Ind}_\gamma(z)$是个整数．

由定理10.7,(1)表明$\mathrm{Ind}_\gamma\in H(\Omega)$．由于在连续映射下连通集的像是连通的([26],定理4.22)，又由于Ind_γ是一个整数值函数，故Ind_γ在Ω的每一个分支上必须是常数．

最后,(2)表明当$|z|$充分大时，$|\mathrm{Ind}_\gamma(z)|<1$，这就推出在$\Omega$的无界分支上$\mathrm{Ind}_\gamma(z)=0$．

评注 如果记(3)式中的积分为$\lambda(t)$，上述的证明说明$2\pi\mathrm{Ind}_\gamma(z)$是当$t$由$\alpha$跑到$\beta$时，$\lambda(t)$的虚部的纯增量，并且与$\gamma(t)-z$幅角的纯增量相同．(我们没有定义"幅角"，将来也不需要它．)如果我们将这增量除以2π，我们就得到"γ环绕z的次数"．这说明为什么常常把指数称为"环绕数"．上述证明的一个优点就是它确立了指数的主要性质，而并不涉及复数的幅角．

10.11 定理 如果γ是中心在a，半径为r的正向圆周，则

$$\mathrm{Ind}_\gamma(z)=\begin{cases}1 & \text{若}|z-a|<r\\ 0 & \text{若}|z-a|>r\end{cases}$$

证明 我们如10.9(a)一节那样取γ，由定理10.10，只要计算出$\mathrm{Ind}_\gamma(a)$就够了．而10.9(2)表明它等于

$$\frac{1}{2\pi i}\int_\gamma\frac{dz}{z-a}=\frac{r}{2\pi}\int_0^{2\pi}(re^{it})^{-1}e^{it}dt=1$$

局部 Cauchy 定理

Cauchy 定理有好几种形式. 它们全都断言，当 γ 为 Ω 内的闭路径或圈，并且当 γ 及 Ω 满足某些拓扑条件时，每个 $f\in H(\Omega)$ 在 γ 上的积分是零. 我们将首先引出它的简单的局部的形式(定理 10.14)，它对许多应用来说是很够的了. 至于更一般的整体的形式，将在稍晚一些时候建立.

10.12 定理 假设 $F\in H(\Omega)$，且 F' 在 Ω 内连续. 则对 Ω 内的每一条闭路径 γ

$$\int_\gamma F'(z)dz=0$$

证明 如果 $[\alpha,\beta]$ 为 γ 的参数区间，因为 $\gamma(\beta)=\gamma(\alpha)$，微积分学基本定理表明

$$\int_\gamma F'(z)dz=\int_\alpha^\beta F'(\gamma(t))\gamma'(t)dt$$
$$=F(\gamma(\beta))-F(\gamma(\alpha))=0$$

系 因为对所有整数 $n\neq -1$，z^n 是 $z^{n+1}/n+1$ 的导数，当 $n=0,1,2,\cdots$ 时，对每一闭路径 γ；当 $n=-2,-3,-4,\cdots$，$0\notin\gamma^*$ 时，对每一闭路径 γ，我们都有

$$\int_\gamma z^n dz=0$$

$n=-1$ 的情况，已在定理 10.10 讨论.

10.13 对三角形的 Cauchy 定理 假设 $\triangle$ 是平面开集 Ω 内的一个闭三角形，$p\in\Omega$，f 在 Ω 上连续，且 $f\in H(\Omega-\{p\})$. 则

(1)
$$\int_{\partial\triangle} f(z)dz=0$$

$\partial\triangle$ 的定义按照第 10.9 节的(c). 稍后一些将看到，我们的假设确实蕴涵着 $f\in H(\Omega)$. 也就是，所排除的点 p 并不是真正要排

除的. 然而，定理的上述形式，在 Cauchy 公式的证明中将是有用的.

证明 我们首先假设$p\in\triangle$. 设a,b,c是$\triangle$的顶点. 设a',b',c'分别是$[b,c],[c,a]$和$[a,b]$的中点. 考虑由三元序组

(2) $$\{a,c',b'\},\{b,a',c'\},\{c,b',a'\},\{a',b',c'\}$$

构成的四个三角形$\triangle^j$. 若J为积分(1)的值，由 10.9(6) 得

(3) $$J=\sum_{j=1}^{4}\int_{\partial\triangle^j}f(z)dz$$

(3)式右边积分中至少有一个绝对值至少是$|J/4|$. 称这个对应的三角形为$\triangle_1$. 用$\triangle_1$代替$\triangle$重复这论证，并继续这一过程，就得出一三角形$\triangle_n$的序列，使得$\triangle\supset\triangle_1\supset\triangle_2\supset\cdots$，且若$L$为$\partial\triangle$的长，则$\partial\triangle_n$的长为$2^{-n}L$，并使得

(4) $$|J|\leqslant 4^n\left|\int_{\partial\triangle_n}f(z)dz\right| \qquad (n=1,2,3,\cdots)$$

三角形$\triangle_n$有(唯一)公共点z_0. 由于$\triangle$是紧的，故$z_0\in\triangle$. 于是f在z_0可微.

设给定$\varepsilon>0$. 则存在$r>0$，使得$|z-z_0|<r$时，

(5) $$\begin{aligned}&|f(z)-f(z_0)-f'(z_0)(z-z_0)|\\&\leqslant\varepsilon|z-z_0|\end{aligned}$$

并且存在n，使得对所有 $z\in\triangle_n$，$|z-z_0|<r$. 对这个n和所有$z\in\triangle_n$，我们同时有$|z-z_0|\leqslant 2^{-n}L$. 由定理 10.12的系

(6) $$\begin{aligned}&\int_{\partial\triangle_n}f(z)dz\\&=\int_{\partial\triangle_n}[f(z)-f(z_0)-f'(z_0)(z-z_0)]dz\end{aligned}$$

于是(5)推出

(7) $$\left|\int_{\partial\triangle_n}f(z)dz\right|\leqslant\varepsilon(2^{-n}L)^2$$

现在(4)式指出$|J|\leqslant\varepsilon L^2$. 因此，若$p\notin\triangle$，则$J=0$.

其次假设p为$\triangle$的一个顶点，例如$p=a$. 若a、b、c共线，则对任何连续函数f，(1)是平凡的. 如若不然，选取点$x\in[a,b]$，$y\in[a,c]$，使x，y均接近a，并且注意f沿$\partial\triangle$的积分是沿三角形$\{a,x,y\}$，$\{x,b,y\}$，$\{b,c,y\}$边界积分之和. 由于后两三角形不包含p，故后面两积分为零. 因此沿$\partial\triangle$的积分是沿$[a,x]$，$[x,y]$，$[y,a]$积分之和. 又由于这些区间可以任意地短及f在$\triangle$上有界，我们再次得到(1).

最后，若p为$\triangle$的任意一点. 应用上述结果于$\{a,b,p\}$，$\{b,c,p\}$及$\{c,a,p\}$，就完成了证明.

10.14 在凸集内的 Cauchy 定理 假设Ω是一个凸开集，$p\in\Omega$，f在Ω上连续，且$f\in H(\Omega-\{p\})$，则对某些$F\in H(\Omega)$，有$f=F'$，因此，对Ω内每一闭路径γ，

$$\int_\gamma f(z)\,dz=0 \tag{1}$$

证明 固定$a\in\Omega$. 由于Ω是凸集，对每个$z\in\Omega$，Ω包含由a到z的直线区间. 于是，我们能定义

$$F(z)=\int_{[a,z]} f(\xi)\,d\xi \qquad (z\in\Omega) \tag{2}$$

对任意z及$z_0\in\Omega$，顶点在a，z_0，z的三角形位于Ω内，因此，由10.13，f沿$[z_0,z]$的积分是$F(z)-F(z_0)$. 固定z_0，若$z\neq z_0$，我们就得出

$$\begin{aligned}\frac{F(z)-F(z_0)}{z-z_0}-f(z_0)&\\ =\frac{1}{z-z_0}\int_{[z_0,z]}[f(\xi)-f(z_0)]\,d\xi&\end{aligned} \tag{3}$$

给定$\varepsilon>0$，f在z_0的连续性表明，存在$\delta>0$，使得当$|\xi-z_0|<\delta$时，有$|f(\xi)-f(z_0)|<\varepsilon$；因此，当$|z-z_0|<\delta$时，(3)式左边的绝

对值小于 ε. 这证明了 $f=F'$. 特别有 $F\in H(\Omega)$. 由定理10.12就可得出(1).

10.15 凸集内的 Cauchy 公式 假设 γ 为凸开集 Ω 内一条闭路径，且 $f\in H(\Omega)$. 如果 $z\in\Omega$ 且 $z\notin\gamma^*$，则

$$f(z)\cdot \mathrm{Ind}_\gamma(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi-z}d\xi \tag{1}$$

当然，最有兴趣的情形是 $\mathrm{Ind}_\gamma(z)=1$.

证明 固定 z，使上述条件成立，并且定义

$$g(\xi)=\begin{cases}\dfrac{f(\xi)-f(z)}{\xi-z} & \text{若 } \xi\in\Omega,\ \xi\neq z\\ f'(z) & \text{若 } \xi=z\end{cases} \tag{2}$$

于是 g 满足定理 10.14 的假设. 因此

$$\frac{1}{2\pi i}\int_\gamma g(\xi)d\xi=0 \tag{3}$$

如果我们把(2)代入(3)，便得到(1).

如果我们取 γ 为一圆周，关于全纯函数能用幂级数表示的这个定理是定理 10.15 容易得到的结论:

10.16 定理 对平面内任意开集 Ω，每一个 $f\in H(\Omega)$ 都在 Ω 内可表示为幂级数.

证明 假设 $f\in H(\Omega)$ 且 $D(a;R)\subset\Omega$. 若 γ 为中心在 a，半径为 $r<R$ 的正向圆周，$D(a;R)$ 的凸性允许我们应用定理 10.15; 由定理 10.11，我们得出

$$f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{\xi-z}d\xi \qquad (z\in D(a;r)) \tag{1}$$

但现在我们可以对 $X=[0,2\pi]$，$\varphi=\gamma$，

$$d\mu(t)=f(\gamma(t))\gamma'(t)dt$$

来应用定理 10.7，并得出结论: 存在序列 $\{c_n\}$，使得

(2) $$f(z)=\sum_{n=0}^{\infty}c_n(z-a)^n \qquad (z\in D(a;r))$$

$\{c_n\}$的唯一性（见定理10.6之系）表明，对每一个 $r<R$（当 a 固定时）得到相同的幂级数. 因此，对每个 $z\in D(a;R)$，表示式(2)是正确的，这就完成了证明.

系 若 $f\in H(\Omega)$，则 $f'\in H(\Omega)$.

证明 结合定理 10.6 及10.16.

Cauchy 定理有一个有用的逆定理：

10.17 Morera 定理 假设 f 是开集 Ω 内的连续复函数，若对每个闭三角形 $\triangle\subset\Omega$，都有

$$\int_{\partial\triangle}f(z)dz=0$$

则 $f\in H(\Omega)$.

证明 设 V 为 Ω 内的凸开集，象在定理 10.14 证明中一样，我们能构造 $F\in H(V)$，使得 $F'=f$. 由于全纯函数的导数是全纯函数（定理 10.16），故对每个凸开集 $V\subset\Omega$，都有 $f\in H(\Omega)$. 因此，$f\in H(\Omega)$.

幂级数表示

每个全纯函数在局部范围内都是一个收敛的幂级数之和这一事实，有大量有趣的推论. 在本节，将展示其中一小部分.

10.18 定理 假设 Ω 是一区域，$f\in H(\Omega)$，且

(1) $$Z(f)=\{a\in\Omega: f(a)=0\}$$

则或者 $Z(f)=\Omega$，或者 $Z(f)$ 在 Ω 内无极限点. 在后一情况下，每个 $a\in Z(f)$ 都对应有唯一的正整数 $m=m(a)$，使得

(2) $$f(z)=(z-a)^m g(z) \qquad (z\in\Omega)$$

这里 $g\in H(\Omega)$，且 $g(a)\neq 0$；而且，$Z(f)$ 最多是可数的.

（我们重提一下，区域就是开连通集．）

整数 m 称为 f 在点 a 所具有的零点的阶．很清楚，当且仅当 f 在 Ω 内恒等于零时 $Z(f)=\Omega$．我们称 $Z(f)$ 为 f 的零点集．当然，对 f 的 α-点集，也就是，$f-\alpha$ 的零点集，也有类似的结果．这里 α 是任意复数．

证明 设 A 为 Ω 内 $Z(f)$ 所有极限点的集．因为 f 是连续的，$A\subset Z(f)$．

固定 $a\in Z(f)$，并选取 $r>0$，使得 $D(a;r)\subset\Omega$．由定理 10.16

$$f(z)=\sum_{n=0}^{\infty}c_n(z-a)^n \qquad (z\in D(a;r)) \tag{3}$$

这里有两种可能．或者所有 c_n 为零，在这种情形下 $D(a;r)\subset A$，且 a 为 A 的内点；或者有最小的整数 m（必须是正数，因 $f(a)=0$），使得 $c_m\neq 0$．在这种情况下，定义

$$g(z)=\begin{cases}(z-a)^{-m}f(z) & (z\in\Omega-\{a\})\\ c_m & (z=a)\end{cases} \tag{4}$$

于是(2)成立．很清楚，$g\in H(\Omega-\{a\})$．但由(3)可以推出

$$g(z)=\sum_{k=0}^{\infty}c_{m+k}(z-a)^k \qquad (z\in D(a;r)) \tag{5}$$

因此 $g\in H(D(a;r))$，于是确实有 $g\in H(\Omega)$．

进而言之，$g(a)\neq 0$，并且 g 的连续性表明，存在 a 的一个邻域，在这个邻域内 g 没有零点．这样，由(2)，a 就是 $Z(f)$ 的孤立点．

若 $a\in A$，这就必然出现第一种情况．于是 A 为开集．若 $B=\Omega-A$，由 A 作为极限点的集的定义，很明显，B 是开集．这样 Ω 就是不相交的开集 A 和 B 的并．由于 Ω 是连通的，我们得出或者 $A=\Omega$，在这种情况时 $Z(f)=\Omega$，或者 $A=\varnothing$．在后一种情况下，在 Ω 每个紧子集上，$Z(f)$ 最多有有限个点，而因为 Ω 是 σ-紧的，$Z(f)$

最多可数.

系 如果 f 和 g 是区域 Ω 内的全纯函数，且对 Ω 内某一个具有极限点的集的所有点 z，有 $f(z)=g(z)$，则对所有的 $z\in\Omega$，$f(z)=g(z)$ 成立.

换句话说，在区域 Ω 内的全纯函数，由 Ω 内任何一个具有极限点的集上的函数值决定. 这是一个重要的唯一性定理.

注 如果我们去掉 Ω 是连通的假设，这定理是不成立的！例如，若 $\Omega=\Omega_0\cup\Omega_1$，而且 Ω_0 及 Ω_1 为不相交的开集，在 Ω_0 内令 $f=0$，在 Ω_1 内令 $f=1$.

10.19 定义 如果 $a\in\Omega$ 而 $f\in H(\Omega-\{a\})$，则 f 称为在点 a 具有孤立奇点. 如果 f 能在 a 点给予定义，使得扩充后的函数在 Ω 内全纯，这个奇点就称为可去的.

10.20 定理 假设 $f\in H(\Omega-\{a\})$，且对某个 $r>0$，f 在 $D'(a;r)$ 内有界，则 f 在 a 具有可去奇点.

注意 $D'(a;r)=\{z:0<|z-a|<r\}$.

证明 定义 $h(a)=0$，且在 $\Omega-\{a\}$ 内，$h(z)=(z-a)^2f(z)$，有界性的假设表明 $h'(a)=0$. 由于 h 在 Ω 其他点明显地可微，我们有 $h\in H(\Omega)$，于是

$$h(z)=\sum_{n=2}^{\infty}c_n(z-a)^n \quad (z\in D(a;r))$$

令 $f(a)=c_2$，我们就得到所求的 f 的全纯开拓. 因此

$$f(z)=\sum_{n=0}^{\infty}c_{n+2}(z-a)^n \quad (z\in D(a;r))$$

10.21 定理 如果 $a\in\Omega$，且 $f\in H(\Omega-\{a\})$，则必然出现下列三种情况之一：

(a) f 在 a 具有可去奇点.

(*b*) 存在复数 $c_1, \cdots, c_m$，这里 m 是一正整数，且 $c_m \neq 0$，使得

$$f(z)-\sum_{k=1}^{m}\frac{c_k}{(z-a)^k}$$

在 a 点具有可去奇点.

(*c*) 若 $r>0, D(a;r)\subset\Omega$，则 $f(D'(a;r))$ 在平面内稠密.

在(*b*)情形下，f 称为在 a 具有 m 阶极点. 函数

$$\sum_{k=1}^{m}c_k(z-a)^{-k}$$

作为 $(z-a)^{-1}$ 的多项式，称为 f 在 a 的主要部分. 很清楚，在这种情形下，当 $z\to a$ 时，$|f(z)|\to\infty$.

在(*c*)的情形下，f 称为在 a 点具有本性奇点. (*c*)的一个等价的陈述是，对每一个复数 w，存在对应的序列 $\{z_n\}$，使得当 $n\to\infty$ 时，$z_n\to a$，且 $f(z_n)\to w$.

证明 假设(*c*)不成立. 则存在 $r>0, \delta>0$，和复数 w，使得在 $D'(a;r)$ 内 $|f(z)-w|>\delta$. 我们记 $D(a;r)$ 为 D，$D'(a;r)$ 为 D'. 定义

$$g(z)=\frac{1}{f(z)-w} \qquad (z\in D') \tag{1}$$

则 $g\in H(D')$，而且 $|g|<1/\delta$. 由定理 10.20，g 可以扩充为在 D 内的全纯函数.

若 $g(a)\neq 0$. (1)表明存在某个 $\rho>0$，f 在 $D'(a;\rho)$ 内有界. 因此由定理 10.20，(*a*)成立.

若 g 在 a 有 $m\geqslant 1$ 阶零点，则定理 10.18表明

$$g(z)=(z-a)^m g_1(z) \qquad (z\in D) \tag{2}$$

此处 $g\in H(D)$，而且 $g_1(a)\neq 0$. 同时由(1)得知，g_1 在 D' 内没有零点. 在 D 内令 $h=1/g_1$，则 $h\in H(D)$，h 在 D 内没有零点，而且

$$f(z)-w=(z-a)^{-m}h(z) \qquad (z\in D') \tag{3}$$

但 h 具有形如

$$(4)\qquad h(z)=\sum_{n=0}^{\infty}b_n(z-a)^n \qquad (z\in D)$$

的展开式，并有 $b_0\neq 0$. 现在，(3)表明(b)成立，且有 $c_k=b_{m-k}$，$k=1,2,\cdots,m$.

这就完成了证明.

现在我们将利用当一个幂级数 $\sum c_n(z-a)^n$ 限制在中心为 a 的圆周上时是一个三角级数这一事实.

10.22 定理 如果

$$(1)\qquad f(z)=\sum_{n=0}^{\infty}c_n(z-a)^n \qquad (z\in D(a;R))$$

且若 $0<r<R$，则

$$(2)\qquad \sum_{n=0}^{\infty}|c_n|^2r^{2n}=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(a+re^{i\theta})|^2d\theta$$

证明 我们有

$$(3)\qquad f(a+re^{i\theta})=\sum_{n=0}^{\infty}c_nr^ne^{in\theta}$$

对 $r<R$，级数(3)在$[-\pi,\pi]$上一致收敛. 因此

$$(4)\qquad c_nr^n=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(a+re^{i\theta})e^{-in\theta}d\theta \quad (n=0,1,2,\cdots)$$

并且可以看出(2)就是 Parseval 公式的一种特款.

下面是几个推论:

10.23 Liouville 定理 每个有界整函数都是常数.

回忆一下，如果函数在整个平面全纯则为整函数.

证明 假设 f 是整的，并且对所有 z 有：$|f(z)|<M$. $f(z)=\sum c_nz^n$. 由定理 10.22，对所有 r，应有

$$\sum_{n=0}^{\infty}|c_n|^2r^{2n}<M^2$$

但这只有对所有 $n\geqslant 1$, $c_n=0$ 时，才可能成立.

10.24 最大模定理 假设 Ω 是区域，$f\in H(\Omega)$，且 $\bar{D}(a;r)\subset\Omega$，则

(1) $$|f(a)|\leqslant\max_{\theta}|f(a+re^{i\theta})|$$

当且仅当 f 在 Ω 内为常数等式成立.

因此，$|f|$ 在 Ω 的任何点都不会有局部的极大值，除非 f 是常数.

证明 假设对所有实数 θ，$|f(a+re^{i\theta})|\leqslant|f(a)|$. 那末，用定理 10.22 的记号，可以得出

$$\sum_{n=0}^{\infty}|c_n|^2r^{2n}\leqslant|f(a)|^2=|c_0|^2$$

因此，$c_1=c_2=c_3=\cdots=0$. 由此推出在 $D(a;r)$ 内，$f(z)=f(a)$. 由于 Ω 是连通的，定理 10.18 表明 f 在 Ω 内是常数.

10.25 定理 若 n 为正整数，而且

$$P(z)=z^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0$$

这里 $a_0,\cdots a_{n-1}$ 均为复数，则 P 在平面内恰好有 n 个零点.

当然，这些零点是根据它们的重数来计算的：一个 m 阶零点，计算为 m 个零点. 这个定理包括了下述的事实：复数域是代数封闭的，也就是，每个复系数的非常数多项式至少有一个复的零点.

证明 选取 $r>1+2|a_0|+|a_1|+\cdots+|a_{n-1}|$. 则

$$|P(re^{i\theta})|>|P(0)| \qquad (0\leqslant\theta\leqslant 2\pi)$$

若 P 没有零点，则函数 $f=1/P$ 将是整的，且对所有 θ，满足 $|f(0)|>|f(re^{i\theta})|$，与最大模定理矛盾. 这样，对某个 z_1，$P(z_1)=0$. 因此，存在 $n-1$ 次多项式 Q，使得 $P(z)=(z-z_1)Q(z)$. 对

n应用有限归纳法即可完成证明.

10.26 定理 (Cauchy 估值) 如果 $f\in H(D(a;R))$, 且对所有 $z\in D(a;R)$, $|f(z)|\leqslant M$, 则

$$(1)\qquad |f^{(n)}(a)|\leqslant\frac{n!M}{R^n}\qquad (n=1,2,3,\cdots)$$

证明 对每个 $r<R$, 级数 10.22(2)每一项都有上界M^2.

若我们取 $a=0, R=1$, 及 $f(z)=z^n$, 则 $M=1, f^n(0)=n!$. 由此看出(1)的结果不能再改进.

10.27 定义 如果对每个紧集 $K\subset\Omega$和每个 $\varepsilon>0$, 对应有 $N=N(K,\varepsilon)$, 使得当 $j>N$时,对所有 $z\in K$,有$|f_j(z)-f(z)|<\varepsilon$. 则称$\Omega$内的函数序列$\{f_j\}$在$\Omega$的紧子集上一致收敛于 f.

例如, 序列$\{z^n\}$在$D(0;1)$的紧子集上一致收敛于零. 但在$D(0;1)$内却不是一致收敛的. 紧子集上一致收敛这一概念, 在与全纯函数的极限运算有关的问题上, 最自然地提了出来. 这个概念有时也采用"几乎一致收敛"一词.

10.28 定理 假设对 $j=1,2,3,\cdots$, $f_j\in H(\Omega)$, 而且在Ω的紧子集上 f_j一致收敛于 f. 则 $f\in H(\Omega)$, 而且在Ω的紧子集上f'_j一致收敛于f'.

证明 由于在Ω内每个紧圆盘上的是一致收敛的, 故 f 连续. 设$\triangle$为 Ω内的三角形, 则$\triangle$是紧的, 于是由 Cauchy 定理

$$\int_{\partial\triangle}f(z)dz=\lim_{j\to\infty}\int_{\partial\triangle}f_j(z)dz=0$$

因此由 Morera 定理推出 $f\in H(\Omega)$.

设K是紧的, $K\subset\Omega$. 则存在 $r>0$, 对所有 $z\in K$, 闭圆盘$\bar{D}(z;r)$的并E是Ω的紧子集. 应用定理 10.26 于 $f-f_j$, 我们有

$$|f'(z)-f'_j(z)|\leqslant r^{-1}\|f-f_j\|_E\qquad (z\in K)$$

这里$\|f\|_E$为$|f|$在E上的上确界. 由于在E上f_j一致收敛于f,

得出在K上f_j'一致收敛于f'.

系 在同样的假设下，当$j\to\infty$时，在每个紧集$K\subset\Omega$和对每个正整数n，有$f_j^{(n)}\to f^{(n)}$.

与实直线上的情况相比较，在那里无限次可微函数的序列能够一致收敛于处处不可微的函数.

开映射定理

若Ω为区域且$f\in H(\Omega)$，则$f(\Omega)$或者是区域或者是一个点.

全纯函数的这个重要性质将以更为详细的形式在定理 10.32 予以证明.

10.29 引理 如果$f\in H(\Omega)$，且g由

$$g(z,w)=\begin{cases}\dfrac{f(z)-f(w)}{z-w} & \text{若 } w\neq z\\ f'(z) & \text{若 } w=z\end{cases}$$

定义于$\Omega\times\Omega$内. 则g在$\Omega\times\Omega$内连续.

证明 g的连续性值得怀疑的那些点$(z,\ w)\in\Omega\times\Omega$只有$z=w$.

固定$a\in\Omega$. 固定$\varepsilon>0$. 则存在$r>0$，使得$D(a;r)\subset\Omega$，并且对所有$\zeta\in D(a;r)$，$|f'(\zeta)-f'(a)|<\varepsilon$. 若$z$及$w$均在$D(a;r)$内，且若

$$\zeta(t)=(1-t)z+tw$$

则对$0\leqslant t\leqslant 1$，有$\zeta(t)\in D(a;r)$，并且

$$g(z,w)-g(a,\ a)=\int_0^1[f'(\zeta(t))-f'(a)]dt$$

对每个t，被积函数的绝对值$<\varepsilon$. 这样$|g(z,w)-g(a,a)|<\varepsilon$. 这就证明了$g$在$(a,a)$连续.

10.30 定理 假设 $\varphi\in H(\Omega)$，$z_0\in\Omega$，而且 $\varphi'(z_0)\neq 0$，则 Ω 包含 z_0 的一个邻域 V，使得

(a) φ 在 V 内是一一的，

(b) $W=\varphi(V)$ 是一开集，且

(c) 如果 ψ：$W\to V$ 由 $\psi(\varphi(z))=z$ 定义，则 $\psi\in H(W)$.

这样，$\varphi: V\to W$ 就有一个全纯的逆映射.

证明 在引理 10.29 中用 φ 代替 f. 应用该引理可证明，Ω 包含 z_0 的一个邻域 V，使得当 $z_1\in V$ 及 $z_2\in V$ 时，有

$$|\varphi(z_1)-\varphi(z_2)|\geqslant\frac{1}{2}|\varphi'(z_0)||z_1-z_2| \tag{1}$$

于是(a)成立. 同时

$$\varphi'(z)\neq 0 \qquad (z\in V) \tag{2}$$

要证明(b)，固定 $\zeta\in V$. 选取 $r>0$，使得 $\bar{D}(\zeta;r)\subset V$. 由(1)，存在 $\delta>0$，使得

$$|\varphi(\zeta+re^{i\theta})-\varphi(\zeta)|>2\delta \qquad (-\pi\leqslant\theta\leqslant\pi) \tag{3}$$

设 α 为不在 $\varphi(V)$ 内的复数，则 $h=1/(\alpha-\varphi)$ 在 V 内全纯. 由于对所有 θ

$$2\delta<|\alpha-\varphi(\zeta)|+|\alpha-\varphi(\zeta+re^{i\theta})|$$

由最大模定理推出

$$\frac{1}{|\alpha-\varphi(\zeta)|}=|h(\zeta)|\leqslant\sup|h(\zeta+re^{i\theta})|\leqslant\frac{1}{2\delta-|\alpha-\varphi(\zeta)|}$$

因此 $|\alpha-\varphi(\zeta)|\geqslant\delta$.

这证明了 $\varphi(V)\supset D(\varphi(\zeta);\delta)$. 因为 ζ 是 V 的任意一点，故 $\varphi(V)$ 是开的.

要证明(c)，固定 $w_1\in W$，则有唯一的一点 $z_1\in V$，$\varphi(z_1)=w_1$. 若 $w\in W$，而且 $\psi(w)=z\in V$，我们有

$$\frac{\psi(w)-\psi(w_1)}{w-w_1}=\frac{z-z_1}{\varphi(z)-\varphi(z_1)}. \tag{4}$$

由(1)，当 $w\to w_1$ 时，$z\to z_1$. 因此(2)推出 $\psi'(w_1)=1/\varphi'(z_1)$. 这样 $\psi\in H(W)$.

10.31 定义 对 $m=1,2,3,\cdots$，我们记“m 次幂函数”$z\to z^m$ 为 π_m.

每个 $w\neq 0$ 都刚好是 m 个不同的 z 值的 $\pi_m(z)$：若 $w=re^{i\theta}$，$r>0$，则 $\pi_m(z)=w$ 当且仅当 $z=r^{1/m}e^{i(\theta+2k\pi)/m}$，$k=1,\cdots,m$.

同时注意每个 π_m 都是一个开映射：若 V 开且不包含零，则由定理 10.30，$\pi_m(V)$ 开. 另一方面，$\pi_m(D(0;r))=D(0;r^m)$.

开映射的复合显然是开的. 特别，若 φ' 没有零点，由定理 10.30，$\pi_m\circ\varphi$ 是开的. 下面的定理(它包含引理 10.29 前面已经指出的开映射定理更详细的叙述)，陈述了一个相反的命题：在一个区域内非常数的全纯函数，除了附加上一个常数外，都局部地有 $\pi_m\circ\varphi$ 的形式.

10.32 定理 假设 Ω 是一区域，$f\in H(\Omega)$，f 不是常数，$z_0\in\Omega$，且 $w_0=f(z_0)$. 设 m 为函数 $f-w_0$ 在点 z_0 的零点的阶数.

则存在 z_0 的邻域 V，$V\subset\Omega$，且存在 $\varphi\in H(V)$，使得

(a) 对所有的 $z\in V$，$f(z)=w_0+[\varphi(z)]^m$，

(b) φ' 在 V 内没有零点，且 φ 为 V 到圆盘 $D(0;r)$ 上的可逆映射.

这样，在 V 内，$f-w_0=\pi_m\circ\varphi$. 由此得出 f 恰好是 $V-\{z_0\}$ 到 $D'(w_0;r^m)$ 上的 m 对一的映射. 并且每个 $w_0\in f(\Omega)$ 都是 $f(\Omega)$ 的一个内点. 因此 $f(\Omega)$ 是开的.

证明 不失一般性，我们可以假设 Ω 是 z_0 的一个凸邻域，而且它如此地小，使得若 $z\in\Omega-\{z_0\}$ 时，$f(z)\neq w_0$，则对某个在 Ω 内无零点的 $g\in H(\Omega)$，有

$$f(z)-w_0=(z-z_0)^m g(z)\qquad(z\in\Omega)\tag{1}$$

因此 $g'/g\in H(\Omega)$. 由定理 10.14，存在 $h\in H(\Omega)$，使 $g'/g=h'$.

在Ω内$g\cdot\exp(-h)$的导数为零. 如果h通过附加上一个适当的常数加以修正,便得出$g=\exp(h)$. 定义

$$\varphi(z)=(z-z_0)\exp\frac{h(z)}{m} \qquad (z\in\Omega) \tag{2}$$

则对所有$z\in\Omega$,(a)成立.

同时,$\varphi(z_0)=0$而$\varphi'(z_0)\neq 0$. 根据定理10.30,现在可以得出满足(b)的开集V的存在性. 这就完成了证明.

下一定理其实已经包含在前述的结论中. 不过,看起来还是值得明显地加以叙述.

10.33 定理 假设Ω是一区域,$f\in H(\Omega)$,且f在Ω内是一一的. 则对每个$z\in\Omega$,$f'(z)\neq 0$,且f的逆是全纯的.

证明 若对某个$z_0\in\Omega$,$f'(z_0)=0$,定理10.32的假设将对某个$m>1$成立. 于是f将是在z_0的某个去心邻域内是m对一的. 现在再应用10.30的(c)即可.

注意定理10.33的逆是不成立的:若$f(z)=e^z$,则对每个z,$f'(z)\neq 0$. 但f在整个复平面内并非一一的.

整体 Cauchy 定理

在陈述并证明这个定理之前(这个定理将取消定理10.14中有关凸区域的限制),给迄今为止曾经是够用的积分工具,再加上一些工具将是很方便的. 实质上,这主要是使我们不再限于沿单一的路径积分,而是代之以考虑路径的有限"和". 在第10.9(c)给出了一个简单的例子.

10.34 链与圈 假设$\gamma_1,\cdots,\gamma_n$是平面上的路径,并令$K=\gamma_1^*\cup\cdots\cup\gamma_n^*$. 每个$\gamma_i$在向量空间$C(K)$上由公式

$$\tilde{\gamma}_i(f)=\int_{\gamma_i} f(z)\,dz \tag{1}$$

导出一个线性泛函$\tilde{\gamma}_i$. 定义

(2) $$\tilde{\Gamma}=\tilde{\gamma}_1+\cdots+\tilde{\gamma}_n$$

显然，对所有 $f\in C(K)$，$\tilde{\Gamma}(f)=\tilde{\gamma}_1(f)+\cdots+\tilde{\gamma}_n(f)$. 关系式(2)提示我们引出“形式和”

(3) $$\Gamma=\gamma_1+\cdots+\gamma_n$$

并定义

(4) $$\int_\Gamma f(z)dz=\tilde{\Gamma}(f)$$

则(3)仅是陈述

(5) $$\int_\Gamma f(z)dz=\sum_{i=1}^{n}\int_{\gamma_i} f(z)dz \qquad (f\in C(K))$$

的一个缩写. 注意(5)式可用来作为它左边的定义.

这样定义的 Γ 称为链.

如果 Ω 是一个开集，对 $1\leqslant i\leqslant n$，$\gamma_i^*\subset\Omega$，且 Γ 由(3)定义，则 Γ 就是 Ω 内的一个链.

若(3)成立，且每个 γ_i 为闭路径，则 Γ 称为圈.

一个链可以用多种方式表示为路径的和. 我们说

$$\gamma_1+\cdots+\gamma_n=\delta_1+\cdots+\delta_k$$

的意思，简单说来是指对于 $\gamma_1^*\cup\cdots\cup\gamma_n^*\cup\delta_1^*\cup\cdots\cup\delta_k^*$ 上的每个连续函数 f，有

$$\sum_i\int_{\gamma_i} f(z)dz=\sum_j\int_{\delta_j} f(z)dz$$

特别，一个圈能很好地表示为不是闭路径的和.

如果(3)成立，我们定义

(6) $$\Gamma^*=\gamma_1^*\cup\cdots\cup\gamma_n^*$$

如果 Γ 是一个圈，且 $\alpha\notin\Gamma^*$，我们定义 α 关于 Γ 的指数为

(7) $$\mathrm{Ind}_\Gamma(\alpha)=\frac{1}{2\pi i}\int_\Gamma\frac{dz}{z-\alpha}$$

这正和定理 10.10 中的一样. 很明显，由(3)可推出

(8) $$\mathrm{Ind}_{\Gamma}(\alpha)=\sum_{i=1}^{n}\mathrm{Ind}_{\gamma_i}(\alpha)$$

如果在(3)中每个 γ_i 都用它的反向路径代替，（见 10.8 节），所得的链将记为 $-\Gamma$. 这样

(9) $$\int_{-\Gamma}f(z)dz=-\int_{\Gamma}f(z)dz \qquad (f\in C(\Gamma^*))$$

特别，如果 Γ 是一个圈，且 $\alpha\notin\Gamma^*$，则 $\mathrm{Ind}_{-\Gamma}(\alpha)=-\mathrm{Ind}_{\Gamma}(\alpha)$.

最后，注意链能通过用其对应泛函的加或减这种明显的方式来进行加法和减法：$\Gamma=\Gamma_1+\Gamma_2$ 这个式子意味着对每个 $f\in C(\Gamma_1^*\cup\Gamma_2^*)$

(10) $$\int_{\Gamma}f(z)dz=\int_{\Gamma_1}f(z)dz+\int_{\Gamma_2}f(z)dz$$

10.35 Cauchy 定理 假设 $f\in H(\Omega)$，这里 Ω 是复平面上的任意开集. 若 Γ 是 Ω 内的一个圈，并对每个不在 Ω 内的 α，满足

(1) $$\mathrm{Ind}_{\Gamma}(\alpha)=0$$

则当 $z\in\Omega-\Gamma^*$ 时有

(2) $$f(z)\cdot\mathrm{Ind}_{\Gamma}(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(w)}{w-z}dw$$

并且

(3) $$\int_{\Gamma}f(z)dz=0$$

若 Γ_0 及 Γ_1 均为 Ω 内的圈，使得对每个不在 Ω 内的 α 有

(4) $$\mathrm{Ind}_{\Gamma_0}(\alpha)=\mathrm{Ind}_{\Gamma_1}(\alpha)$$

则

(5) $$\int_{\Gamma_0}f(z)dz=\int_{\Gamma_1}f(z)dz$$

证明 设

$$g(z,w)=\begin{cases}\dfrac{f(w)-f(z)}{w-z} & \text{若 } w\neq z\\ f'(z) & \text{若 } w=z\end{cases} \tag{6}$$

是在$\Omega\times\Omega$内定义的函数. g 在$\Omega\times\Omega$内是连续的(引理 10.29). 因此我们能定义

$$h(z)=\frac{1}{2\pi i}\int_\Gamma g(z;w)dw \qquad (z\in\Omega) \tag{7}$$

对 $z\in\Omega-\Gamma^*$, Cauchy 公式(2)显然等价于断言

$$h(z)=0 \tag{8}$$

要证明(8), 我们首先证明 $h\in H(\Omega)$. 注意 g 在$\Omega\times\Omega$ 的每一个紧子集上一致连续. 若 $z\in\Omega$, $z_n\in\Omega$, 且 $z_n\to z$, 便可得出对 $w\in\Gamma^*$(Ω的一个紧子集), 有 $g(z_n,w)$一致地趋于 $g(z,w)$. 这证明了 h 在Ω内连续. 设$\triangle$是Ω内一个闭三角形, 则

$$\int_{\partial\triangle}h(z)dz=\frac{1}{2\pi i}\int_\Gamma\left(\int_{\partial\triangle}g(z,w)dz\right)dw \tag{9}$$

对每个 $w\in\Omega$, $z\to g(z,w)$在Ω内是全纯的. (在 $z=w$ 时的奇点是可去的.)因此(9)式右边里面的积分对每个 $w\in\Gamma^*$都为零. 这时, Morera 定理表明 $h\in H(\Omega)$.

其次, 我们设Ω_1是使 $\mathrm{Ind}_\Gamma(z)=0$ 的所有复数 z 的集, 并且定义

$$h_1(z)=\frac{1}{2\pi i}\int_\Gamma\frac{f(w)}{w-z}dz \qquad (z\in\Omega_1) \tag{10}$$

如果 $z\in\Omega\cap\Omega_1$, 由Ω_1的定义, 显然有 $h_1(z)=h(z)$. 因此存在一函数 $\varphi\in H(\Omega\cup\Omega_1)$, 它限制在$\Omega$上时是 h, 限制在Ω_1上时是 h_1.

我们的假设(1)表明 Ω_1 包含 Ω 的余集, 这样, φ 是一个整函数. 由 $\mathrm{Ind}_\Gamma(z)$在Ω_1为零, 故Ω_1也包含 Γ^*的余集的无界分支. 因此

$$\lim_{|z|\to\infty}\varphi(z)=\lim_{|z|\to\infty}h_1(z)=0 \tag{11}$$

这时，由 Liouville 定理推出，对每个 z，$\varphi(z)=0$. 这就证明了(8)，因此也证明了(2).

为了由(2)导出(3)，取$a\in\Omega-\Gamma^*$，并定义$F(z)=(z-a)f(z)$. 则因为 $F(a)=0$，故

$$(12)\qquad \frac{1}{2\pi i}\int_\Gamma f(z)dz=\frac{1}{2\pi i}\int_\Gamma \frac{F(z)}{z-a}dz=F(a)\cdot \mathrm{Ind}_\Gamma(a)=0$$

最后，把(3)应用于圈 $\Gamma=\Gamma_1-\Gamma_0$，由(4)可得出(5). 这就完成了证明.

10.36 评注

(*a*) 如果 γ 是凸区域Ω内的闭路径，且 $a\notin\Omega$，应用定理 10.14 于 $f(z)=(z-\alpha)^{-1}$，可以证明 $\mathrm{Ind}_\gamma(\alpha)=0$. 如果$\Omega$是凸的，则$\Omega$内每一个圈都满足定理 10.35 的假设(1). 这表明定理 10.35 是定理 10.14 及 10.15 的推广.

(*b*) 定理 10.35 的最后部分表明，在什么情况下沿一个圈的积分能用沿另一个圈的积分代替而不改变其积分值. 例如设Ω为去掉三个不相交闭圆盘 D_i 的平面. 若 Γ，γ_1，γ_2，γ_3 为Ω内正向圆周，使得Γ围绕 $D_1\cup D_2\cup D_3$ 和 γ_i 围绕 D_i，但 $j\neq i$ 时，γ_i 不围绕D_j，则对每个 $f\in H(\Omega)$，有

$$\int_\Gamma f(z)dz=\sum_{i=1}^{3}\int_{\gamma_i} f(z)dz$$

(*c*) 为了应用定理 10.35，可以期望会有一种合理而有效的方法，来寻求一个点关于一条闭路径的指数. 下面一个定理对实际中所出现的所有路径都能做到这一点. 它实质上是说，当一条路径“由右到左”地交叉时，指数增加 1. 如果我们回忆一下，当 α 在 γ^* 的余集W的无界分支内时，$\mathrm{Ind}_\gamma(\alpha)=0$. 假若$W$仅有有限多个分支，并且 γ 穿越它们时，没有一段弧会穿过一次以上. 我们就能成功地决定W的其它分支中的 $\mathrm{Ind}_\gamma(\alpha)$.

10.37 定理 假设 γ 为平面上一条闭路径，具有参数区间 $[\alpha, \beta]$. 假设 $\alpha<u<v<\beta$, a 和 b 为复数，$|b|=r>0$，并且

(i) $\gamma(u)=a-b$, $\gamma(v)=a+b$,

(ii) $|\gamma(s)-a|<r$ 当且仅当 $u<s<v$,

(iii) $|\gamma(s)-a|=r$ 当且仅当 $s=u$ 或 $s=v$.

进一步假设 $D(a;r)-\gamma^*$ 是两个区域 D_+ 及 D_- 之并，它们按照 $a+bi\in\bar{D}_+$ 及 $a-bi\in\bar{D}_-$ 作为标志，则当 $z\in D_+$, $w\in D_-$ 时，有

$$\mathrm{Ind}_\gamma(z)=1+\mathrm{Ind}_\gamma(w)$$

当 $\gamma(t)$ 由 $a-b$ 穿越 $D(a;r)$ 到 $a+b$ 时，D_- 是在这条路径的“右边”而 D_+ 是在这条路径的“左边”.

证明 为了书写简单，改变 γ 的参数，使得 $u=0, v=\pi$. 定义

$$C(s)=a-be^{is} \quad (0\leqslant s\leqslant 2\pi)$$

$$f(s)=\begin{cases} C(s) & (0\leqslant s\leqslant \pi) \\ \gamma(2\pi-s) & (\pi\leqslant s\leqslant 2\pi) \end{cases}$$

$$g(s)=\begin{cases} \gamma(s) & (0\leqslant s\leqslant \pi) \\ C(s) & (\pi\leqslant s\leqslant 2\pi) \end{cases}$$

$$h(s)=\begin{cases} \gamma(s) & (\alpha\leqslant s\leqslant 0 \text{ 或 } \pi\leqslant s\leqslant \beta) \\ C(s) & (0\leqslant s\leqslant \pi) \end{cases}$$

由于 $\gamma(0)=C(0)$ 及 $\gamma(\pi)=C(\pi)$，所以 f, g, h 都是闭路径.

如果 $E\subset\bar{D}(a;r)$, $|\xi-a|=r$, 且 $\xi\notin E$, 则 E 位于圆盘 $D(2a-\xi;2r)$ 之内，而这个圆盘不包含 ξ. 应用这结果于 $E=g^*$, $\xi=a-bi$，则看出[由评注 10.36(a)]，$\mathrm{Ind}_g(a-bi)=0$. 由于 $\bar{D}_-$ 是连通的，且 D_- 与 g^* 不交，便得出

(1) $$\mathrm{Ind}_g(w)=0 \qquad \text{若 } w\in D_-$$

相同的理由证明

(2) $$\mathrm{Ind}_f(z)=0 \qquad \text{若 } z\in D_+$$

我们得到结论

$$\begin{aligned}\mathrm{Ind}_\gamma(z)&=\mathrm{Ind}_h(z)=\mathrm{Ind}_h(w)\\&=\mathrm{Ind}_C(w)+\mathrm{Ind}_\gamma(w)=1+\mathrm{Ind}_\gamma(w)\end{aligned}$$

由于 $h=\gamma+f$，第一个等式由(2)得出．因为 z 及 w 均在 $D(a;r)$ 内，而这是一个与 h^* 不相交的连通集，故第二个等式成立．由于 $h+g=C+\gamma$，第三个等式由(1)得出．第四个等式则是定理 10.11 的一个推论．这就完成了证明．

我们现在转到与 Cauchy 定理有关的另一个拓扑概念的简短讨论．

10.38 同伦 假设 γ_0 及 γ_1 均为拓扑空间 X 内的闭曲线，两者都具有参数区间 $I=[0,1]$．我们称 γ_0 及 γ_1 是 X-同伦的，假如存在由单位正方形 $I^2=I\times I$ 到 X 内的一个连续映射 H，使得对所有 $s\in I$ 和 $t\in I$，都有

(1) $$H(s,0)=\gamma_0(s),\quad H(s,1)=\gamma_1(s),\quad H(0,t)=H(1,t)$$

令 $\gamma_t(s)=H(s,t)$．则(1)定义连接 γ_0 及 γ_1 的、X 内的一个单参数闭曲线族 γ_t．直观地说，这意味着在 X 内，γ_0 能够连续地变形到 γ_1．

若 γ_0 是 X-同伦到一常数映射 γ_1（也就是，γ_1^* 仅由一点组成），我们称 γ_0 在 X 内是零伦的．如果 X 是连通的，且 X 内每条闭曲线都是零伦的，则 X 称为单连通的．

例如，每个凸区域 Ω 都是单连通的．为了看出这个结论，令 γ_0 为 Ω 内的闭曲线，固定 $z_1\in\Omega$，并定义

(2) $$H(s,t)=(1-t)\gamma_0(s)+tz_1\quad(0\leqslant s\leqslant 1,\ 0\leqslant t\leqslant 1)$$

定理 10.40 将证明 Cauchy 定理 10.35 的条件(4)当 Γ_0 及 Γ_1 是 Ω-同伦的闭路径时成立．作为它一种特殊情况，注意若 Ω 是单连通，对 Ω 内每条闭路径 Γ，定理 10.35 的条件(1)均成立．

10.39 引理 如果 γ_0 及 γ_1 都是具有参数区间 $[0,1]$ 的闭路径，α 为一复数，且

(1) $$|\gamma_1(s)-\gamma_0(s)|<|\alpha-\gamma_0(s)|\qquad(0\leqslant s\leqslant 1)$$

则 $\mathrm{Ind}_{\gamma_1}(\alpha)=\mathrm{Ind}_{\gamma_0}(\alpha)$.

证明 首先注意由(1)推出 $\alpha\notin\gamma_0^*$ 及 $\alpha\notin\gamma_1^*$. 因此能定义 $\gamma=(\gamma_1-\alpha)/(\gamma_0-\alpha)$. 于是

$$\frac{\gamma'}{\gamma}=\frac{\gamma_1'}{\gamma_1-\alpha}-\frac{\gamma_0'}{\gamma_0-\alpha} \tag{2}$$

且由(1)得$|1-\gamma|<1$. 因此$\gamma^*\subset D(1;1)$. 这就推出$\mathrm{Ind}_\gamma(0)=0$. 现在,(2)式沿[0, 1]的积分给出了所求结果.

10.40 定理 如果 Γ_0 及 Γ_1 都是区域 Ω 内 Ω-同伦的闭路径,并且 $\alpha\notin\Omega$,则

$$\mathrm{Ind}_{\Gamma_1}(\alpha)=\mathrm{Ind}_{\Gamma_0}(\alpha) \tag{1}$$

证明 由定义,存在连续映射 $H: I^2\to\Omega$,使得

$$H(s,0)=\Gamma_0(s),\quad H(s,1)=\Gamma_1(s),\quad H(0,t)=H(1,t) \tag{2}$$

由于 I^2 是紧的,于是 $H(I^2)$也是紧的. 因此,存在 $e>0$,使得

$$\text{当}(s,t)\in I^2\text{时},\ |\alpha-H(s,t)|>2e \tag{3}$$

由于H一致连续,存在正数 n,使得

$$\text{当}|s-s'|+|t-t'|\leqslant 1/n\text{时},\ |H(s,t)-H(s',t')|<e \tag{4}$$

若 $i-1\leqslant ns\leqslant i$ 且 $i=1,\cdots,n$,由

$$\gamma_k(s)=H\left(\frac{i}{n},\frac{k}{n}\right)(ns+1-i)+H\left(\frac{i-1}{n},\frac{k}{n}\right)(i-ns) \tag{5}$$

定义多边形闭路径 $\gamma_0,\cdots,\gamma_n$. 由(4)及(5)得

$$|\gamma_k(s)-H(s,k/n)|<e\qquad(k=0,\cdots,n;\ 0\leqslant s\leqslant 1) \tag{6}$$

特别,取 $k=0$ 和 $k=n$,

$$|\gamma_0(s)-\Gamma_0(s)|<e,\quad|\gamma_n(s)-\Gamma_1(s)|<e \tag{7}$$

由(6)和(3)

$$|\alpha-\gamma_k(s)|>e\qquad(k=0,\cdots,n;\ 0\leqslant s\leqslant 1) \tag{8}$$

另一方面,(4)和(5)也给出

$$|\gamma_{k-1}(s)-\gamma_k(s)|<e\qquad(k=1,\cdots,n;\ 0\leqslant s\leqslant 1) \tag{9}$$

现由(7)、(8)、(9)并将引理 10.39 应用 $n+2$ 次，便得出 α 关于路径 $\Gamma_0, \gamma_0, \gamma_1, \cdots, \gamma_n, \Gamma_1$ 的每一个都具有相同的指数．这就证明了定理．

注：上述证明中若 $\Gamma_t(s)=H(s,t)$，因为 H 并没有假设可微，所以每个 Γ_t 乃是一条闭曲线，而不一定是一条路径．基于这个理由，才引出了路径 γ_k．另一个(或者是更合适的)克服这一困难的方法是，把指数的定义扩充到闭曲线上．这将在习题 28 中略述．

残数计算

10.41 定义 一函数 f 称为开集 Ω 内的亚纯函数，如果存在一个集 $A\subset\Omega$，使得

(*a*) A 在 Ω 内无极限点．

(*b*) $f\in H(\Omega-A)$．

(*c*) f 在 A 每一点具有极点．

注意并没有排除 $A=\varnothing$ 的可能．这样，每个 $f\in H(\Omega)$ 在 Ω 内也都是亚纯的．

同时要注意由(*a*)可推出 Ω 内没有一个紧子集能包含 A 的无限多个点，因此 A 最多是可数的．

如果 f 及 A 如上所述，$a\in A$，且

$$Q(z)=\sum_{k=1}^{m}c_k(z-a)^{-k} \tag{1}$$

如定理10.21所定义的一样，是 f 在 a 的主要部分(也就是，若 $f-Q$ 具有可去奇点 a)．则数 c_1 称为 f 在 a 的残数：

$$c_1=\operatorname{Res}(f;a) \tag{2}$$

如果 Γ 是一个圈，且 $a\notin\Gamma^*$，由(1)推出：

$$\frac{1}{2\pi i}\int_{\Gamma}Q(z)dz=c_1\operatorname{Ind}_{\Gamma}(\alpha)=\operatorname{Res}(Q;a)\operatorname{Ind}_{\Gamma}(a) \tag{3}$$

这是下列定理的一个很特殊的情况，并且将在该定理的证明中用到.

10.42 残数定理 假设 f 是 Ω 内的亚纯函数. 令 A 为 Ω 内的使 f 具有极点的点集. 若 Γ 为 $\Omega-A$ 内的一个圈，使得

(1) $$\text{对所有 } \alpha\notin\Omega, \operatorname{Ind}_\Gamma(\alpha)=0$$

则

(2) $$\frac{1}{2\pi i}\int_\Gamma f(z)dz=\sum_{a\in A}\operatorname{Res}(f;a)\operatorname{Ind}_\Gamma(a)$$

证明 设 $B=\{a\in A;\ \operatorname{Ind}_\Gamma(a)\neq 0\}$. 令 W 为 Γ^* 的余集，则 $\operatorname{Ind}_\Gamma(z)$ 在 W 的每个分支 V 上是常数. 若 V 是无界的，或者当 V 与 Ω^c 相交时，(1)推出对每个 $z\in V$, $\operatorname{Ind}_\Gamma(z)=0$. 由于 A 在 Ω 内无极限点，我们得出 B 是一个有限集.

因此(2)中的和式虽然形式上是无限的，而实际上是有限的.

令 $a_1,\cdots,a_n$ 是 B 中的点，$Q_1,\cdots,Q_n$ 为 f 在 $a_1,\cdots,a_n$ 的主要部分，并令 $g=f-(Q_1+\cdots+Q_n)$. (若 $B=\varnothing$，这种可能性并没有排除在外，则 $g=f$.) 令 $\Omega_0=\Omega-(A-B)$. 由于 g 在 $a_1,\cdots a_n$ 具有可去奇点，应用定理 10.35 于函数 g 及开集 Ω_0，可得

(3) $$\int_\Gamma g(z)dz=0$$

因此

$$\frac{1}{2\pi i}\int_\Gamma f(z)dz=\sum_{k=1}^{n}\frac{1}{2\pi i}\int_\Gamma Q_k(z)dz=\sum_{k=1}^{n}\operatorname{Res}(Q_k;a_k)\operatorname{Ind}_\Gamma(a_k)$$

且由于 f 及 Q_k 在 a_k 具有相同的残数，我们便得到(2).

我们用残数定理两个典型的应用来结束这一章. 第一个涉及到全纯函数的零点. 第二个是某些积分的计算.

10.43 定理 假设 γ 是区域 Ω 内的闭路径，使得对不在 Ω 内的每个 α, $\operatorname{Ind}_\gamma(\alpha)=0$. 同时假设对每个 $\alpha\in\Omega-\gamma^*$, $\operatorname{Ind}_\gamma(\alpha)=0$ 或

1, 并设具有 $\mathrm{Ind}_\gamma(\alpha)=1$ 的所有 α 的集为 Ω_1.

对任一个 $f\in H(\Omega)$, 令 N_f 为 f 在 Ω 内的按重数计算的零点的个数.

(*a*) 若 $f\in H(\Omega)$, 且 f 在 γ^* 上没有零点, 则

$$N_f=\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz=\mathrm{Ind}_\Gamma(0) \tag{1}$$

此处 $\Gamma=f\circ\gamma$.

(*b*) 若同时有 $g\in H(\Omega)$, 且

$$|f(z)-g(z)|<|f(z)| \quad \text{对所有 } z\in\gamma^* \tag{2}$$

则 $N_g=N_f$.

(*b*) 这一部分通常称为 Rouche 定理. 它说明如果两个全纯函数, 如同(2)所明确表示出来的那样, 在 Ω_1 的边界上互相接近, 则在 Ω_1 内具有相同的零点个数.

证明 令 $\varphi=f'/f$. 它是 Ω 内的一个亚纯函数. 若 $a\in\Omega$, 而且 f 在 a 具有 $m=m(a)$ 阶零点, 则 $f(z)=(z-a)^m h(z)$. 此处 h 及 $1/h$ 均在 a 的某邻域 V 全纯. 在 $V-\{a\}$ 内

$$\varphi(z)=\frac{f'(z)}{f(z)}=\frac{m}{z-a}+\frac{h'(z)}{h(z)} \tag{3}$$

这样

$$\mathrm{Res}(\varphi;a)=m(a) \tag{4}$$

令 $A=\{a\in\Omega_1: f(a)=0\}$. 如果我们把关于 γ 的指数的假设与残数定理结合起来, 就得到

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz=\sum_{a\in A}\mathrm{Res}(\varphi;a)=\sum_{a\in A}m(a)=N_f$$

这就证明了(1)的一半. 另一半是直接计算得出的结果:

$$\mathrm{Ind}_\Gamma(0)=\frac{1}{2\pi i}\int_\Gamma \frac{dz}{z}=\frac{1}{2\pi i}\int_0^{2\pi}\frac{\Gamma'(s)}{\Gamma(s)}ds$$

$$=\frac{1}{2\pi i}\int_0^{2\pi}\frac{f'(\gamma(s))}{f(\gamma(s))}\gamma'(s)ds=\frac{1}{2\pi i}\int_\gamma\frac{f'(z)}{f(z)}dz$$

这里 γ 的参数区间取为 $[0,2\pi]$.

其次,(2)表明 g 在 γ^* 上没有零点. 因此用 g 代替 f 时,(1)成立. 令 $\Gamma_0=g\circ\gamma$,则由(1)、(2)及引理 10.39 得出

$$N_g=\mathrm{Ind}_{\Gamma_0}(0)=\mathrm{Ind}_\Gamma(0)=N_f.$$

10.44 问题 对实数 t,当 $A\to\infty$ 时,求

(1) $$\int_{-A}^{A}\frac{\sin x}{x}e^{itx}dx$$

的极限.

解 因为 $z^{-1}\cdot\sin z\cdot e^{itz}$ 是整函数,由 Cauchy 定理,它沿 $[-A,\ A]$的积分等于沿路径 Γ_A 的积分,其中 Γ_A 沿实轴 $-A$到 -1,沿单位圆下半圆周由 -1 到 1,沿实轴由 1 到 A 得出. Γ_A 避开了原点,因此我们可以通过恒等式

$$2i\sin z=e^{iz}-e^{-iz}$$

看出(1)等于 $\varphi_A(t+1)-\varphi_A(t-1)$. 这里

(2) $$\frac{1}{\pi}\varphi_A(s)=\frac{1}{2\pi i}\int_{\Gamma_A}\frac{e^{isz}}{z}dz$$

用两种方法把 Γ_A 补充成为闭路径:第一种,用由 A 到 $-Ai$ 到 $-A$ 的半圆周;第二种,用 A 到 Ai 到 $-A$ 的半圆周. 函数 e^{isz}/z 在 $z=0$ 具有一个极点,在这点它的残数为 1. 于是得到

(3) $$\frac{1}{\pi}\varphi_A(s)=\frac{1}{2\pi}\int_{-\pi}^{0}\exp(isAe^{i\theta})d\theta$$

和

(4) $$\frac{1}{\pi}\varphi_A(s)=1-\frac{1}{2\pi}\int_0^{\pi}\exp(isAe^{i\theta})d\theta$$

注意到

(5) $$|\exp(isAe^{i\theta})|=\exp(-As\sin\theta)$$

并且这是<1的，且若 s 与$\sin\theta$具有相同符号，当 $A\to\infty$时，它趋于零．因此，控制收敛定理表明若 $s<0$，则(3)的积分趋于零，若 $s>0$，则(4)的积分趋于零．这样

$$\lim_{A\to\infty}\varphi_A(s)=\begin{cases}\pi & \text{若 } s>0\\ 0 & \text{若 } s<0\end{cases} \tag{6}$$

若我们应用(6)于 $s=t+1$ 及 $s=t-1$，我们就得到

$$\lim_{A\to\infty}\int_{-A}^{A}\frac{\sin x}{x}e^{itx}dx=\begin{cases}\pi & \text{若 } -1<t<1\\ 0 & \text{若 } |t|>1\end{cases} \tag{7}$$

又由于 $\varphi_A(0)=\frac{\pi}{2}$，故当 $t=\pm1$ 时，(7)中的极限是$\frac{\pi}{2}$．

注意(7)给出$\sin x/x$的 Fourier 变式．我们把对照反演定理来验证这结果作为一个习题．

习　题

1　在这章内暗中使用了下列事实：如果 A 及 B 为平面的不相交子集，A 是紧的而 B 是闭的，则存在 $\delta>0$，使得对所有 $\alpha\in A$ 及 $\beta\in B$，有 $|\alpha-\beta|\geqslant\delta$．用任意度量空间代替平面证明这个结果．

2　在 10.8 节末尾出现了路径长度的定义．这是否与第 8 章习题 10 给出的定义相一致？

3　若 f 和 g 均为整函数，且对每个 z，$|f(z)|\leqslant|g(z)|$，你能推出什么结果？

4　假设 f 为整函数，且对所有的 z，

$$|f(z)|\leqslant A+B|z|^k$$

此处 A、B、和 k 都是正数．证明 f 必然是一个多项式．

5　假设$\{f_n\}$是Ω内一致有界的全纯函数序列，使得对每个 $z\in\Omega$，$\{f_n\}$收敛．证明在Ω的每个紧子集上，这个收敛性是一致的．

提示：把控制收敛定理应用于 f_n-f_m 的 Cauchy 公式．

6　存在区域 Ω，使得 $\exp(\Omega)=D(1;1)$．证明 exp 在Ω内是一一的，但有许多这样的 Ω．固定一个，并对$|z-1|<1$，定义$\log z$为使$e^w=z$的 w，有

$w\in\Omega$. 证明 $\log'(z)=\frac{1}{z}$. 在

$$\frac{1}{z}=\sum_{n=0}^{\infty}a_n(z-1)^n$$

中求出系数 a_n，并由此求出展开式

$$\log z=\sum_{n=0}^{\infty}c_n(z-1)^n$$

的系数 c_n. 在其它哪些圆盘上能够这样做？

7 若 $f\in H(\Omega)$，f 的导数的 Cauchy 公式

$$f^{(n)}(z)=\frac{n!}{2\pi i}\int_\Gamma\frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta\qquad(n=1,2,3,\cdots)$$

在关于 z 和 Γ 的某些条件下是正确的. 陈述出这些条件，并证明这个公式.

8 假设 P 和 Q 为多项式，Q 的次数比 P 至少高 2 次，且有理函数 $R=P/Q$ 在实轴上无极点. 证明 R 沿 $(-\infty,\infty)$ 的积分是 R 在上半平面残数和的 $2\pi i$ 倍. (以沿合适的半圆周的积分代替沿 $(-A,A)$ 的积分，并应用残数定理.) 对下半平面类似的陈述应是怎样？用这方法计算:

$$\int_{-\infty}^{\infty}\frac{x^2}{1+x^4}dx$$

9 用习题 8 所述的方法，对实数 t 计算 $\int_{-\infty}^{\infty}\frac{e^{itx}}{1+x^2}dx$. 对照 Fourier 变式的反演定理核对你的答案.

10 设 γ 为正向单位圆周，计算

$$\frac{1}{2\pi i}\int_\gamma\frac{e^z-e^{-z}}{z^4}dz$$

11 假设 α 为一复数，$|\alpha|\neq1$. 通过沿单位圆周对 $(z-\alpha)^{-1}(z-1/\alpha)^{-1}$ 积分来计算

$$\int_0^{2\pi}\frac{d\theta}{1-2\alpha\cos\theta+\alpha^2}$$

12 计算

$$\int_{-\infty}^{+\infty}\left(\frac{\sin x}{x}\right)^2e^{itx}dx\quad(\text{对实数 }t)$$

13 计算

$$\int_0^{\infty}\frac{dx}{1+x^n}\quad(n=2,3,4,\cdots)$$

[对偶数 n，可用习题 8 的方法．然而，可以选取一条不同的路径：由 0 到 R 到 $R\exp(2\pi i/n)$ 到 0，它能简化计算，同时也能对奇数 n 进行计算．]

答：$(\pi/n)/\sin(\pi/n)$．

14　假设 Ω_1 和 Ω_2 都是平面区域，f 和 g 分别在 Ω_1 和 Ω_2 内定义，并且都是非常数的复函数，且 $f(\Omega_1)\subset\Omega_2$．令 $h=g\circ f$．如果 f 和 g 是全纯的，我们知道 h 也是全纯的．假设我们知道 f 和 h 是全纯的，关于 g 我们能得出什么结论？若知道 g 和 h 是全纯的，又怎么样？

15　假设 Ω 是一个区域，$\varphi\in H(\Omega)$，φ' 在 Ω 内没有零点．$f\in H(\varphi(\Omega))$，$g=f\circ\varphi$，$z_0\in\Omega$ 且 $w_0=\varphi(z_0)$．证明如果 f 在 w_0 有 m 阶零点，则 g 在 z_0 也有 m 阶零点．如果 φ' 在 z_0 有 k 阶零点，结论应该怎样改变？

16　假设 μ 为测度空间 X 上的复测度，Ω 是平面内的开集，φ 为 $\Omega\times X$ 上的有界函数，使得对每个 $z\in\Omega$，$\varphi(z,t)$ 是 t 的可测函数，且 $\varphi(z,t)$ 对每个 $t\in X$，在 Ω 内是全纯的．对 $z\in\Omega$，定义

$$f(z)=\int_X\varphi(z,t)\,d\mu(t)$$

证明 $f\in H(\Omega)$．提示：证明对每个紧集 $K\subset\Omega$，对应一个常数 $M<\infty$，使得

$$\left|\frac{\varphi(z,t)-\varphi(z_0,t)}{z-z_0}\right|<M\qquad(z\text{ 和 }z_0\in K,\ t\in X)$$

17　决定下列函数有定义且全纯的区域：

$$f(z)=\int_0^1\frac{dt}{1+tz},\quad g(z)=\int_0^\infty\frac{e^{tz}}{1+t^2}dt,\quad h(z)=\int_{-1}^1\frac{e^{tz}}{1+t^2}dt$$

提示：或者用习题 16，或者结合使用 Morera 定理和 Fubini 定理．

18　假设 $f\in H(\Omega)$，$\bar{D}(a;r)\subset\Omega$，$\gamma$ 为中心在 a，半径为 r 的正向圆周，且 f 在 γ^* 上没有零点．对 $p=0$，积分

$$\frac{1}{2\pi i}\int_\gamma\frac{f'(z)}{f(z)}z^p\,dz$$

等于 f 在 $D(a;r)$ 内的零点数．对 $p=1,2,3,\cdots$，这些积分的值是多少？（利用 f 的零点）．如果用任意的 $\varphi\in H(\Omega)$ 代替 z^p，答案又是什么？

19　假设 $f\in H(U)$，$g\in H(U)$，且 f 和 g 在 U 内都没有零点．若

$$\frac{f'}{f}\left(\frac{1}{n}\right)=\frac{g'}{g}\left(\frac{1}{n}\right)\qquad(n=1,2,3,\cdots)$$

求 f 和 g 之间另一个简单关系．

20 假设Ω为一区域，对$n=1,2,3,\cdots$，$f_n\in H(\Omega)$，没有任何一个函数f_n在Ω内有零点，且在Ω的紧子集上$\{f_n\}$一致收敛于f. 证明：或者f在Ω内没有零点，或者对所有$z\in\Omega$，$f(z)=0$.

如果Ω'是一个包含每一个$f_n(\Omega)$的区域，且f不是常数，证明$f(\Omega)\subset\Omega'$.

21 假设$f\in H(\Omega)$，Ω包含闭单位圆盘，且$|z|=1$时，$|f(z)|<1$. f在这圆盘内必须有多少个不动点？就是说，方程$f(z)=z$在圆盘内有多少解？

22 假设$f\in H(\Omega)$，Ω包含闭单位圆盘，$|z|=1$时，$|f(z)|>2$，且$f(0)=1$. f在单位圆盘内是否一定有一零点？

23 假设$P_n(z)=1+z/1!+\cdots+z^n/n!$，$Q_n(z)=P_n(z)-1$，此处$n=1,2,3,\cdots$. 对大的$n$，关于$P_n$及$Q_n$的零点位置你能说些什么？要尽可能地指明.

24 证明下列 Rouché 定理的一般形式：设Ω为平面上紧集K的内部. 假设f和g在K上连续，在Ω内全纯，且对所有$z\in K-\Omega$，$|f(z)-g(z)|<|f(z)|$，则f和g在Ω内具有相同数目的零点.

25 设A为圆环区域$\{z:r_1<|z|<r_2\}$，这里r_1及r_2都是给定的正数.

(a) 证明 Cauchy 公式

$$f(z)=\frac{1}{2\pi i}\left(\int_{\gamma_1}+\int_{\gamma_2}\right)\frac{f(\zeta)}{\zeta-z}d\zeta$$

在下列条件下是正确的：$f\in H(A)$，

$$r_1+\varepsilon<|z|<r_2-\varepsilon$$

且

$$\gamma_1(t)=(r_1+\varepsilon)e^{-it},\quad \gamma_2(t)=(r_2-\varepsilon)e^{it}\quad(0\leqslant t\leqslant 2\pi)$$

(b) 运用(a)证明每个$f\in H(A)$都能分解为和式$f=f_1+f_2$，这里f_1在$\overline{D}(0;r_1)$的外部全纯，而$f_2\in H(D(0;r_2))$. 如果我们要求当$|z|\to\infty$时，$f_1(z)\to 0$，证明这个分解式是唯一的.

(c) 用这分解式使每个$f\in H(A)$，对应于在A内收敛于f的“Laurent 级数”

$$\sum_{-\infty}^{\infty}c_nz^n$$

证明对每个f仅有一个这样的级数，并证明在A的紧子集上，这级数一致收敛于f.

(*d*) 若 $f\in H(A)$ 且 f 在 A 内有界，证明 f_1 和 f_2 这两部分也都是有界的.

(*e*) 上述结论有那些能扩充到 $r_1=0$ 的情形（或者 $r_2=\infty$，或两者同时出现）？

(*f*) 上述结论有那些能扩充到界于有限多个（多于两个）圆周的区域？

26 要求展开函数

$$\frac{1}{1-z^2}+\frac{1}{3-z}$$

为形如 $\sum_{-\infty}^{\infty}c_nz^n$ 的级数.

这样的展开式有多少个？它们中的每一个在什么区域内是正确的？对每个这样的展开式，明确地求出系数 c_n.

27 假设 Ω 是由不等式 $a<y<b$ 决定的水平带形，假设 $f\in H(\Omega)$，并且对所有的 $z\in\Omega$，$f(z)=f(z+1)$. 证明 f 在 Ω 内具有 Fourier 展开式

$$f(z)=\sum_{-\infty}^{\infty}c_ne^{2\pi ins}$$

且对每个 $\varepsilon>0$，这个级数在 $\{z:a+\varepsilon\leqslant y\leqslant b-\varepsilon\}$ 内一致收敛.

提示：映射 $z\to e^{2\pi iz}$ 将 f 变为圆环内的一个函数.

找出一个积分公式，使系数 c_n 能按这个公式由 f 计算出来.

28 假设 Γ 是平面内一条闭曲线，具有参数区间 $[0,2\pi]$，取 $\alpha\notin\Gamma^*$. 用三角多项式 Γ_n 一致逼近 Γ. 证明当 m 和 n 充分大时，$\mathrm{Ind}_{\Gamma_n}(\alpha)=\mathrm{Ind}_{\Gamma_m}(\alpha)$. 定义这公共值为 $\mathrm{Ind}_\Gamma(\alpha)$. 证明这结果不依赖于 $\{\Gamma_n\}$ 的选择. 证明引理 10.39 现在对闭曲线也真确，并用以给出定理 10.40 的一个不同的证明.

第十一章　调和函数

Cauchy-Riemann 方程

11.1　算子 ∂ 和 $\bar{\partial}$　设 f 是定义在平面开集 Ω 内的复函数，把 f 看作是将 Ω 映到 R^2 内的变换，且设 f 在某点 $z_0\in\Omega$ 有 8.22 的意义下的微分．为简单起见，设 $z_0=f(z_0)=0$．这样，f 的可微性假定等价于两个复数 α 和 β（在 $z_0=0$ 处 f 关于 x 和 y 的偏导数）的存在，使得

$$f(z)=\alpha x+\beta y+\eta(z)z \qquad (z=x+iy) \tag{1}$$

此处当 $z\to 0$ 时，$\eta(z)\to 0$．

由于 $2x=z+\bar{z}$ 和 $2iy=z-\bar{z}$，故(1)可写为

$$f(z)=\frac{\alpha-i\beta}{2}z+\frac{\alpha+i\beta}{2}\bar{z}+\eta(z)z \tag{2}$$

的形式．这就启发我们引进微分算子

$$\partial=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right),\quad \bar{\partial}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) \tag{3}$$

现在(2)变成

$$\frac{f(z)}{z}=(\partial f)(0)+(\bar{\partial}f)(0)\cdot\frac{\bar{z}}{z}+\eta(z)\quad (z\neq 0) \tag{4}$$

对实数 z，$\bar{z}/z=1$；对纯虚数 z，$\bar{z}/z=-1$．因此，当且仅当 $(\bar{\partial}f)(0)=0$ 时，$f(z)/z$ 的极限为零，并且得到全纯函数的如下特征：

11.2　定理　设 f 是 Ω 内的复函数，在 Ω 的每一点上 f 可微，则 $f\in H(\Omega)$ 当且仅当对每一点 $z\in\Omega$，Cauchy-Riemann 方程

(1) $$(\bar{\partial}f)(z)=0$$

成立. 在那种情况下,我们有

(2) $$f'(z)=(\partial f)(z) \qquad (z\in\Omega)$$

如果 $f=u+iv$, u 和 v 是实的,则(1)分解成一对方程

$$u_x=v_y,\quad u_y=-v_x$$

此处的下标表示对所指的变量求偏导数. 这些就是全纯函数的实部和虚部所必须满足的 Cauchy-Riemann 方程.

11.3 Laplace 算子 设 f 是平面开集 Ω 内的复函数,使得在 Ω 内每一点 f_{xx} 和 f_{yy} 存在. 则 f 的 Laplace 算子值定义为

(1) $$\Delta f=f_{xx}+f_{yy}$$

如果 f 在 Ω 内连续,且在 Ω 的每一点

(2) $$\Delta f=0$$

则称 f 在 Ω 内是调和的.

由于实函数的 Laplace 算子是实的(如果存在的话),显然,一个复函数当且仅当它的实部和虚部都在 Ω 内调和时, f 在 Ω 内是调和的.

注意,假定 $f_{xy}=f_{yx}$,则

(3) $$\Delta f=4\partial\bar{\partial}f$$

当 f 具有连续二阶导数时,这个假定是成立的.

如果 f 是全纯函数,则 $\bar{\partial}f=0$,且具有任意阶的连续导数,因而(3)式表明:

11.4 定理 全纯函数是调和的.

现在我们转过来关注调和函数的积分表示. 它与全纯函数的 Cauchy 公式有着密切关系. 其中它将证明每一个实调和函数局部地是一个全纯函数的实部. 同时它将指出关于开圆盘内一些全纯函数类的边界状况的信息.

Poisson 积分

11.5 Poisson 核 这是指的函数

$$(1)\qquad P_r(t)=\sum_{-\infty}^{\infty}r^{|n|}e^{int}\qquad(0\leqslant r<1,\ t\text{ 为实数})$$

我们可以把 $P_r(t)$看作是两个变量 r 和 t 的函数，或者以 r 为指标，把 $P_r(t)$看作是 t 的函数族.

如果 $z=re^{i\theta}(0\leqslant r<1,\ \theta$ 为实数)，则用 5.24 节做过的简单计算便可证明.

$$(2)\qquad P_r(\theta-t)=\mathrm{Re}\left[\frac{e^{it}+z}{e^{it}-z}\right]=\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}$$

从(1)看出

$$(3)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(t)dt=1\qquad(0\leqslant r<1)$$

从(2)得到 $P_r(t)>0,\quad P_r(t)=P_r(-t)$,

$$(4)\qquad P_r(t)<P_r(\delta)\qquad(0<\delta<|t|\leqslant\pi)$$

而且

$$(5)\qquad \lim_{r\to 1}P_r(\delta)=0\qquad(0<\delta\leqslant\pi)$$

11.6 记号 今后，U 表示开单位圆盘，即$U=D(0;1)$，而 T 表示单位圆周，即复平面内 U 的边界.

如果 f 是 T 上的函数而 μ 是 T 上的一个测度，用

$$(1)\qquad \int_{\alpha}^{\alpha+2\pi}f(t)d\mu(t)$$

代替$\int_T f d\mu$ 经常是合意的，此处 α 是一个适当选取的实数. (1)理解为沿半开区间

$$[\alpha,\alpha+2\pi)$$

积分. 当然(1)的严格解释没有什么意义，f 和 μ 是定义在 T 上而

不是定义在实轴的一个区间上. 然而，如果 f 和 μ 定义在 T 上，且对 $\alpha \leqslant t < \alpha + 2\pi$, $E \subset [\alpha, \alpha + 2\pi)$，定义

(2) $$\tilde{f}(t) = f(e^{it}), \quad \tilde{\mu}(E) = \mu(\varphi(E))$$

此处 $\varphi(t) = e^{it}$，则

(3) $$\int_{\alpha}^{\alpha+2\pi} \tilde{f}(t)\, d\tilde{\mu}(t) = \int_T f\, d\mu$$

是正确的. 因此，记号(1)相当于将 $\tilde{f}$ 和 f、并将 $\tilde{\mu}$ 和 μ 视为相同，而且不应当引起混乱.

关于(1)优于 $\int_T f d\mu$ 的地方. 简单地说就是，形如(1)的积分处理起来要比较容易.

如同 4.23 节那样，我们继续把空间 $L^p(T)$ 和 $C(T)$ 与 R^1 上以 2π 为周期的相应的函数空间看作是同一的.

11.7 Poisson积分 如果 $f \in L^1(T)$ 且

(1) $$F(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(t)\, dt$$

则在 U 内这样定义的函数 F 称为 f 的 Poisson 积分. 我们有时将关系式(1)缩写为

(2) $$F = P[f]$$

类似地，在 T 上的复 Borel 测度 μ 的 Poisson 积分 $F = P[d\mu]$ 定义为

(3) $$F(re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\, d\mu(t)$$

如果把每一个 $f \in L^1(T)$ 对应于其不定积分 $\mu(E) = \int_E f(t)\, dt$，我们看出形式(1)的函数 F 组成由(3)定义的那类函数的一个子类.

如果 μ 是实的，则公式 11.5(2)表明 $P[d\mu]$ 是

(1) $$\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\mu(t) \qquad (z=re^{i\theta};\ z\in U)$$

的实部. 但根据定理 10.7,(4)定义了 U 内的一个全纯函数. 因此 $P[d\mu]$ 是一个调和函数. 由于调和函数的线性组合(关于常系数的)是调和函数,所以我们证明了下面的定理:

11.8 定理 对 T 上的每一个复 Borel 测度 μ,它的 Poisson 积分 $P[d\mu]$ 是 U 内的调和函数,因而对每一个 $f\in L^1(T)$, $P[f]$ 在 U 内是调和的.

下面的定理表明,连续函数的 Poisson 积分在接近 U 的边界时表现得特别理想.

11.9 定理 如果 $f\in C(T)$,并且在闭单位圆盘 $\bar{U}$ 上, Hf 由

(1) $$(Hf)(re^{i\theta})=\begin{cases}f(e^{i\theta}) & \text{当 } r=1\\ P[f](re^{i\theta}) & \text{当 } 0\leqslant r<1\end{cases}$$

定义,则 $Hf\in C(\bar{U})$.

证明 由于 $P_r(t)>0$,所以对每一个 $g\in C(T)$,公式 11.5(3)表明

(2) $$|P[g](re^{i\theta})|\leqslant\|g\|_T \qquad (0\leqslant r<1)$$

因此

(3) $$\|Hg\|_{\bar{U}}=\|g\|_T \qquad (g\in C(T))$$

(如同 5.22 节那样,我们用记号 $\|g\|_E$ 表示 $|g|$ 在集 E 上的上确界.)

如果

(4) $$g(e^{i\theta})=\sum_{n=-N}^{N}c_ne^{in\theta}$$

是任意三角多项式,则从 11.5(1)得到

(5) $$(Hg)(re^{i\theta})=\sum_{n=-N}^{N}c_nr^{|n|}e^{in\theta}$$

所以 $Hg\in C(\bar{U})$.

最后，存在三角多项式 g_k，使得当 $k\to\infty$ 时，$\|g_k-f\|_T\to 0$（参看 4.24 节）. 由(3)得到，当 $k\to\infty$ 时，

(6) $$\|Hg_k-Hf\|_{\bar{U}}=\|H(g_k-f)\|_{\bar{U}}\to 0$$

这就是说，函数 $Hg_k\in C(\bar{U})$ 在 $\bar{U}$ 上一致收敛于 Hf. 因此 $Hf\in C(\bar{U})$.

注：这个定理提供了一个边值问题(Dirichlet 问题)的解：在 T 上给出一个连续函数 f，要求在 U 内找出“边界值为 f”的调和函数 F. 借助于 f 的 Poisson 积分，定理提出了一个解答，而且更精确地表述了 f 和 F 之间的关系. 与这个存在性定理对应的唯一性定理包含在下面的结果中.

11.10 定理 设 u 是闭单位圆盘 $\bar{U}$ 上的连续实函数，且 u 在 U 内调和，则(在 U 内) u 是限制在 T 上的 Poisson 积分，且 u 是全纯函数

(1) $$f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}u(e^{it})\,dt \qquad (z\in U)$$

的实部.

证明 定理 10.7 证明了 $f\in H(U)$. 如果 $u_1=\operatorname{Re} f$，则(1)表明 u_1 是 u 的边界值的 Poisson 积分. 而我们一旦证实了 $u=u_1$，本定理也就得到证明.

设 $h=u-u_1$，则 h 在 $\bar{U}$ 上连续(应用定理11.9 于u_1)，h 在 U 内调和，并且对于 T 的所有点，$h=0$. 假设(这将引出矛盾)对某点 $z_0\in U$，$h(z_0)>0$. 固定 ε，使得 $0<\varepsilon<h(z_0)$，同时定义

(2) $$g(z)=h(z)+\varepsilon|z|^2 \qquad (z\in\bar{U})$$

则 $g(z_0)\geqslant h(z_0)>\varepsilon$. 由于 $g\in C(\bar{U})$ 且对于 T 的所有各点有 $g=\varepsilon$，所以存在一点 $z_1\in U$，在这点 g 有一个局部极大值. 这就推出在 z_1，$g_{xx}\leqslant 0$ 和 $g_{yy}\leqslant 0$. 但(2)表明 g 的 Laplace 算子值是 $4\varepsilon>0$. 这就得出了矛盾.

因此 $u-u_1\leqslant 0$. 相同的论证表明 $u_1-u\leqslant 0$, 因而 $u=u_1$, 并且完成了证明.

我们以下涉及到的, 将是对于固定的 θ, 当 $r\to 1$ 时 $P[d\mu](re^{i\theta})$ 的变化状况. 换言之, 我们准备研究测度的 Poisson 积分的径向极限(如果它们存在的话). 下面的引理包含了这一研究的计算部分.

11.11 引理 设 μ 是 T 上的实 Borel 测度. 固定 θ, 设

(1) $$J(\theta;s)=\{e^{it}:\theta-s<t<\theta+s\}$$

于是 $J(\theta;\ s)$ 是以 $e^{i\theta}$ 为中心长为 $2s$ 的开圆弧, 并且假定存在一个 δ, $0<\delta<\pi$, 和一个实数 A, 使得 $0<s<\delta$ 时,

(2) $$\mu(J(\theta;s))<2sA$$

如果 $F=P[d\mu]$, 则由这些条件可以推出

(3) $$F(re^{i\theta})<A+|A|P_r(\delta)+\frac{1}{2\pi}P_r(\delta)\|\mu\| \quad (0\leqslant r<1)$$

此处 $\|\mu\|=|\mu|(T)$ 是 μ 的全变差.

证明 已知

(4) $$F(re^{it})=\frac{1}{2\pi}\int_{\theta-\pi}^{\theta+\pi}P_r(\theta-t)d\mu(t)$$

将这积分分成两部分. 如果 $\delta\leqslant|\theta-t|\leqslant\pi$, 则

$$P_r(\theta-t)\leqslant P_r(\delta)$$

因而沿这个区域的积分不大于(3)的最后一项. 剩下沿 $I(\delta)=(\theta-\delta,\theta+\delta)$ 的积分, 应用 Fubini 定理, 可以通过分部积分法进行估计. 我们求

(5) $$P'_r(s)dsd\mu(t)$$

(微分是对于 s 来求的)沿三角形

(6) $$\{(s,t):\theta-s<t<\theta+s,0<s<\delta\}$$

的积分, 得到

(7) $$\int_0^\delta \mu(I(s))P_r'(s)ds=\int_{I(\delta)}[P_r(\delta)-P_r(\theta-t)]d\mu(t)$$

先对 t 进行积分得到(7)的左边;先对 s 进行积分得到(7)的右边. 因为根据假设(注意, 如同 11.6 节那样, 我们认为 μ 和 $\tilde{\mu}$ 是同一的), 有 $\mu(I(s))<2sA$,又由于在$(0,\pi)$上$P_r'(s)<0$(从不等式11.5(4)得到), 所以从(7)可以断定

$$\begin{aligned}\int_{I(\delta)}P_r(\theta-t)d\mu(t)&=P_r(\delta)\mu(I(\delta))+\int_0^\delta\mu(I(s))[-P_r'(s)]ds\\&<2A\left[\delta P_r(\delta)-\int_0^\delta sP_r'(s)ds\right]\\&=2A\int_0^\delta P_r(s)ds\\&=2\pi A-2A\int_\delta^\pi P_r(s)ds\\&\leqslant 2\pi A+2\pi|A|P_r(\delta)\end{aligned}$$

这就得出式(3).

11.12 定理 设 μ 是 T 上的实 Borel 测度, $J(\theta;\ s)$ 与引理 11.11 中的相同. 定义

(1) $$(\bar{D}\mu)(\theta)=\limsup_{s\to 0}\frac{\mu(J(\theta;s))}{2s}$$

类似地, 用 lim 和 liminf 代替 limsup 而定义$(D\mu)(\theta)$和$(\underline{D}\mu)(\theta)$. 设

(2) $$F(re^{i\theta})=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)d\mu(t)\qquad(0\leqslant r<1)$$

则对每个 θ,

(3) $$(\underline{D}\mu)(\theta)\leqslant\liminf_{r\to 1}F(re^{i\theta})\leqslant\limsup_{r\to 1}F(re^{i\theta})\leqslant(\bar{D}\mu)(\theta)$$

同时

(4) $$\lim_{r\to 1} F(re^{i\theta}) = (D\mu)(\theta)$$

存在且对几乎所有的 θ 是有限的(关于 Lebesgue 测度).

注意,$(D\mu)(\theta)$在这里定义为 μ 在 θ 处的对称导数,而且对于在第八章中研究过的更一般意义下不是可微的那些点 θ,它仍然有可能是存在的.

证明 固定 θ,并设 $A>(\bar{D}\mu)(\theta)$. 则存在 $\delta>0$ 使得引理 11.11 的假设成立. 当 $r\to 1$ 时,$P_r(\delta)\to 0$,于是根据引理 11.11 得到(3)式中最后面的不等式. 如果用 $-\mu$ 代替 μ 并应用最后一个不等式,则得到(3)式中第一个不等式,而(4)是定理 8.6 在一维情况下的推论.

系 如果 $f\in L^1(T)$,且 $F=P[f]$,则

$$\lim_{r\to 1} F(re^{i\theta}) = f(e^{i\theta}) \quad \text{a. e.}$$

由于 f 几乎处处等于它的不定积分的导数,所以从(4)得到上式.

11.13 到现在为止,我们只研究了单位圆盘 $U=D(0;\ 1)$的情况. 显然,通过一个简单的变量代换,以前的论述可以转移到任意圆盘上. 因此我们只扼要讲一些结果:

如果 u 是圆盘 $D(a;R)$边界上的连续实函数,且在 $D(a;\ R)$内由Poisson 积分

(1) $$u(a+re^{i\theta}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2-r^2}{R^2-2Rr\cos(\theta-t)+r^2} u(a+Re^{it})dt$$

定义 u,则 u 在 $\bar{D}(a;R)$上连续而在 $D(a;R)$内调和.

如果 u 在开集 Ω 内调和(而且是实的),又 $\bar{D}(a;\ R)\subset\Omega$,则在 $D(a;R)$内 u 满足(1),且存在一个定义在 $D(a;\ R)$内,其实部为 u 的全纯函数 f,在纯虚的附加常数范围内,这个 f 唯一地确定. 因为如果在同一区域内的两个全纯函数有相同的实部,则它们的差

必为常数(开映射定理或 Cauchy-Riemann 方程的推论).

我们把这些概括地说成每一个实调和函数局部地都是全纯函数的实部. 因此,每一个调和函数具有任意阶的连续偏导数.

Poisson 积分同样提供了关于调和函数序列的信息.

11.14 Harnack 定理 设$\{u_n\}$是区域Ω内的调和函数序列.

(*a*) 如果$u_n \to u$在Ω的紧子集上是一致的,则u在Ω内调和.

(*b*) 如果$u_1 \leqslant u_2 \leqslant u_3 \leqslant \cdots$,则或者$\{u_n\}$在$\Omega$的紧子集上一致收敛,或者对每一个$z \in \Omega$, $u_n(z) \to \infty$.

证明 为了证明(*a*),假设$\bar{D}(a;R) \subset \Omega$,且在 Poisson 积分 11.13(1)中用u_n代替u. 由于$u_n \to u$在$\bar{D}(a;R)$的边界上是一致的,所以我们断定在$D(a;R)$内u满足 11.13(1).

在(*b*)的证明中,我们可以假定$u_1 \geqslant 0$(否则用$u_n - u_1$代替u_1). 设$u = \sup u_n$, $A = \{z \in \Omega : u(z) < \infty\}$,且$B = \Omega - A$. 选取$\bar{D}(a;R) \subset \Omega$. 对$0 \leqslant r < R$, Poisson 核满足不等式

$$\frac{R-r}{R+r} \leqslant \frac{R^2-r^2}{R^2-2rR\cos(\theta-t)+r^2} \leqslant \frac{R+r}{R-r}$$

因此

$$\frac{R-r}{R+r}u_n(a) \leqslant u_n(a+re^{it}) \leqslant \frac{R+r}{R-r}u_n(a)$$

用u代替u_n,不等式同样成立. 由此可见,对所有$z \in D(a;R)$, $u(z) = \infty$,或者对所有$z \in D(a;R)$, $u(z) < \infty$.

因此A和B都是开集;又由于Ω是连通的,所以或者有$A = \varnothing$(这种情况下就没什么要证的),或者$A = \Omega$. 在后一情况下,单调收敛定理表明在Ω的每一圆盘内, Poisson 公式对u成立. 因此u在Ω内调和. 当连续函数序列单调收敛于一个连续函数时,在

紧集上收敛一定是一致的([26] 定理 7.13). 这便完成了定理的证明.

平均值性质

11.15 定义 如果对每一个 $z\in\Omega$, 对应一个序列$\{r_n\}$, 使得 $r_n>0$, 当 $n\to\infty$ 时, $r_n\to 0$, 且

$$(1)\qquad u(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}u(z+rne^{it})dt \qquad (n=1,2,3,\cdots)$$

则称在开集 Ω 内的连续函数 u 具有平均值性质. 换句话说, $u(z)$ 等于 u 在以 z 为中心、半径为 r_n 的圆周上的平均值.

注意, Poisson 公式表明, 对每一个调和函数 u, 且对每个 r, 使得 $\bar{D}(z;r)\subset\Omega$, (1)是成立的. 所以调和函数满足一个比刚才定义的还要强些的平均值性质. 因此下述定理可能会使人感到意外.

11.16 定理 如果连续函数 u 在开集Ω内具有平均值性质, 则 u 在 Ω 内调和.

证明 为证此定理, 只须对实的 u 证明就够了. 固定 $\bar{D}(a;R)\subset\Omega$. Poisson 积分给出一个在 $\bar{D}(a;R)$上连续、在 $D(a;R)$ 内调和、而在 $D(a;R)$的边界上与 u 重合的函数 h. 令 $v=u-h$, 且令 $m=\sup\{v(z):z\in\bar{D}(a;R)\}$. 假设 $m>0$, 并且设E是使 $v(z)=m$ 的所有 $z\in\bar{D}(a;R)$的集. 因为在 $D(a;R)$的边界上 $v=0$, 所以E是 $D(a;R)$的一个紧子集. 因此存在一个 $z_0\in E$, 使得对所有 $z\in E$,

$$|z_0-a|\geqslant|z-a|$$

对于所有足够小的 r, 以 z_0 为中心, r为半径的圆至少有一半在 E 的外部, 所以对应的 v 的平均值都小于 $m=v(z_0)$. 但 v 具有平均值性质, 因而得出了矛盾. 这样, $m=0$, 故 $v\leqslant 0$. 对 $-v$ 可应用相同的论证. 因此有 $v=0$, 或在 $D(a;R)$内 $u=h$; 又由于 $\bar{D}(a;R)$是

Ω 内一个任意的闭圆盘. 所以 u 在 Ω 内调和.

定理 11.16 导出了全纯函数的一个反射定理. 所谓上半平面 Π^+ 是指使 $y>0$ 的所有 $z=x+iy$ 的集;而下半平面 Π^- 则是由虚部为负的所有 z 组成.

11.17 定理(Schwarz 反射原理) 设 L 是实轴上一条线段, Ω^+ 是 Π^+ 的区域,且每个 $t\in L$ 是一个开圆盘 D_t 的中心,使得 $\Pi^+\cap D_t$ 位于 Ω^+ 内. 令 Ω^- 是 Ω^+ 的反射:

(1) $$\Omega^-=\{z:\bar{z}\in\Omega^+\}$$

设 $f=u+iv$ 在 Ω^+ 内全纯,且对收敛于 L 的某个点的 Ω^+ 内的每一个序列 $\{z_n\}$,有

(2) $$\lim_{n\to\infty} v(z_n)=0$$

则在 $\Omega^+\cup L\cup\Omega^-$ 内有一个全纯函数 F,在 Ω^+ 内使得 $F(z)=f(z)$;这个 F 满足关系

(3) $$F(\bar{z})=\overline{F(z)} \qquad (z\in\Omega^+\cup L\cup\Omega^-)$$

定理断定 f 能扩展为关于实轴对称的一个区域内的全纯函数,且(3)指出 F 保持这个对称性. 注意,连续性假设(2)只是对于 f 的虚部提出的.

证明 设 $\Omega=\Omega^+\cup L\cup\Omega^-$, 当 $z\in L$ 时,定义 $v(z)=0$; 当 $z\in\Omega^-$ 时,定义 $v(z)=-v(\bar{z})$. 由此我们把 v 扩展到 Ω 上. 这就立即得出,在 Ω 内 v 连续且具有平均值性质. 所以根据定理 11.16, v 在 Ω 内调和.

因此 v 在局部上是一个全纯函数的虚部. 这就意味着对每个圆盘 D_t 对应一个 $f_t\in H(D_t)$, 使得 $\operatorname{Im} f_t=v$. 在附加一个实常数的范围内,每个 f_t 由 v 所确定. 由于 $f-f_t$ 在区域 $D_t\cap\Pi^+$ 内是常数,若这个常数选择得使之对某个 $z\in D_t\cap\Pi^+$ 有 $f_t(z)=f(z)$, 则对所有 $z\in D_t\cap\Pi^+$, 结论同样成立. 我们假定对函数 f_t 是作过

这样修正的.

由于在 L 上 $v=0$，所以 f_t 在 t 处的所有各阶导数都是实数，故 f_t 按 $z-t$ 的幂级数展开式只有实系数. 由此可见

$$f_t(\bar{z})=\overline{f_t(z)} \qquad (z\in D_t) \tag{4}$$

其次，设 $D_s\cap D_t\neq\varnothing$，则在 $D_t\cap D_s\cap \Pi^+$ 内 $f_t=f=f_s$；又由于 $D_t\cap D_s$ 是连通的，所以定理 10.18 表明

$$f_t(z)=f_s(z) \qquad (z\in D_t\cap D_s) \tag{5}$$

因此定义

$$F(z)=\begin{cases} f(z) & \text{当 } z\in\Omega^+ \\ f_t(z) & \text{当 } z\in D_t \\ \overline{f(z)} & \text{当 } z\in\Omega^- \end{cases} \tag{6}$$

是相容的，而尚待证明的是 F 在 Ω^- 内全纯. 如果 $D(a;\ r)\subset\Omega^-$，则 $D(\bar{a};r)\subset\Omega^+$，所以对每个 $z\in D(a;r)$，有

$$f(\bar{z})=\sum_{n=0}^{\infty}c_n(\bar{z}-\bar{a})^n \tag{7}$$

因此

$$F(z)=\sum_{n=0}^{\infty}\bar{c}_n(z-a)^n \qquad (z\in D(a;r)) \tag{8}$$

这就完成了证明.

正调和函数

11.18 显然，在单位圆周 T 上的每一个正有限 Borel 测度 μ 的 Poisson 积分 $P[d\mu]$ 是开圆盘 U 内的正调和函数. 问题是，每个 U 内的正调和函数是否都能按这种方式得到？

如下的观察将导致一个肯定的回答. 如果 $h=P[d\mu]$，此处 μ 是 T 上任意一个复 Borel 测度，则 Fubini 定理表明

(1)
$$\int_{-\pi}^{\pi}|h(re^{i\theta})|d\theta$$
$$\leqslant\int_{-\pi}^{\pi}\left[\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)d\theta\right]d|\mu|(t)$$
$$=\|\mu\|$$

此处$\|\mu\|$是μ在T上的全变差. 因此, 式(1)的第一个积分在$[0,1)$上是r的有界函数, 这是所有 Poisson 积分必须满足的条件.

现在, 如果h是正调和函数, 则$|h|=h$, 所以对于每一个$r\in[0,1)$, 式(1)的第一个积分等于$2\pi h(0)$. 因此, 正调和函数满足我们刚才找出的必要条件, 并且导出如下更强的定理.

11.19 定理 映射$\mu\to P[d\mu]$是T上所有复 Borel 测度的空间和所有满足增长条件

(1)
$$\sup_{0\leqslant r<1}\int_{-\pi}^{\pi}|h(re^{i\theta})|d\theta<\infty$$

的U内调和函数h的空间之间的一个线性一一对应.

T上所有正有限 Borel 测度集因此与U内所有正调和函数集构成一一对应.

系 每一个满足(1)的U内调和函数h关于 Lebesgue 测度在T的几乎所有各点都有径向极限.

证明 显然, 映射$\mu\to P[d\mu]$是线性的. 第 11.18 节的讨论证实, 对于每一个$P[d\mu]$, (1)是成立的.

为了证明$\mu\to P[d\mu]$是一一的, 设在U内$P[d\mu]=0$, 我们必须证明$\mu=0$. 选取$f\in C(T)$, 定理 11.9 表明

$$\int_{-\pi}^{\pi}f(\theta)d\mu(\theta)=\lim_{r\to1}\int_{-\pi}^{\pi}d\mu(\theta)\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)P_r(\theta-t)dt$$
$$=\lim_{r\to1}\int_{-\pi}^{\pi}f(t)dt\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)d\mu(\theta)$$
$$=\lim_{r\to1}\int_{-\pi}^{\pi}f(t)P[d\mu](re^{it})dt=0$$

因此对每个 $f\in C(T)$, $\int f d\mu=0$. 由定理 6.19 的唯一性结论可推出 $\mu=0$.

我们着手证明定理的最重要部分: 每一个满足(1)的 U 内调和函数都是 T 上某个复 Borel 测度 μ 的 $P[d\mu]$. 这一证明依赖于"弱收敛"的概念. 在习题 18 中, 它将在更一般的情况下作出定义.

令 h 为 U 内的调和函数, 满足

$$\int_{-\pi}^{\pi}|h(re^{i\theta})|d\theta\leqslant M<\infty \qquad (0\leqslant r<1) \tag{2}$$

对于 $0<s<1$, 定义

$$\Lambda_s f=\int_{-\pi}^{\pi}h(se^{it})f(t)dt \qquad (f\in C(T)) \tag{3}$$

每个 Λ_s 因此是 $C(T)$ 上的一个线性泛函, 其范数 $\|\Lambda_s\|\leqslant M$. 设 $\{f_j\}$ 是 $C(T)$ 的一个可数稠密子集(例如, 取系数在复平面的某个可数稠密子集内的所有三角多项式). 对每个 j, 数 $\Lambda_s f_j$ 组成一个有界集. 因此, 对角线过程(参看[26]p. 145)确保存在一个序列 $\{s_n\}$, 使 $s_n\to 1$, 且对每个 f_j,

$$\lim_{n\to\infty}\Lambda_{s_n}f_j \tag{4}$$

存在.

对每个 $f\in C(T)$, 我们断言 $\{\Lambda_{s_n}f\}$ 是 Cauchy 序列: 固定 $f\in C(T)$, $\varepsilon>0$, 选取 f_j 使 $\|f-f_j\|_\infty<\varepsilon/M$, 然后选取这样大的 N, 使 $n>N$、$m>N$ 时,

$$|\Lambda_{s_n}f_j-\Lambda_{s_m}f_j|<\varepsilon \tag{5}$$

因为

$$\begin{aligned}|\Lambda_{s_n}f-\Lambda_{s_m}f|\leqslant{}&|\Lambda_{s_n}(f-f_j)|+|\Lambda_{s_n}f_j-\Lambda_{s_m}f_j|\\&+|\Lambda_{s_m}(f_j-f)|\end{aligned}$$

若 $n>N$ 和 $m>N$, 则 $|\Lambda_{s_n}f-\Lambda_{s_m}f|<3\varepsilon$. 因此对每个 $f\in C(T)$,

$$Af=\lim_{n\to\infty}A_{s_n}f \tag{6}$$

存在，且(6)定义了T上一个线性泛函，其范数$\|A\|\leqslant M$. Riesz 表示定理 6.19 现在引出T上一个复 Borel 测度μ，使得关系式

$$\lim_{n\to\infty}\int_{-\pi}^{\pi}f(t)h(s_ne^{it})\,dt=\int_{-\pi}^{\pi}f(t)d\mu(t) \tag{7}$$

对每个$f\in C(T)$成立；这可以从(3)和(6)得到.

对每个n，$h(s_nre^{i\theta})$在包含$\bar{U}$的圆盘$D(0;1/s_n)$内调和. 因此在U内这个函数是它限制于T上的 Poisson 积分. 换言之，

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)h(s_ne^{it})dt=h(s_nre^{i\theta})\qquad(0\leqslant r<1) \tag{8}$$

如果在(8)中令$n\to\infty$并应用(7)，则得

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)d\mu(t)=h(re^{i\theta})\qquad(0\leqslant r<1,\ -\pi\leqslant\theta\leqslant\pi) \tag{9}$$

这便是所期望的h的表示式：$h=P[d\mu]$.

根据原先确立的唯一性，如果h是实的，则μ也是实的；如果$h>0$，则每个A_s是T上一个正线性泛函，因之A亦复如此. 而μ的正性可由 Riesz 表示定理 2.14 得到.

这便完成了定理的证明，而系则是定理 11.12 的一个推论.

这个定理当然有许多应用. 我们现在提出的一个应用表明对于U内的有界全纯函数存在一个 Cauchy 公式，其积分路径被移动到T上. 其它的应用见第十七章.

11.20 定义 令H^∞为U内所有有界全纯函数的空间，由

$$\|f\|_\infty=\sup_{z\in U}|f(z)|$$

赋范. 如同前面一样，$L^\infty(T)$是T上所有本性有界函数(等价类)的空间，由本性上确界范数（相对于 Lebesgue 测度）赋范. 当

$g\in L^\infty(T)$时，用$\|g\|_\infty$表示$|g|$的本性上确界.

11.21 定理 对于每一个$f\in H^\infty$，对应于一个几乎处处由

(1) $$f^*(e^{it})=\lim_{r\to 1}f(re^{it})$$

定义的函数$f^*\in L^\infty(T)$. 则等式$\|f\|_\infty=\|f^*\|_\infty$成立. 对每个$z\in U$，Cauchy 公式

(2) $$f(z)=\frac{1}{2\pi i}\int_\gamma \frac{f^*(\xi)}{\xi-z}d\xi$$

成立，其中γ是正向单位圆周：

$$\gamma(t)=e^{it},\quad 0\leqslant t\leqslant 2\pi.$$

照这种方式得到的函数$f^*\in L^\infty(T)$正好是满足关系式

(3) $$\frac{1}{2\pi}\int_{-\pi}^{\pi}f^*(e^{it})e^{-int}dt=0$$

$$(n=-1,-2,-3,\cdots)$$

的那些函数.

每个$f\in H^\infty$在T的几乎所有各点都有径向极限这一点，曾由 Fatou 在把 Lebesgue 积分用于全纯函数研究的一个最早期的应用中证明.

证明 f^*几乎处处存在可从定理 11.19 的系得到，而由(1)显然有$\|f^*\|_\infty\leqslant\|f\|_\infty$.

如果$z\in U$，且$|z|<r<1$，设$\gamma_r(t)=re^{it}(0\leqslant t\leqslant 2\pi)$，则

(4) $$\begin{aligned}f(z)&=\frac{1}{2\pi i}\int_{\gamma_r}\frac{f(\xi)}{\xi-z}d\xi\\&=\frac{r}{2\pi}\int_{-\pi}^{\pi}\frac{f(re^{it})}{re^{it}-z}e^{it}dt\end{aligned}$$

令$\{r_n\}$为一个序列，$r_n\to 1$，并应用 Lebesgue 控制收敛定理到(4)中最后一个积分上，结果得

(5) $$f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f^*(e^{it})}{1-ze^{it}}dt$$

这与(2)相同，Cauchy 定理给出

$$\int_{\gamma_r} f(\xi)\xi^n d\xi = 0 \qquad (n=0,1,2,\cdots) \tag{6}$$

与上面类似，通过取极限证实 f^* 满足(3)．因此我们可以把(5)写成 Poisson 积分：如果 $z=re^{i\theta}$，则

$$\begin{aligned} f(z) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(e^{it}) \sum_{n=0}^{\infty} r^n e^{in(\theta-t)}\,dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(e^{it}) \sum_{n=-\infty}^{\infty} r^{|n|} e^{in(\theta-t)}\,dt \qquad (7) \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta-t) f^*(e^{it})\,dt \end{aligned}$$

据此，我们断定 $\|f\|_\infty \leqslant \|f^*\|_\infty$，所以这两个范数相等．

最后，如果 $f^* \in L^\infty(T)$ 使得(3)成立，且对所有 $z\in U$，由(5)定义 f，则(5)表明 $f\in H(U)$，(3)蕴涵着 Cauchy 积分 (5)等于 Poisson 积分(7)．因而 f 是有界的，并且根据定理 11.12 的系，f 的表示式作为 f^* 的 Poisson 积分指明了(1)几乎处处成立．

根据上述 Cauchy 公式容易得到一个唯一性定理．

11.22 定理 设 $f\in H^\infty$，J 是 T 的子弧，且在 J 上几乎处处有 $f^*(e^{it})=0$，则对所有 $z\in U$，$f(z)=0$．

（在稍后的定理 15.19 中，将得到一个相当强的论断．也可参看定理 17.18 和 17.19 节）．

证明 令 n 为正整数，使 J 的长度大于 $2\pi/n$，令 $\eta=\exp\{2\pi i/n\}$，且设

$$g(z) = \prod_{k=1}^{n} f(\eta^k z) \qquad (z\in U) \tag{1}$$

由于 f 有界，且在 J 上几乎处处有 $f^*=0$，所以看出在 T 上几乎处处有 $g^*=0$，且 $g\in H^\infty$．由于 g 是 g^* 的 Cauchy 积分，故对所有

$z\in U, g(z)=0$. 如果 U 内 f 的零点集 $Z(f)$至多是可数的，则由于 $Z(g)$是由 $Z(f)$经旋转得到的 n 个集的并，所以 $Z(g)$也至多是可数的. 但 $Z(g)=U$. 因此根据定理 10.18, $f=0$.

习　题

1 设 u 和 v 是平面区域 Ω 内的实调和函数. 在什么条件下 uv 是调和的？(注意答案强烈地依赖于本问题是就实函数提出的这一事实.) 证明 u^2 在 Ω 内不是调和的，除非 u 为常数. 对哪些 $f\in H(\Omega)$, $|f|^2$ 是调和的？

2　设 f 是区域 Ω 内的复函数，且 f 和 f^2 均在 Ω 内调和. 证明：或者 f 或者 $\bar{f}$ 在 Ω 内是全纯的.

3　如果 u 是区域 Ω 内的调和函数. 对于使 u 的梯度为零的点集你能说些什么？(这个集就是使 $u_x=u_y=0$ 的点集).

4　证明调和函数的每个偏导数是调和的.

通过直接计算验证对每个固定的 t, $P_r(\theta-t)$ 是 $re^{i\theta}$ 的一个调和函数. 通过证明 $P[d\mu]$的每个偏导数等于对应的核的偏导数的积分，推证(不涉及全纯函数) T 上每个有限 Borel 测度 μ 的 Poisson 积分 $P[d\mu]$在 U 内调和.

5　设 $f\in H(\Omega)$，且 f 在 Ω 内没有零点. 通过计算 $\log|f|$的 Laplace 算子值证明 $\log|f|$在 Ω 内调和. 有没有较容易的方法？

6　设 $f\in H(U)$，此处 U 是开单位圆盘，f 在 U 内是一一的，$\Omega=f(U)$，且 $f(z)=\sum c_n z^n$. 证明 Ω 的面积是

$$\pi\sum_{n=1}^{\infty}n|c_n|^2$$

提示：f 的 Jacobi 行列式是 $|f'|^2$.

7　(a) 如果 $f\in H(\Omega)$，对 $z\in\Omega$. $f(z)\neq 0$，且 $-\infty<\alpha<\infty$. 通过证明公式

$$\partial\bar{\partial}(\psi\circ(f\bar{f}))=(\varphi\circ|f|^2)\cdot|f'|^2$$

其中 ψ 在 $(0,\infty)$ 上是二次可微的，且

$$\varphi(t)=t\psi''(t)+\psi'(t)$$

来证明

$$\Delta(|f|^\alpha)=\alpha^2|f|^{\alpha-2}|f'|^2$$

(b) 设 $f\in H(\Omega)$，而 Φ 是定义域为 $f(\Omega)$的复函数，它有连续二阶偏导

数. 证明

$$\Delta[\Phi\circ f]=[(\Delta\Phi)\circ f]\cdot|f'|^2$$

证明在 $\Phi(w)=\Phi(|w|)$ 的特殊情况下，上式就是(a)的结果.

8 利用引理 11.11 的证明方法可以引出一个一般的定理，它使得对形如 $\int_I f d\mu$ 的积分应用分部积分法成为合法的，此处 μ 是实直线的线段 I 上的一个 Borel 测度. 叙述并证明这样的一个定理.

9 设 u 是区域 Ω 内的一个 Lebesgue 可测函数，且 u 局部地属于 L^1. 这意味着 $|u|$ 在 Ω 的任何紧子集上的积分是有限的. 如果 u 满足如下形式的平均值性质：当 $\overline{D}(a;r)\subset\Omega$ 时，

$$u(a)=\frac{1}{\pi r^2}\iint_{D(a;r)}u(x,y)\,dxdy$$

证明 u 是调和的.

10 设 $I=[a,b]$ 是实轴上的一个区间，φ 是 I 上的连续函数，且

$$f(z)=\frac{1}{2\pi i}\int_a^b\frac{\varphi(t)}{t-z}dt\qquad(z\notin I)$$

证明对每个实数 x

$$\lim_{\varepsilon\to 0}[f(x+i\varepsilon)-f(x-i\varepsilon)]\qquad(\varepsilon>0)$$

存在，并用 φ 求出它.

如果我们只假定 $\varphi\in L^1$，则对结果有怎样的影响？在 φ 具有左极限和右极限的点 x 处会怎么样？

11 设 $I=[a,b]$，Ω 是一个区域，$I\subset\Omega$，f 在 Ω 内连续，且 $f\in H(\Omega-I)$. 证明实际上 $f\in H(\Omega)$.

用另一些集代替 I，使之可以推断同样的结论.

12 设 Ω 是区域，K 是 Ω 的紧子集，$z_0\in\Omega$. 证明存在正数 α 和 β（依赖 z_0，K 和 Ω），使得对 Ω 内每个正调和函数及对所有 $z\in K$，

$$\alpha u(z_0)\leqslant u(z)\leqslant\beta u(z_0)$$

如果 $\{u_n\}$ 是 Ω 内一个正调和函数序列，且 $u_n(z_0)\to 0$，描述 $\{u_n\}$ 在 Ω 的剩余部分的状况，对 $u_n(z_0)\to\infty$ 作同样的叙述. 说明 $\{u_n\}$ 的正性这一假定对这些结果是本质的.

13 设 u 是 U 内的正调和函数且 $u(0)=1$. $u\left(\frac{1}{2}\right)$ 可以有多大？可以有多小？试给出最好的可能的界限.

14 设 μ 是 T 上不恒为零的正 Borel 测度，且 μ 是奇异的(相对于 Lebesgue 测度)．如果 $u=P[d\mu]$，证明至少有一个 θ，当 $r\to 1$ 时 $u(re^{i\theta})\to\infty$．提示：定理 8.10.

15 设 u 是 U 内的正调和函数，且对每个 $e^{i\theta}\neq 1$，当 $r\to 1$ 时 $u(re^{i\theta})\to 0$．证明存在常数 c，使得

$$u(re^{i\theta})=cP_r(\theta)$$

16 这里是 U 内不恒为零但其所有的径向极限为零的调和函数的例子：

$$u(z)=\mathrm{Im}\left[\left(\frac{1+z}{1-z}\right)^2\right]$$

证明 u 不是 T 上任何测度的 Poisson 积分，并且不是 U 内两个正调和函数的差．

17 令 Φ 为 U 内使 $u(0)=1$ 的所有正调和函数的集．证明 Φ 是凸集并找出 Φ 的极值点．(称凸集 Φ 内的点 x 为 Φ 的一个极值点，如果对于在 Φ 内的任何线段，其两个端点都与 x 不同时，x 就不会落在这个线段上．)提示：如果 C 是凸集，该集的元素是 T 上全变差为1的正 Borel 测度，证明 C 的极值点恰好是那些 $\mu\in C$，使得 μ 的支集仅由 T 的一个点组成．

18 令 X^* 为 Banach 空间 X 的对偶空间．如果对每个 $x\in X$，当 $n\to\infty$ 时，$\Lambda_n x\to\Lambda x$，则称 X^* 内的序列 $\{\Lambda_n\}$ 弱收敛于 $\Lambda\in X^*$．注意，当按 X^* 范数 $\Lambda_n\to\Lambda$ 时，则一定在弱收敛意义下 $\Lambda_n\to\Lambda$．(参看第五章习题 8.)逆命题不一定正确．例如，在 $L^2(T)$ 上，泛函 $f\to\hat{f}(n)$ 弱趋于 0 (根据 Bessel 不等式)，但这些泛函的每一个都具有范数 1.

(a) 如果 $\{\Lambda_n\}$ 弱收敛，证明 $\{\|\Lambda_n\|\}$ 必定有界．

(b) 设 X 是一个可分的 Banach 空间，且 $\{\Lambda_n\}$ 是 X^* 内的一个序列，使得 $\{\|\Lambda_n\|\}$ 有界．证明 $\{\Lambda_n\}$ 包含一个弱收敛子序列．(其证明与定理 11.19 的特殊极为相似．)

19 如果 $h=P[d\mu]$，且 $f\in C(T)$，证明

$$\lim_{r\to 1}\int_{-\pi}^{\pi} f(t)h(re^{it})\,dt=\int_{-\pi}^{\pi} f\,d\mu$$

(把这个结果与定理 11.19 的证明相比较．)

20 设 $f\in L^p(T)$，$1\leqslant p<\infty$，且 $F=P[f]$．证明

$$\lim_{r\to 1}\int_{-\pi}^{\pi}|F(re^{i\theta})-f(e^{i\theta})|^p d\theta=0$$

(与定理 9.5 和 9.10 作比较．第三章习题 17 在这里也可以应用．)

21 如果 $1 \leqslant p < \infty$ 和 $f \in L^p(R^1)$，证明 $(f * h_\lambda)(x)$ 是上半平面内 $x+i\lambda$ 的一个调和函数. (h_λ 在 9.7 节中定义；它是半平面内的 Poisson 核.)

22 设 Ω 是一个区域，对 $n=1, 2, 3, \cdots$，$f_n \in H(\Omega)$，u_n 是 f_n 的实部，$\{u_n\}$ 在 Ω 的紧子集上一致收敛，且对至少一个点 $z \in \Omega$，$\{f_n(z)\}$ 收敛. 证明 $\{f_n\}$ 在 Ω 的紧子集上一致收敛.

第十二章　最大模原理

引言

12.1　最大模定理(10.24)断定常数是其绝对值在域Ω的任意点具有局部极大值的唯一的全纯函数.

我们复述如下：如果K是有界域Ω的闭包，若f在K上连续而在Ω内全纯，则对每一$z\in\Omega$

(1)　　$$|f(z)|\leqslant\|f\|_{\partial\Omega}$$

如果在一点$z\in\Omega$等式成立，则f是常数.

[(1)的右边是$|f|$在Ω的边界$\partial\Omega$上的上确界.]

因为如果在某点$z\in\Omega$，$|f(z)|\geqslant\|f\|_{\partial\Omega}$，则$|f|$在$K$上的最大值实际上在$\Omega$的某点达到.（由于$K$是紧的，此最大值在$K$的某点达到.）根据定理10.24，所以$f$是常数.

等式$\|f\|_\infty=\|f^*\|_\infty$是定理11.21的一部分，它蕴涵

(2)　　$$|f(z)|\leqslant\|f^*\|_\infty\qquad(z\in U,f\in H^\infty(U))$$

类似于(1)的陈述，这就是说(大略地说)，$|f(z)|$不大于f的边界值的上确界.但这时不需要f在$\bar{U}$上连续，而只要在U上有界就够了.

本章包括最大模定理的进一步的推广及一些相当显著的应用，并用一个定理作为结束，它指出最大值性质"几乎"完全刻划了全纯函数类.

Schwarz引理

这是通常给如下定理的一个称呼.　我们仍然使用定义11.20

中规定的记号.

12.2 定理 设 $f\in H^\infty$, $\|f\|_\infty\leqslant 1$, 且 $f(0)=0$, 则

(1) $$|f(z)|\leqslant|z| \quad (z\in U)$$

(2) $$|f'(0)\leqslant 1$$

如果对一个 $z\in U-\{0\}$, (1)的等号成立, 或者(2)的等号成立, 则 $f(z)=\lambda z$, 此处 λ 是常数, $|\lambda|=1$.

用几何语言来说, 这一假设就是说 f 是保持原点不动的 U 到 U 内的一个全纯映射. 结论部分则是说或者 f 是一个旋转, 或者 f 将每一个 $z\in U-\{0\}$ 变到比原来更靠近原点的地方.

证明 由于 $f(0)=0$, 所以 $f(z)/z$ 在 $z=0$ 有一个可去奇点. 因此存在一个 $g\in H(U)$, 使得 $f(z)=zg(z)$. 由于 $1/|z|$ 在接近 U 的边界时是有界的, 所以 $g\in H^\infty$. 其径向极限满足关系式 $|g^*|=|f^*|$. 因此 $\|g^*\|_\infty\leqslant 1$. 根据定理 11.21, $\|g\|_\infty\leqslant 1$, 这时, 最大模定理表明: 当且仅当 g 是常数时, 对某个 $z\in U$, $|g(z)|=1$. 既然 $f'(0)=g(0)$, 这就完成了证明.

注: 我们没有真正使用径向极限在 T 上几乎处处存在; 我们能够在圆盘 $D(0;\ 1-\varepsilon)$ 上对 g 应用定理 10.24 并令 $\varepsilon\to 0$ 而得到最后结果.

藉助如下把 U 映到 U 上的映射, 可以得到 Schwarz 引理的许多变形.

12.3 定义 对任意 $\alpha\in U$, 定义

$$\varphi_\alpha(z)=\frac{z-\alpha}{1-\bar{\alpha}z}$$

12.4 定理 固定 $\alpha\in U$. 则 φ_α 是将 T 映到 T 上, U 映到 U 上和将 α 映为 0 的一个一一映射. φ_α 的逆映射为 $\varphi_{-\alpha}$. 我们有

(1) $$\varphi'_\alpha(0)=1-|\alpha|^2,$$

$$\varphi_a'(\alpha)=\frac{1}{1-|\alpha|^2}$$

证明 除去位于 $\bar{U}$ 外面的一个极点 $1/\bar{\alpha}$ 外，φ_a 在整个平面内全纯. 用直接的代换证得

(2) $$\varphi_{-a}(\varphi_a(z))=z$$

因此，φ_a 是一、一的，而且 φ_{-a} 是它的逆映射. 由于对实数 t，

(3) $$\left|\frac{e^{it}-\alpha}{1-\bar{\alpha}e^{it}}\right|=\left|\frac{e^{it}-\alpha}{e^{-it}-\bar{\alpha}}\right|=1$$

(z 和 $\bar{z}$ 有相同的绝对值)，所以 φ_a 把 T 映射到 T 内；对 φ_{-a} 有相同的结论；因而 $\varphi_a(T)=T$. 根据最大模定理得到 $\varphi_a(U)\subset U$，且考虑到 φ_{-a}，证得实际上有 $\varphi_a(U)=U$.

12.5 一个极值问题 设 α 和 β 是复数，$|\alpha|<1$ 和 $|\beta|<1$. 如果 f 满足条件：$f\in H^\infty$，$\|f\|_\infty\leqslant 1$ 和 $f(\alpha)=\beta$，问 $|f'(\alpha)|$ 能取到多大？

为解决这个问题，设

(1) $$g=\varphi_\beta\circ f\circ\varphi_{-a}$$

由于 φ_{-a} 和 φ_β 把 U 映到 U 上，我们看出 $g\in H^\infty$ 和 $\|g\|_\infty\leqslant 1$；同样 $g(0)=0$. 从 f 过渡到 g 就把我们的问题归结到 Schwarz 引理，并给出 $|g'(0)|\leqslant 1$. 对(1)，链式法则给出

(2) $$g'(0)=\varphi_\beta'(\beta)f'(\alpha)\varphi_{-a}'(0)$$

如果我们使用 12.4(1)的等式，便得到不等式

(3) $$|f'(\alpha)|\leqslant\frac{1-|\beta|^2}{1-|\alpha|^2}$$

由于在(3)中等号可以成立，这就解决了我们的问题. 而这当且仅当 $|g'(0)|=1$ 时就会发生. 此时 g 是一个旋转(定理 12.2)，所以对某个常数 λ，$|\lambda|=1$，有

(4) $$f(z)=\varphi_{-\beta}(\lambda\varphi_a(z))\qquad(z\in U)$$

应当强调解的一个值得注意的特性. 我们对 f 靠近 U 的边界

时的性态没有提出一些光滑性条件(如在 $\bar{U}$ 上连续). 然而, 在所述的限制之下使 $|f'(\alpha)|$ 取得最大值的函数实际上是有理函数. 也应注意这些极值函数是映 U 到 U 上的(不只是到 U 内), 并且映射是一一的. 这一考察可以为第十四章中 Riemann 映射定理的证明提供一个启示.

现在, 我们只打算指出这一极值问题如何用来刻划把 U 映到 U 上的一一全纯映射.

12.6 定理 设 $f\in H(U)$, f 是一一的, $f(U)=U$, $\alpha\in U$, 且 $f(\alpha)=0$, 则有一个常数 λ, $|\lambda|=1$, 使得

(1) $$f(z)=\lambda\varphi_\alpha(z) \qquad (z\in U)$$

换言之, 我们通过把映射 φ_α 和一个旋转复合起来以得到 f.

证明 设 g 为 f 的逆映射, 由 $g(f(z))=z$ $(z\in U)$ 确定. 由于 f 是一一的, f' 在 U 内没有零点. 根据定理 10.33, 所以 $g\in H(U)$. 由链式法则, 有

(2) $$g'(0)f'(\alpha)=1$$

把 12.5 的解应用于 f 和 g, 引出不等式

(3) $$|f'(\alpha)|\leqslant\frac{1}{1-|\alpha|^2}, \qquad |g'(0)|\leqslant 1-|\alpha|^2$$

根据(2), 式(3)中的等式必定成立. 如同我们在前述问题(此处 $\beta=0$)中观察到那样, 这就保证了 f 满足(1).

Phragmen-Lindelöf 方法

12.7 对于有界区域 Ω 我们在 12.1 节中看到, 如果 f 在 Ω 的闭包上连续且 $f\in H(\Omega)$, 则最大模定理推出

(1) $$\|f\|_\Omega=\|f\|_{\partial\Omega}$$

对于无界域它就不再真确了.

请看一个例子, 设

(2) $$\Omega=\left\{z=x+iy:\ -\frac{\pi}{2}<y<\frac{\pi}{2}\right\}$$

Ω是以平行线 $y=\pm\pi/2$ 为界的开带域，它的边界 $\partial\Omega$ 是这两条直线的并. 设

(3) $$f(z)=\exp(\exp(z))$$

对实数 x,

(4) $$f\left(x\pm\frac{\pi i}{2}\right)=\exp(\pm ie^x)$$

由于 $\exp(\pi i/2)=i$, 所以对 $z\in\partial\Omega$, $|f(z)|=1$. 但当 x 沿位于Ω内的正实轴趋于 ∞, 即 $x\to\infty$ 时, $f(z)$非常迅速地趋于 ∞.

在上句中"非常" 是个关键性的词. 由 Phragmen 和 Lindelöf 发现的一种方法使之有可能证明如下类型的定理: 如果 $f\in H(\Omega)$ 且 $|f|<g$, 此处, 在Ω内当 $z\to\infty$ 时, $g(z)$"慢慢"地趋于∞. ("慢" 的含义是与Ω有关的), 则 f 实际上在Ω内有界, 并且根据最大模定理, 常常包含关于 f 的进一步结论.

我们不打算用一个包罗万象的定理来描述这个方法, 而仅仅通过两种情况来指出它是怎样做的. 这两种情况中, Ω都是一个带域. 第一种情况下, f 假定是有界的, 而定理将改进这个界限. 第二种情况下, 对 f 提出一个增长条件以便排除(3)中的函数. 鉴于以后的应用, 定理 12.8 中的Ω将是垂直的带形.

但是, 首先让我们提出另一个同样是很有意思的例子: 设 f 是一个整函数, 且对所有 z

(5) $$|f(z)|<1+|z|^{1/2}$$

则 f 是常数.

由于 Cauchy 估值 10.26 表明, 对 $n=1,2,3,\cdots$, $f^{(n)}(0)=0$, 从而立即得到上面的结果.

12.8 定理 设

(1)
$$\Omega=\{x+iy:a<x<b\},$$
$$\overline{\Omega}=\{x+iy:a\leqslant x\leqslant b\}$$

f 在 $\overline{\Omega}$ 上连续，$f\in H(\Omega)$，又设对所有 $z\in\Omega$ 和某个固定的 $B<\infty$，$|f(z)|<B$. 如果

(2)
$$M(x)=\sup\{|f(x+iy)|:-\infty<y<\infty\}$$
$$(a\leqslant x\leqslant b)$$

则实际上有

(3)
$$M(x)^{b-a}\leqslant M(a)^{b-x}M(b)^{x-a}\qquad(a<x<b)$$

注：结论(3)蕴涵着不等式 $|f|<B$ 可以用 $|f|<\max(M(a),M(b))$ 代替，所以 $|f|$ 在 Ω 内不大于 $|f|$ 在 Ω 的边界上的上确界.

如果把定理应用到以直线 $x=\alpha$ 和 $x=\beta$ 为界的带域上，此处 $a\leqslant\alpha<\beta\leqslant b$，则结论可按以下方式来叙述：

系 在定理的假设下，$\log M$ 是 (a,b) 上的一个凸函数.

证明 我们首先假定 $M(a)=M(b)=1$. 在这种情况下，我们需要证明对所有 $z\in\Omega$，$|f(z)|\leqslant 1$.

对每个 $\varepsilon>0$，我们定义一个辅助函数

(4)
$$h_\varepsilon(z)=\frac{1}{1+\varepsilon(z-a)}\qquad(z\in\overline{\Omega})$$

由于在 $\overline{\Omega}$ 内 $\mathrm{Re}\{1+\varepsilon(z-a)\}=1+\varepsilon(x-a)\geqslant 1$，故在 $\overline{\Omega}$ 内 $|h_\varepsilon|\leqslant 1$，所以

(5)
$$|f(z)h_\varepsilon(z)|\leqslant 1\qquad(z\in\partial\Omega)$$

同样，$|1+\varepsilon(z-a)|\geqslant\varepsilon|y|$，因此

(6)
$$|f(z)h_\varepsilon(z)|\leqslant\frac{B}{\varepsilon|y|}\qquad(z=x+iy\in\overline{\Omega})$$

设 R 是用直线 $y=\pm B/\varepsilon$ 从 $\overline{\Omega}$ 割出的矩形. 由(5)和(6)，在

∂R 上 $|fh_\varepsilon|\leqslant 1$，因此根据最大模定理，在 R 上 $|fh_\varepsilon|\leqslant 1$. 但(6)表明在 $\overline{\Omega}$ 的剩余部分上 $|fh_\varepsilon|\leqslant 1$. 因此，对所有 $z\in\Omega$ 和所有 $\varepsilon>0$，$|f(z)h_\varepsilon(z)|\leqslant 1$. 如果我们固定 $z\in\Omega$，然后令 $\varepsilon\to 0$，则得到期望的结果 $|f(z)|\leqslant 1$.

现在我们转向一般情况，设

$$g(z)=M(a)^{(b-z)/(b-a)}M(b)^{(z-a)/(b-a)} \tag{7}$$

此处，对 $M>0$ 及复数 w，M^w 由

$$M^w=\exp(w\log M) \tag{8}$$

定义，且 $\log M$ 是实数. 则在 $\overline{\Omega}$ 内 g 是整函数，g 没有零点，$1/g$ 有界，

$$|g(a+iy)|=M(a)\qquad |g(b+iy)|=M(b) \tag{9}$$

从而 f/g 满足我们原先的假定. 因此在 Ω 内 $|f/g|\leqslant 1$，而这便给出(3). (参看习题 7.)

12.9 定理 设

$$\Omega=\left\{x+iy:|y|<\frac{\pi}{2}\right\}, \tag{1}$$

$$\overline{\Omega}=\left\{x+iy:|y|\leqslant\frac{\pi}{2}\right\}$$

设 f 在 $\overline{\Omega}$ 上连续，$f\in H(\Omega)$，则存在常数 $\alpha<1$，$A<\infty$，使得

$$|f(z)|<\exp\{A\exp(\alpha|x|)\}\qquad (z=x+iy\in\Omega) \tag{2}$$

且

$$\left|f\left(x\pm\frac{\pi i}{2}\right)\right|\leqslant 1\qquad (-\infty<x<\infty) \tag{3}$$

则对所有 $z\in\Omega$，$|f(z)|\leqslant 1$.

注意，如同由函数 $\exp(\exp z)$ 所指出的那样，当 $\alpha=1$ 时，是无法得到这个结论的.

证明 选取 $\beta>0$, 使 $\alpha<\beta<1$. 对 $\varepsilon>0$, 定义

(4) $$h_\varepsilon(z)=\exp\{-\varepsilon(e^{\beta z}+e^{-\beta z})\}$$

对$z\in\overline{\Omega}$

(5) $$\begin{aligned}\mathrm{Re}[e^{\beta z}+e^{-\beta z}]&=(e^{\beta x}+e^{-\beta x})\cos\beta y\\&\geqslant\delta(e^{\beta x}+e^{-\beta x})\end{aligned}$$

此处因$|\beta|<1$, 故 $\delta=\cos(\beta\pi/2)>0$. 因此

(6) $$|h_\varepsilon(z)|\leqslant\exp\{-\varepsilon\delta(e^{\beta x}+e^{-\beta x})\}<1 \qquad (z\in\overline{\Omega})$$

可见在 $\partial\Omega$ 上$|fh_\varepsilon|\leqslant1$, 同时

(7) $$|f(z)h_\varepsilon(z)|\leqslant\exp\{Ae^{\alpha|x|}-\varepsilon\delta(e^{\beta x}+e^{-\beta x})\} \qquad (z\in\overline{\Omega})$$

固定 $\varepsilon>0$. 由于 $\varepsilon\delta>0$ 和 $\beta>\alpha$, 所以当 $x\to\pm\infty$ 时, (7)中的指数趋于 $-\infty$. 因而存在一个 x_0, 使得对所有 $x>x_0$(7)的右边小于1. 由于在顶点为$\pm x_0\pm(\pi i/2)$的矩形的边界上$|fh_\varepsilon|\leqslant1$, 故最大模定理表明, 实际上在这个矩形上, $|fh_\varepsilon|\leqslant1$. 因此, 对每个 $\varepsilon>0$ 在Ω的每一点, $|fh_\varepsilon|\leqslant1$. 当$\varepsilon\to0$时, 对每一个 z, $h_\varepsilon(z)\to1$, 所以我们断定对所有的 $z\in\Omega$, $|f(z)|\leqslant1$.

一个插值定理

12.10 凸性定理 12.8 有时能用于证明某些线性变换关于某些L^p-范数是有界的. 我们只考虑此种类型的一个特殊情况, 而不作完整的一般性的讨论.

设X是具有正测度μ的测度空间, 又设$\{\psi_n\}$ $(n=1,2,3,\cdots)$是 $L^2(\mu)$内的一个规范正交函数集, 我们记住这里所指的意思是:

(1) $$\int_X\psi_n\bar{\psi}_m d\mu=\begin{cases}1 & \text{当 } m=n\\0 & \text{当 } m\neq n\end{cases}$$

我们同样假定$\{\psi_n\}$是 $L^\infty(\mu)$内的有界序列: 存在一个 $M<\infty$ 使得

(2) $$|\psi_n(x)|\leqslant M \qquad (n=1,2,3,\cdots;x\in X)$$

则对任意 $f\in L^p(\mu)$，此处 $1\leqslant p\leqslant 2$，积分

(3) $$\hat{f}(n)=\int_X f\bar{\psi}_n d\mu \qquad (n=1,2,3,\cdots)$$

存在并且在所有正整数集上定义了一个函数 $\hat{f}$.

现在有两个很容易的定理：对 $f\in L^1(\mu)$，(2)给出

(4) $$\|\hat{f}\|_\infty\leqslant M\|f\|_1$$

而对 $f\in L^2(\mu)$，则 Bessel 不等式给出

(5) $$\|\hat{f}\|_2\leqslant\|f\|_2$$

此处范数按通常的方式定义：

(6) $$\|f\|_p=\left[\int|f|^p d\mu\right]^{1/p},\quad \|\hat{f}\|_q=[\Sigma|\hat{f}(n)|^q]^{1/q}$$

并且 $\|\hat{f}\|_\infty=\sup_n|\hat{f}(n)|$.

由于 $(1,\infty)$ 和 $(2,2)$ 都是共轭指数对，所以人们可以推测，当 $f\in L^p(\mu)$ 和 $1<p<2$，$q=p/(p-1)$ 时，$\|\hat{f}\|_q$ 是有限的. 这结论确实是正确的，并且可以在前面的平凡情形 $p=1$ 和 $p=2$ 之间，通过“插值”来证明.

12.11 Hausdorff-Young 定理 在上面的假设下，如果 $1\leqslant p\leqslant 2$ 及 $f\in L^p(\mu)$，则不等式

(1) $$\|\hat{f}\|_q\leqslant M^{(2-p)/p}\|f\|_p$$

成立.

证明 我们首先证明定理的一个简化形式.

固定 p，$1<p<2$. 设 f 是一个简单复函数，使得 $\|f\|_p=1$，又设 $b_1,\cdots,b_N$ 是复数，使得 $\Sigma|b_n|^p=1$. 我们的目标是证明不等式

(2) $$\left|\sum_{n=1}^{N} b_n\hat{f}(n)\right|\leqslant M^{(2-p)/p}$$

设 $F=|f|^p$，又设 $B_n=|b_n|^p$，则有一个函数 φ 和复数 β_1，$\beta_2,\cdots,\beta_N$，使得

$$f=F^{1/p}\varphi,\quad |\varphi|=1,\quad \int_X F d\mu=1 \tag{3}$$

和

$$b_n=B_n^{1/p}\beta_n,\quad |\beta_n|=1,\quad \sum_{n=1}^{N}B_n=1 \tag{4}$$

如果我们利用这些关系式和第 12.10 节给出的 $\hat{f}(n)$ 的定义，则得到

$$\sum_{n=1}^{N}b_n\hat{f}(n)=\sum_{n=1}^{N}B_n^{1/p}\beta_n\int_X F^{1/p}\varphi\bar{\psi}_n d\mu \tag{5}$$

在(5)中现在用 z 代替 $1/p$，且对任意复数 z，定义

$$\Phi(z)=\sum_{n=1}^{N}B_n^z\beta_n\int_X F^z\varphi\bar{\psi}_n d\mu \tag{6}$$

记住 $A>0$ 时，$A^z=\exp(z\log A)$；如果 $A=0$，我们约定 $A^z=0$. 由于 F 是简单函数，$F\geqslant 0$ 及 $B_n\geqslant 0$，我们看到 Φ 就是这种指数函数的有限的线性组合，所以，对任意有限数 a 和 b，Φ 在

$$\{z:a\leqslant \mathrm{Re}(z)\leqslant b\}$$

上是有界整函数. 我们将取 $a=\frac{1}{2}$ 和 $b=1$，并在这带形域的边缘上估计 Φ，然后应用定理 12.8 去估计 $\Phi(1/p)$.

对 $-\infty<y<\infty$ 定义

$$c_n(y)=\int_X F^{1/2}F^{iy}\varphi\bar{\psi}_n d\mu \tag{7}$$

Bessel 不等式给出

$$\sum_{n=1}^{N}|c_n(y)|^2\leqslant\int_X|F^{1/2}F^{iy}\varphi|^2 d\mu \tag{8}$$

$$=\int_X |F|\,d\mu=1$$

又 Schwarz 不等式表明

$$(9)\qquad \left|\Phi\left(\frac{1}{2}+iy\right)\right|=\left|\sum_{n=1}^{N}B_n^{1/2}B_n^{iy}\beta_n c_n\right|$$

$$\leqslant\left\{\sum_{n=1}^{N}B_n\cdot\sum_{n=1}^{N}|c_n|^2\right\}^{1/2}\leqslant 1$$

由于 $\|\psi_n\|_\infty\leqslant M$，从(3)，(4)，和(6)很容易得到估计值

$$(10)\qquad |\Phi(1+iy)|\leqslant M\qquad(-\infty<y<\infty)$$

现在从式(9)，(10)和定理 12.8 可以断定

$$(11)\qquad |\Phi(x+iy)|\leqslant M^{2x-1}\left(\frac{1}{2}\leqslant x\leqslant 1,\ -\infty<y<\infty\right)$$

由于 $x=1/p$ 和 $y=0$，这就给出所期望的不等式(2).

整个证明现在不难完成了. 首先，注意到由于在任意测度空间上的任意函数的 L^q 范数等于它作为 L^p 上的一个线性泛函时的范数，于是

$$(12)\qquad \{\Sigma|\hat{f}(n)|^q\}^{1/q}$$

$$=\sup\left|\sum_{n=1}^{N}b_n\hat{f}(n)\right|$$

这里的上确界是通过使 $\Sigma|b_n|^p=1$ 的所有 $\{b_1,\cdots,b_N\}$ 来取得的. 因此，对每一个简单复函数 $f\in L^p(\mu)$，(2)表明

$$(13)\qquad \left\{\sum_{n=1}^{N}|\hat{f}(n)|^q\right\}^{1/q}\leqslant M^{(2-p)/p}\|f\|_p$$

如果 $f\in L^p(\mu)$，则有简单函数 f_j，使得当 $j\to\infty$ 时，$\|f_j-f\|_p\to 0$. 因为 $\psi_n\in L^q(\mu)$，故对每一个 n，$\hat{f}_j(n)\to\hat{f}(n)$. 这样，由于

(13)对每个 f_j 成立，所以对 f 同样成立. 由于 N 是任意的，因而最后得到(1).

最大模定理的逆定理

现在，我们回到本章引言中提到过的定理上.

字母 j 表示恒等函数：$j(z)=z$.

对每个 $z\in\overline{U}$ 指定数值为 1 的函数用 1 表示.

12.12 定理 设 M 是闭单位圆盘 $\overline{U}$ 上的连续复函数的向量空间，它具有如下性质：

(*a*) $1\in M$.

(*b*) 如果 $f\in M$，则亦有 $jf\in M$.

(*c*) 如果 $f\in M$，则 $\|f\|_U=\|f\|_T$.

则每一个 $f\in M$ 在 U 内全纯.

注意(*c*)是最大模定理稍弱的形式；(*c*) 只是断定 $|f|$ 在 $\overline{U}$ 上的全局极大值在它边界 T 上的某点达到，但 (*c*) 并没有预先排除 $|f|$ 在 U 内的局部极大值的存在性.

证明 根据(*a*)和(*b*)，M 包含所有多项式. 连同(*c*)一起便表明 M 满足定理 5.25 的假设. 因此每一个 $f\in M$ 在 U 内是调和的，我们将用 (*b*) 去证明每一个 $f\in M$ 实际上满足 Cauchy-Riemann 方程.

设 ∂ 和 $\bar{\partial}$ 是第 11.1 节中引进的微分算子. 乘积的微分法则给出

$$(\partial\bar{\partial})(fg)=f\cdot(\partial\bar{\partial}g)+(\partial f)\cdot(\bar{\partial}g)+(\bar{\partial}f)\cdot(\partial g)$$
$$+(\partial\bar{\partial}f)\cdot g$$

固定 $f\in M$，并取 $g=j$. 则 $fj\in M$. 因此 f 和 fj 是调和的，所以 $\partial\bar{\partial}f=0$ 和 $(\partial\bar{\partial})(fj)=0$. 同样，$\bar{\partial}j=0$ 和 $\partial j=1$. 因而上面的等式简化为 $\bar{\partial}f=0$. 所以 $f\in H(U)$.

这个结果将在下面的证明中使用.

12.13 Rado 定理 假定 $f\in C(\overline{U})$，Ω 是使 $f(z)\neq 0$ 的所有 $z\in U$ 的集，且 f 在 Ω 内全纯，则 f 在 U 内全纯.

特别，定理断定 $U-\Omega$ 至多是可数的，除非 $\Omega=\varnothing$.

证明 设 $\Omega\neq\varnothing$. 我们首先证明 Ω 在 U 内稠密. 如果不是这样，则存在 $\alpha\in\Omega$ 和 $\beta\in U-\overline{\Omega}$，使得 $2|\beta-\alpha|<1-|\beta|$. 选择 n 使得 $2^n|f(\alpha)|>\|f\|_T$. 对 $z\in\overline{\Omega}$，定义 $h(z)=(z-\beta)^{-n}f(z)$. 如果 $z\in U\cap\partial\Omega$，则 $f(z)=0$. 因此 $h(z)=0$. 如果 $z\in T\cap\partial\Omega$，则

$$|h(z)|\leqslant(1-|\beta|)^{-n}\|f\|_T<|\alpha-\beta|^{-n}|f(\alpha)|$$
$$=|h(\alpha)|$$

这与最大模定理发生矛盾.

因此 Ω 在 U 内稠密.

其次，设 M 是在 Ω 内全纯的所有 $g\in C(\overline{U})$ 的向量空间. 固定 $g\in M$. 当 $n=1,2,3,\cdots$，在 $U\cap\partial\Omega$ 上 $fg^n=0$. 因此，对每一个 $\alpha\in\Omega$，由最大模定理推出

$$|f(\alpha)||g(\alpha)|^n\leqslant\|fg^n\|_{\partial\Omega}$$
$$=\|fg^n\|_T\leqslant\|f\|_T\|g\|_T^n$$

如果我们取 n 次根，然后令 $n\to\infty$，则对每一个 $\alpha\in\Omega$ 看出 $|g(\alpha)|\leqslant\|g\|_T$. 由于 Ω 在 U 内稠密，所以 $\|g\|_U=\|g\|_T$.

由此可见，M 满足定理 12.12 的假设. 因为 $f\in M$，所以 f 在 U 内全纯.

习 题

1 不要使用 f 边界值的知识，证明定理 12.2.

2 设 $f\in H(\Pi^-)$，此处 Π^+ 是上半平面，且 $|f|\leqslant 1$. 问 $|f'(i)|$ 能取得多大？找出此极值函数.（比较 12.5 节的讨论）.

3 设 $f\in H(\Omega)$. 在什么条件下，$|f|$ 在 Ω 内有局部极小值？

4 (a) 设 Ω 是一区域，D 是圆盘，$\overline{D}\subset\Omega$，$f\in H(\Omega)$，$f$ 不是常数，而在 D

的边界上$|f|$是常数. 证明在D内f至少有一个零点.

(*b*) 找出所有整函数f, 使得当$|z|=1$时就有$|f(z)|=1$.

5 设Ω是一个有界区域, $\{f_n\}$是在$\overline{\Omega}$上连续而在Ω内全纯的函数序列, 且$\{f_n\}$在Ω的边界上一致收敛. 证明$\{f_n\}$在$\overline{\Omega}$上一致收敛.

6 设$f\in H(\Omega)$, Γ是Ω内的一个圈, 使得对所有$\alpha\notin\Omega$, $\mathrm{Ind}_\Gamma(z)=0$, 对每一个$\zeta\in\Gamma^*$, $|f(\zeta)|\leqslant 1$, 且$\mathrm{Ind}_\Gamma(z)\neq 0$. 证明$|f(z)|\leqslant 1$.

7 在定理 12.8 的证明中, 暗中假定了$M(a)>0$和$M(b)>0$. 证明当$M(a)=0$时这个定理也成立, 因而对所有$z\in\Omega$, $f(z)=0$.

8 如果$0<R_1<R_2<\infty$, 令$A(R_1, R_2)$表示环域

$$\{z: R_1<|z|<R_2\}$$

则存在一个垂直带形域使指数函数把它映射到$A(R_1, R_2)$上. 用此去证明 Hadamard 三圆周定理: 如果$f\in H(A(R_1, R_2))$, 又

$$M(r)=\max_\theta |f(re^{i\theta})| \qquad (R_1<r<R_2)$$

且如果$R_1<a<r<b<R_2$, 则

$$\log M(r)\leqslant\frac{\log(b/r)}{\log(b/a)}\log M(a)+\frac{\log(r/a)}{\log(b/a)}\log M(b)$$

[换言之, $\log M(r)$是$\log r$的一个凸函数.]对哪些f能使这个不等式成为等式?

9 设Π为开的右半平面 (当且仅当$\mathrm{Re}z>0$时$z\in\Pi$). 设f在Π的闭包($\mathrm{Re}z\geqslant 0$)上连续, $f\in H(\Pi)$, 且有常数$A<\infty$和$\alpha<1$使得对于所有$z\in\Pi$,

$$|f(z)|<A\exp(|z|^\alpha)$$

而且对所有实数y, $|f(iy)|\leqslant 1$. 证明在Π内$|f(z)|\leqslant 1$.

证明当$\alpha=1$时结论是错误的.

如果用通过原点的, 其角度不等于π的两条射线为界的区域来代替Π, 则结果应如何修正?

10 设Π为开的右半平面. $f\in H(\Pi)$, 对所有$z\in\Pi$, $|f(z)|<1$, 且存在α, $-\pi/2<\alpha<\pi/2$, 使得当$r\to\infty$时

$$\frac{\log|f(re^{i\alpha})|}{r}\longrightarrow-\infty$$

证明$f=0$.

提示: 设$g_n(z)=f(z)e^{nz}$, $n=1, 2, 3, \cdots$. 将习题 9 应用到由$-\pi/2<\theta<\alpha$, $\alpha<\theta<\pi/2$所定义的两个角域上. 推断每个g_n在Π内有界, 且对所有

n, 在Π内$|g_n|<1$.

11 设Γ是无界域Ω的边界，$f\in H(\Omega)$，f在$\Omega\cup\Gamma$上连续，且有常数$B<\infty$和$M<\infty$使在Γ上$|f|\leqslant M$而在Ω内$|f|\leqslant B$. 证明在Ω内实际上有$|f|\leqslant M$.

提示: 证明，不失一般性本题目可进一步假设$U\cap\Omega=\varnothing$. 固定$z_0\in\Omega$，令n为一个大的整数，令V为以0为中心的一个大圆盘，且在包含z_0的$V\cap\Omega$的分支内把最大模定理应用到函数$f^n(z)/z$上.

12 设f是整函数. 如果有一个连续映射γ把$[0,1)$映到复平面内，使得当$t\to1$时，$\gamma(t)\to\infty$和$f(\gamma(t))\to\alpha$，则说α是f的一个渐近值. [在复平面内"当$t\to1$时，$\gamma(t)\to\infty$"是指对每个$R<\infty$，有一个对应的$t_R<1$，使得当$t_R<t<1$时，便有$|\gamma(t)|>R$]. 证明每一个非常数的整函数以∞作为一个渐近值.

提示: 令$E_n=\{z:|f(z)|>n\}$. 根据习题11，E_n的每个分支是无界的(如何证明?)且包含E_{n+1}的一个分支.

13 证明exp恰好有两个渐近值: 0和∞. 关于sin和cos怎样? 注: 对所有复数z，$\sin z$和$\cos z$定义为

$$\sin z=\frac{e^{iz}-e^{-iz}}{2i},\quad \cos z=\frac{e^{iz}+e^{-iz}}{2}$$

14 如果f是整函数且α不在f的值域内，证明α是f的一个渐近值，除非f是常数.

15 设$f\in H(U)$. 证明在U内有一个序列$\{z_n\}$使得$|z_n|\to1$且$\{f(z_n)\}$有界.

16 设Ω是有界区域，$f\in H(\Omega)$，且对Ω内每一个收敛于Ω的一个边界点的序列$\{z_n\}$

$$\limsup_{n\to\infty}|f(z_n)|\leqslant M$$

证明对所有$z\in\Omega$，$|f(z)|\leqslant M$.

第十三章　有理函数逼近

预备知识

13.1　Riemann 球面　用附加一个称为 ∞ 的新点来紧化复平面，对全纯函数的研究常常是方便的. 由此而产生的集 S^2（Riemann 球面，即 R^2 和 $\{\infty\}$ 的并）在下述意义上予以拓扑化. 对任意 $r>0$，令 $D'(\infty;r)$ 为使 $|z|>r$ 的所有复数 z 的集，设 $D(\infty;r)=D'(\infty;r)\cup\{\infty\}$，并且断言当且仅当 S^2 的子集是圆盘 $D(a;r)$ 的并时，则该子集是开集，这里的 a 是 S^2 的任意点而 r 是任意正数. 在 $S^2-\{\infty\}$ 上当然给出通常的平面拓扑. 容易看出，S^2 与球面是同胚的（因此用这个记号）. 事实上，S^2 映到 R^3 内的单位球面上的同胚 φ 能明显地表示为：设 $\varphi(\infty)=(0,0,1)$，而对所有的复数 $re^{i\theta}$，设

$$\varphi(re^{i\theta})=\left(\frac{2r\cos\theta}{r^2+1},\frac{2r\sin\theta}{r^2+1},\frac{r^2-1}{r^2+1}\right). \tag{1}$$

我们留给读者去作出由(1)确定的几何图形.

如果 f 在 $D'(\infty;r)$ 内全纯，则称 f 有一个孤立奇点 ∞. 在 $D'\left(0,\frac{1}{r}\right)$ 内定义函数 $\tilde{f}(z)=f\left(\frac{1}{z}\right)$，则 f 在 ∞ 的奇点的性质与 $\tilde{f}$ 在 0 点的性质相同.

因此，如果 f 在 $D'(\infty;r)$ 内有界，则 $\lim\limits_{z\to\infty}f(z)$ 存在，并且是一个复数（如同我们把定理 10.20 应用到 $\tilde{f}$ 时所看到的那样）. 我们把 $f(\infty)$ 定义为这个极限，因此在 $D(\infty;r)$ 内得到一个称为全纯的函数；注意此函数是用 $\tilde{f}$ 在 0 点近傍的特性而不是用 f 在 ∞ 处的

可微性来定义的.

如果 $z=0$ 是 $\tilde{f}$ 的 m 阶极点,则称∞是 f 的 m 阶极点;f 在∞处的主部就是一个通常的m 次多项式(与定理 10.21 相比较),如果我们从 f 减去这个多项式,就得到一个以∞为可去奇点的函数.

最后,如果 $z=0$ 为 $\tilde{f}$ 的本性奇点,则称∞为 f 的本性奇点.例如:每一个不是多项式的整函数在∞处有一个本性奇点.

稍后,在本章中,我们将遇到"$S^2-\Omega$是连通的"这个条件,此处 Ω 是平面内的一个开集.注意它与"Ω 相对于平面的余集是连通的"这一条件是不等价的.例如,如果 Ω 是由所有复数 $z=x+iy, 0<y<1$ 组成,则 Ω 相对于平面的余集有两个分支,但$S^2-\Omega$是连通的.

13.2 有理函数 根据定义,有理函数 f 是两个多项式 P 和 Q 的商:$f=P/Q$. 由定理 10.25得到,每一个非常数值的多项式是次数为 1 的一些因子的乘积.我们可以假定 P 和 Q没有这样的公共因子,则 f 在Q 的每一零点处有一极点(f 的极点与 Q 的零点同阶).如果减去对应的主部,则得到一个有理函数,它唯一的奇点是∞,因而是一个多项式.

因此,每一个有理函数 $f=P/Q$ 具有形如

$$f(z)=A_0(z)+\sum_{j=1}^{k}A_j((z-a_j)^{-1}) \tag{1}$$

的表达式,此处 $A_0, A_1, \cdots, A_k$ 是多项式,$A_1, \cdots, A_k$ 没有常数项,且 $a_1, \cdots, a_k$ 是 Q 的不同的零点;我们称(1)为 f 的部分分式分解.

我们转向某些拓扑方面的讨论.已知平面内的每个开集是紧集的可数并(例如闭圆盘).然而,为了方便起见,我们要求这些紧集满足某些附加性质:

13.3 定理 平面内的每一个开集 Ω 是紧集序列$\{K_n\}, n=1,$

2, 3, …, 的并, 使得

(a) 当 $n=1, 2, 3, \cdots$ 时, K_n 落在 K_{n+1} 的内部.

(b) Ω 的每一个紧子集落在某个 K_n 内.

(c) 当 $n=1, 2, 3, \cdots$ 时, S^2-K_n 的每一个分支包含 $S^2-\Omega$ 的一个分支.

粗略地说, 性质(c)是指: K_n 除去那些强加在它上面的 Ω 的洞之外, 它是没有洞的. 注意 Ω 没有假定是连通的. 根据定义, 集 E 的内部是 E 的最大开子集.

证明 当 $n=1, 2, 3, \cdots$ 时, 设

$$V_n=D(\infty; n)\cup\bigcup_{a\notin\Omega}D\left(a;\ \frac{1}{n}\right) \tag{1}$$

且 $K_n=S^2-V_n$. [当然在(1)内 $a\neq\infty$.]则 K_n 是 Ω 的有界闭子集(因此是紧集), 且 $\Omega=\cup K_n$. 如果 $z\in K_n$, 且 $r=n^{-1}-(n+1)^{-1}$, 则容易验证 $D(z; r)\subset K_{n+1}$. 这便给出(a). 因此, Ω 是 K_n 的内部 W_n 的并. 如果 K 是 Ω 的一个紧子集, 则对某个 N, $K\subset W_1\cup\cdots\cup W_N$. 因此 $K\subset K_N$.

最后, (1)中的每个圆盘与 $S^2-\Omega$ 相交; 每个圆盘是连通的; 因而 V_n 的每个分支与 $S^2-\Omega$ 相交; 由于 $V_n\supset S^2-\Omega$, 所以没有 $S^2-\Omega$ 的分支能够相交 V_n 的两个分支. 这就给出(c).

13.4 定向区间的集 令 Φ 为平面内定向区间的有限类. 对每一点 p, 令 $m_I(p)[m_E(p)]$ 是起点[终点]为 p 的 Φ 的元的个数. 对每一个 p, 如果 $m_I(p)=m_E(p)$, 则称 Φ 是平衡的.

如果 Φ 是平衡的(且非空), 则可以作出如下的结构.

选择 $\gamma_1=[a_0, a_1]\in\Phi$. 设 $k\geqslant 1$, 且 Φ 的不同元 $\gamma_1, \cdots\gamma_k$ 已经按这样的方法选取, 当 $1\leqslant i\leqslant k$ $\gamma_i=[a_{i-1}, a_i]$. 如果 $a_k=a_0$, 则停止进行. 如果 $a_k\neq a_0$, 且恰好区间 $\gamma_1, \cdots\gamma_k$ 中有 r 个以 a_k 作为终点, 则这些区间中只有 $r-1$ 个以 a_k 作为起点; 由于 Φ 为平衡的,

所以Φ至少包含另外一个区间，比如说γ_{k+1}，以a_k为起点. 由于Φ为有限的，那末，比如说到第n步，我们最终都会回到a_0.

这样，$\gamma_1, \cdots, \gamma_n$(按此顺序)连成一条闭路径.

Φ的剩余元素仍然组成适用上面结构的一个平衡类. 由此可见，Φ的元能如此计算，致使它们形成有限多条闭路径. 这些路径的和是一个圈. 因此得到下面的结论：

如果$\Phi=\{\gamma_1, \cdots \gamma_N\}$是有向区间的一个平衡类，且

$$\Gamma=\gamma_1+\gamma_2+\cdots+\gamma_N$$

则Γ是一个圈.

13.5 定理 如果K是平面开集$\Omega(\neq\varnothing)$的一个紧子集，则在$\Omega-K$内有一个圈，使得对每一个$f\in H(\Omega)$和对每一个$z\in K$，Cauchy公式

$$f(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta-z}d\zeta \tag{1}$$

成立.

证明 由于K是紧的且Ω是开集，则存在一个$\eta>0$，使得从K的任意点到Ω外部的任意点的距离至少为2η. 在平面内由水平线和垂直线构造一种格子，使得任意两条邻接的水平线之间的距离为η，且对垂直线也一样. 设$Q_1, \cdots, Q_m$是以η为边长的正方形(闭2-胞腔)，这些正方形由格子组成且与K相交. 则当$\nu=1, \cdots, m$时，$Q_\nu\subset\Omega$.

如果a_ν为Q_ν的中心而$a_\nu+b$为它的一个顶点，令$\gamma_{\nu k}$为定向区间

$$\gamma_{\nu k}=[a_\nu+i^k b, a_\nu+i^{k+1}b] \tag{2}$$

且定义

$$\partial Q_\nu=\gamma_{\nu 1}+\gamma_{\nu 2}+\gamma_{\nu 3}+\gamma_{\nu 4} \qquad (\nu=1, \cdots, m) \tag{3}$$

则容易验证(例如，作为定理10.37的特殊场合，或者用定理10.11

和 10.40.)

$$\text{(4)}\qquad \mathrm{Ind}_{\partial Q_r}(\alpha)=\begin{cases}1 & \text{当 } \alpha \text{ 在 } Q_r \text{ 的内部}\\ 0 & \text{当 } \alpha \text{ 不在 } Q_r \text{ 内}\end{cases}$$

令 Σ 为所有 $\gamma_{rk}(1\leqslant r\leqslant m, 1\leqslant k\leqslant 4)$ 的类. 显然, Σ 是平衡的. 除去 Σ 中那些其反向亦属于 Σ(参看 10.1 节). 设 Φ 为 Σ 剩余元素的类, 则 Φ 是平衡的, 令 Γ 是如同 13.4 节那样由 Φ 构造出来的圈.

如果某个 Q_r 的一条边 E 与 K 相交, 则以 E 为边界的两个正方形都与 K 相交. 因此, Σ 包含互相反向而范围为 E 的两个有向区间. 这些区间在 Φ 内不出现, 所以 Γ 是 $\Omega-K$ 内的一个圈.

如果 α 不在任何 Q_r 的边界上, 则由 Σ 选出的 Φ 结构同样表明

$$\text{(5)}\qquad \mathrm{Ind}_{\Gamma}(\alpha)=\sum_{r=1}^{m}\mathrm{Ind}_{\partial Q_r}(\alpha)$$

因此, (4)蕴涵着

$$\text{(6)}\qquad \mathrm{Ind}_{\Gamma}(\alpha)=\begin{cases}1 & \text{当 } \alpha \text{ 在某个 } Q_r \text{ 的内部}\\ 0 & \text{当 } \alpha \text{ 不在 } Q_r \text{ 内}\end{cases}$$

如果 $z\in K$, 则 $z\notin\Gamma^*$, 且 z 是某个 Q_r 内部的一个极限点. 由于在 Γ^* 的余集的每个分支内, (6)的左边是常数, 所以(6)给出

$$\text{(7)}\qquad \mathrm{Ind}_{\Gamma}(z)=\begin{cases}1 & \text{当 } z\in K\\ 0 & \text{当 } z\notin\Omega\end{cases}$$

根据 Cauchy 定理 10..35 便得到(1).

Runge 定理

本节主要目标是定理 13.9. 我们从一个稍为不同的说法开始, 其中着重的是在一个紧集上的一致逼近.

13.6 定理 设 K 是平面内的一个紧集, 且 $\{\alpha_j\}$ 是一个集, 对于 S^2-K 的每个分支它都包含其中的一个点. 如果 Ω 为开集, Ω

$\supset K$, $f\in H(\Omega)$, 且 $\varepsilon>0$, 则存在一个有理函数 R, 其所有极点在预定的集$\{\alpha_j\}$内, 使得对于每一个 $z\in K$

(1) $$|f(z)-R(z)|<\varepsilon$$

注意 S^2-K 至多有可数多个分支. 同时注意在 S^2-K 的无界分支内选定的点最好是∞; 事实上, 这恰好是最有趣的选择.

证明 我们考虑其元素为 K 上的连续复函数, 具有上确界范数的 Banach 空间 $C(K)$. 设 M 为 $C(K)$的子空间, 这个子空间由全部极点在$\{\alpha_j\}$内的那些有理函数在 K 上的限制所组成. 定理断言 f 在 M 的闭包内. 根据定理 5.19(Hahn-Banach 定理的推论), 这就等于说, 每一个在 M 上为零的在 $C(K)$上的有界线性泛函在 f 处亦为零, 因而 Riesz 表示定理(定理 6.19)指出必须要证明下面的结论:

如果 μ 为 K 上的复 Berel 测度, 对其极点只在集$\{\alpha_j\}$内的每一个有理函数 R, 满足

(2) $$\int_K R\,d\mu=0$$

且若 $f\in H(\Omega)$, 则同样有

(3) $$\int_K f\,d\mu=0$$

让我们就这样假设 μ 满足(2). 定义

(4) $$h(z)=\int_K \frac{d\mu(\zeta)}{\zeta-z} \qquad (z\in S^2-K)$$

根据定理 10.7(使 $X=K$, $\varphi(\zeta)=\zeta$), 则 $h\in H(S^2-K)$.

设 V_j 为 S^2-K 的包含 α_j 的分支, 且设 $D(\alpha_j;\ r)\subset V_j$. 如果 $\alpha_j\neq\infty$且 z 固定在 $D(\alpha_j;r)$内, 则

(5) $$\frac{1}{\zeta-z}=\lim_{N\to\infty}\sum_{n=0}^{N}\frac{(z-\alpha_j)^n}{(\zeta-\alpha_j)^{n+1}}$$

对 $\zeta\in K$ 是一致的. 式(5)右边每个函数对(2)是适用的. 因此, 对

所有 $z\in D(\alpha_j;r)$, $h(z)=0$. 根据唯一性定理 10.18, 这就推出, 对所有 $z\in V_j$, $h(z)=0$.

如果 $\alpha_j=\infty$, 用

$$\frac{1}{\zeta-z}=-\lim_{N\to\infty}\sum_{n=0}^{N}z^{-n-1}\zeta^n \qquad (\zeta\in K,\ |z|>r) \tag{6}$$

代替(5), 则在 $D(\infty;r)$内可以再一次推出 $h(z)=0$, 因而在 V_j 内 $h(z)=0$. 因此由(2)我们已经证明

$$h(z)=0 \qquad (z\in S^2-K) \tag{7}$$

和定理 13.5 一样, 现在在 $\Omega-K$ 内选取一个圈 Γ, 且将 f 的 Cauchy 积分表示对 μ 进行积分. 应用Fubini 定理(由于我们是处理 Berel 测度和紧空间上的连续函数, 所以是合法的)并结合(7)式, 给出

$$\begin{aligned}\int_K f d\mu &= \int_K d\mu(\zeta)\left[\frac{1}{2\pi i}\int_\Gamma \frac{f(w)}{w-\zeta}dw\right]\\ &=\frac{1}{2\pi i}\int_\Gamma f(w)dw\int_K\frac{d\mu(\zeta)}{w-\zeta}\\ &=\frac{-1}{2\pi i}\int_\Gamma f(w)h(w)dw=0\end{aligned}$$

最后的等式依赖于 $\Gamma^*\subset\Omega-K$ 的事实, 此处 $h(w)=0$.

因此(3)成立, 同时完成了定理的证明.

下面的特殊情况是特别有趣的.

13.7 定理 设K是平面内的紧集,S^2-K连通,且$f\in H(\Omega)$, 此处Ω为包含 K 的某个开集. 则有一个多项式序列 $\{P_n\}$, 使得 $P_n(z)\longrightarrow f(z)$在 K上是一致的.

证明 由于 S^2-K现在只有一个分支, 所以只需要找一个点 α_j 以便于应用定理 13.6, 而我们可以取 $\alpha_j=\infty$.

13.8 评注 对于使S^2-K为非连通的平面内的每一个紧集 K, 前述结果不成立. 因为, 在那种情况下, S^2-K有一个有界分支

V. 选取 $\alpha\in V$, 令 $f(z)=(z-\alpha)^{-1}$, 且令 $m=\max\{|z-\alpha|: z\in K\}$. 设 P 是一个多项式, 使得对所有 $z\in K$, $|P(z)-f(z)|<\frac{1}{m}$. 则

$$|(z-\alpha)P(z)-1|<1 \qquad (z\in K) \tag{1}$$

特别, 如果 z 在 V 的边界上, 则(1)成立; 由于 V 的闭包是紧的, 所以最大模定理表明对每一个 $z\in V$, (1)成立; 取 $z=\alpha$ 便得到 $1<1$. 因此一致逼近是不可能的.

同样的论证表明, 在定理 13.6 中, 没有一个 α_j 可以被省去.

现在我们应用先前的逼近定理在开集内作逼近. 我们强调指出, 在定理 13.6 和 13.7 中, 没有假定 K 是连通的, 并且在如下的定理中, 将不假定 Ω 是连通的.

13.9 定理 令 Ω 为平面内的开集, 令 A 是一个集, 它在 $S^2-\Omega$ 的每个分支内有一个点, 且设 $f\in H(\Omega)$. 则存在极点只在 A 内的有理函数的一个序列 $\{R_n\}$, 使得 $R_n\longrightarrow f$ 在 Ω 的紧子集上是一致的.

在 $S^2-\Omega$ 为连通的特殊情况下, 可以取 $A=\{\infty\}$, 并因此得到多项式 P_n, 使得 $P_n\longrightarrow f$ 在 Ω 的紧子集上是一致的.

注意, $S^2-\Omega$ 可以有不可数个分支; 例如, $S^2-\Omega=\{\infty\}\cup C$, 此处 C 是 Cantor 集.

证明 选取 Ω 内具有定理 13.3 中所规定性质的紧集 K_n 的一个序列. 暂时固定 n. 由于 S^2-K_n 的每个分支包含 $S^2-\Omega$ 的一个分支, 故 S^2-K_n 的每个分支包含 A 的一点, 所以定理 13.6 给出极点在 A 内的一个有理函数 R_n, 使得 .

$$|R_n(z)-f(z)|<\frac{1}{n} \qquad (z\in K_n) \tag{1}$$

现在, 如果 K 为 Ω 内任意紧集, 则存在一个 N, 对所有 $n\geqslant N$, 使得 $K\subset K_n$. 从(1)得到

(2) $$|R_n(z)-f(z)|<\frac{1}{n} \qquad (z\in K, n\geqslant N)$$

这便完成了定理的证明.

Mittag-Leffler 定理

现在将用 Runge 定理来证明,能够构造出具有任意预先指定极点的亚纯函数.

13.10 定理 设Ω是平面内的开集,$A\subset\Omega$,A在Ω内没有极限点,且对每个$\alpha\in A$,对应有一个正整数$m(\alpha)$和一个有理函数

$$P_\alpha(z)=\sum_{j=1}^{m(\alpha)}c_{j,\alpha}(z-\alpha)^{-j}$$

则在Ω内存在一个亚纯函数f,它在每个$\alpha\in A$处的主要部份是P_α且在Ω内没有其他极点.

证明 如同定理 13.3 那样,我们选取Ω内紧集的一个序列:对$n=1,2,3,\cdots$,K_n在K_{n+1}的内部,Ω的每一个紧子集在某个K_n内,且S^2-K_n的每一个分支包含$S^2-\Omega$的一个分支.设$A_1=A\cap K_1$,且对$n=2,3,4,\cdots$,$A_n=A\cap(K_n-K_{n-1})$.由于$A_n\subset K_n$且A在Ω内没有极限点(因而在K_n内没有极限点),所以每个A_n是一个有限集.设

(1) $$Q_n(z)=\sum_{\alpha\in A_n}P_\alpha(z) \qquad (n=1,2,3,\cdots)$$

由于每个A_n为有限,所以每个Q_n是一个有理函数.当$n\geqslant 2$时,Q_n的极点在K_n-K_{n-1}内.特别,Q_n在包含K_{n-1}的一个开集内是全纯的.现在从定理 13.6 得到,存在一个其所有极点在$S^2-\Omega$内的有理函数R_n,使得

(2) $$|R_n(z)-Q_n(z)|<2^{-n} \qquad (z\in K_{n-1})$$

我们断言

$$f(z)=Q_1(z)+\sum_{n=2}^{\infty}(Q_n(z)-R_n(z)) \qquad (z\in\Omega) \tag{3}$$

具有所想要的性质.

固定 N. 在 K_N 上, 我们有

$$f=Q_1+\sum_{n=2}^{N}(Q_n-R_n)+\sum_{N+1}^{\infty}(Q_n-R_n) \tag{4}$$

根据(2), (4)中最后和式的每一项在 K_N 上小于 2^{-n}; 因此在 K_N 上, 此级数一致收敛于在 K_N 内部全纯的一个函数. 由于每个 R_n 的极点在 Ω 的外部, 所以

$$f-(Q_1+\cdots+Q_N)$$

在 K_N 内部全纯. 因此, 在 K_N 的内部 f 恰好具有规定的主要部分. 由于 N 是任意的, 因而在 Ω 内 f 具有同样的性质.

单连通区域

我们现在将概述单连通区域的一些性质(参看 10.38 节), 这些性质阐明它在全纯函数理论中起的重要作用. 在这些性质中, (*a*)和(*b*)称为 Ω 的内拓扑性质; (*c*)和(*d*) 涉及 Ω 嵌入 s^2 内的方式; 性质(*e*)到(*h*)按特征来说是分析性的; (*i*)是关于环 $H(\Omega)$ 的代数陈述. Riemann 映射定理 14.8 是单连通区域另一个十分重要的性质. 事实上, 我们将用它去证明 $(i)\to(a)$.

13.11 定理 对于一个平面区域 Ω, 下面九个条件中的每一个蕴涵着其余的各个条件.

(*a*) Ω 同胚于开单位圆盘 U.

(*b*) Ω 是单连通的.

(*c*) 对 Ω 内每一条闭路径 γ 和对每一个 $\alpha\in S^2-\Omega$, $\mathrm{Ind}_\gamma(\alpha)=0$.

(*d*) $S^2-\Omega$ 是连通的.

(e) 每一个 $f\in H(\Omega)$ 能用多项式在 Ω 的紧子集上一致逼近.

(f) 对每一个 $f\in H(\Omega)$ 和在 Ω 内每一条闭路径 γ,

$$\int_\gamma f(z)dz=0$$

(g) 每一个 $f\in H(\Omega)$ 对应一个 $F\in H(\Omega)$, 使得 $F'=f$.

(h) 如果 $f\in H(\Omega)$ 且 $1/f\in H(\Omega)$, 则存在一个 $g\in H(\Omega)$, 使得

$$f=\exp(g).$$

(i) 如果 $f\in H(\Omega)$ 且 $1/f\in H(\Omega)$, 则存在一个 $\varphi\in H(\Omega)$, 使得 $f=\varphi^2$.

断言(h)是 f 有一个 Ω 内的"全纯的对数" g; (i)断言 f 有一个 Ω 内的"全纯的平方根" φ; 而(f)断言, 对单连通区域内的每条闭路径, Cauchy 定理成立.

在第十六章中, 我们将看到单值性定理还描述了单连通区域的另一特性.

证明 (a)蕴涵(b). 这就是说, Ω 是同胚于 U 的, 意味着有一个将 Ω 映到 U 上的连续的一一映射 ψ, 其逆 ψ^{-1} 亦为连续. 如果 γ 是 Ω 内具有参数区间 $[0,1]$ 的一条闭曲线, 设

$$H(s,t)=\psi^{-1}(t\psi(\gamma(s)))$$

则 H: $I^2\longrightarrow\Omega$ 是连续的; $H(s,0)=\psi^{-1}(0)$ 是常数; $H(s,1)=\gamma(s)$; 且因 $\gamma(0)=\gamma(1)$ 有 $H(0,t)=H(1,t)$. 因此 Ω 是单连通的.

(b)蕴涵(c). 如果(b)成立且 γ 是 Ω 内一条闭路径, 则 γ 是 Ω-同伦于一条不变的路径(由"单连通"的定义). 因此, 根据定理 10.40, (c)成立.

(c)蕴涵(d). 设(d)不成立. 则 $S^2-\Omega$ 是 S^2 的一个非连通的闭子集. 如 10.1 节所指出的, 得知 $S^2-\Omega$ 是两个非空不交的闭集 H 和 K 的并. 令 H 包含 ∞. 令 W 为相对于平面的 H 的余集. 则

$W=\Omega\cup K$. 由于 K 是紧的，所以定理 13.5(令 $f=1$)表明有 $W-K=\Omega$ 内的一个圈 Γ，对 $z\in K$ 使得 $\mathrm{Ind}_\Gamma(z)=1$. 由于 $K\neq\emptyset$，所以(c)不成立.

(d)蕴涵(e). 这是定理 13.9 的一部分.

(e)蕴涵(f). 选择 $f\in H(\Omega)$，设 γ 为 Ω 内一条闭路径，且选取在 γ^* 上一致收敛于 f 的多项式 P_n. 由于对所有 n，$\int_\gamma P_n(z)dz=0$，所以我们断定(f)成立.

(f)蕴涵(g). 设(f)成立，固定 $z_0\in\Omega$，且设

$$F(z)=\int_{\Gamma(z)}f(\zeta)d\zeta \qquad (z\in\Omega) \tag{1}$$

此处 $\Gamma(z)$ 是 Ω 内从 z_0 到 z 的任意路径. 这便定义了 Ω 内的一个函数 F. 因若 $\Gamma_1(z)$ 是从 z_0 到 z 的另一条路径(在 Ω 内)，则 Γ 之后跟着 Γ_1 的反向路径形成 Ω 内的一条闭路径，f 沿这条闭路的积分为零. 所以若用 $\Gamma_1(z)$ 代替 $\Gamma(z)$，则(1)不受影响. 现在论证 $F'=f$. 固定 $a\in\Omega$. 存在 $r>0$，使得 $D(a;r)\subset\Omega$. 对 $z\in D(a;r)$，我们可以通过 f 沿着由区间 $[a,z]$ 得到的路径 $\Gamma(a)$ 的积分来计算 $F(z)$. 因而，对 $z\in D'(a;r)$

$$\frac{F(z)-F(a)}{z-a}=\frac{1}{z-a}\int_{[a,z]}f(\zeta)d\zeta \tag{2}$$

如同定理 10.14 的证明一样，由 f 在 a 处的连续性推出 $F'(a)=f(a)$.

(g)蕴涵(h). 如果 $f\in H(\Omega)$ 且 f 在 Ω 内没有零点，则 $f'/f\in H(\Omega)$，且由(g)推出存在 $g\in H(\Omega)$，使得 $g'=f'/f$. 我们能够对 g 加上一个常数，使得对某个 $z_0\in\Omega$ 有 $\exp\{g(z_0)\}=f(z_0)$. g 的选择表明，在 Ω 内 fe^{-g} 的导数为零. 因此 fe^{-g} 为常数(因 Ω 为连通)，且由此得到 $f=e^g$.

(*h*)蕴涵(*i*). 由(*h*), $f=e^g$. 设 $\varphi=\exp\left(\frac{1}{2}g\right)$.

(*i*)蕴涵(*a*). 如果Ω为整个平面，则Ω是同胚于U的：将 z 映为 $z/(1+|z|)$.

如果Ω是满足(*i*)的平面的一个真子区域，则确实存在把Ω映到U上的一个全纯的同胚(保形映射). 此结论就是 Riemann 映射定理，它是下一章的主要对象. 因此，Riemann 映射定理的证明与定理 13.11 的证明将一道完成(参看定理 14.8 陈述后面的注).

(*h*)在每个单连通区域内成立这个事实具有如下的推论(此结论也可以用很初等的工具去证明).

13.12 定理 如果$f\in H(\Omega)$，此处Ω为平面内任意开集，且f在Ω内没有零点，则 $\log|f|$在Ω内调和.

证明 对每一个圆盘 $D\subset\Omega$，对应一个函数 $g\in H(D)$，在 D 内使得 $f=e^g$. 如果 $u=\mathrm{Re}\,g$，则 u 在 D 内调和，且 $|f|=e^u$. 因此，$\log|f|$在Ω内的每一个圆盘内调和，这就得出所希望的论断.

习　题

1　证明在 S^2 上的每一个亚纯函数是有理函数.

2　令 $\Omega=\{z:|z|<1$ 和 $|2z-1|>1\}$，且设 $f\in H(\Omega)$.

(*a*) 是否必定存在一个多项式 P_n 的序列，使得 $P_n\to f$ 在Ω的紧子集上是一致的？

(*b*) 是否必定存在这样一个在Ω内一致收敛于 f 的序列？

(*c*) 如果我们对 f 要求的条件更多些，即 f 在包含Ω的闭包的某个开集内全纯，则对(*b*)的回答有改变吗？

3　有没有多项式 P_n 的序列，使得对 $n=1,2,3,\cdots$，$P_n(0)=1$，但当 $n\to\infty$时，对每一个 $z\neq0$，$P_n(z)\to0$？

4　有没有多项式 P_n 的序列，使得

$$\lim_{n\to\infty}P_n(z)=\begin{cases}1 & \text{当 } \mathrm{Im}\,z>0,\\ 0 & \text{当 } z \text{ 为实数},\\ -1 & \text{当 } \mathrm{Im}\,z<0?\end{cases}$$

5　对 $n=1, 2, 3, \cdots$，令 Δ_n 为 U 内的闭圆盘，且令 L_n 为 $U-\Delta_n$ 内的与 U 的每一条半径相交的一段弧（$[0, 1]$ 的同胚像）．存在多项式 P_n，在 Δ_n 上十分小而在 L_n 上或多或少是任意的．证明 $\{\Delta_n\}$，$\{L_n\}$ 和 $\{P_n\}$ 可以这样选择，使得级数 $f=\sum P_n$ 定义一个在 T 的任意点没有径向极限的函数 $f\in H(U)$．换言之，没有实数 θ，使得 $\lim_{r\to 1} f(re^{i\theta})$ 存在．

6　下面是这种函数的另一种构造法．令 $\{n_k\}$ 为整数序列，使得 $n_1>1$ 和 $n_{k+1}>2kn_k$．定义

$$h(z)=\sum_{k=1}^{\infty} 5^k z^{n_k}$$

证明若 $|z|<1$，则级数收敛．并证明有一个常数 $c>0$，对所有使 $|z|=1-(1/n_m)$ 的 z，$|h(z)|>c\cdot 5^m$．［提示：对那样的 z，在所定义的 $h(z)$ 级数中第 m 项必远远大于所有其他项的和］．

因而 h 没有有限的径向极限．

同时证明 h 在 U 内必有无限多个零点（与第十二章习题 15 作比较）．事实上，证明对每一个复数 α 都对应有无限多个 $z\in U$，使 $h(z)=\alpha$．

7　证明在定理 13.9 中，我们不必假设 A 与 $S^2-\Omega$ 的每个分支相交．只要假定 A 的闭包与 $S^2-\Omega$ 的每个分支相交就够了．

8　对 Ω 为全平面的情况，通过直接论证而不求助于 Runge 定理，证明 Mittag-Leffler 定理．

9　设 Ω 为单连通区域，$f\in H(\Omega)$，f 在 Ω 内没有零点，且 n 为正整数．证明存在一个 $g\in H(\Omega)$，使得 $g^n=f$．

10　设 Ω 为一个区域，$f\in H(\Omega)$，且 $f\not\equiv 0$．证明：当且仅当 f 在 Ω 内对每一个正整数 n 有全纯的 n 次方根时，f 在 Ω 内有一个全纯的对数．

11　设 $f_n\in H(\Omega)$ $(n=1, 2, 3, \cdots)$，f 是 Ω 内的复函数，且对每一个 $z\in\Omega$，$f(z)=\lim_{n\to\infty} f_n(z)$．证明 Ω 有一个稠密开子集 V，f 在 V 上全纯．提示：设 $\varphi=\sup|f_n|$．利用 Baire 定理证明：Ω 内每一个圆盘包含一个在其上 φ 为有界的圆盘．应用第十章习题 5．（在一般情况下，$V\neq\Omega$．与习题 3 和 4 比较．）

第十四章　保形映射

角的保持性

14.1　定义　每一个复数 $z\neq 0$ 可以决定一个从原点发出的方向，这方向是由在单位圆周上的点

$$A[z]=\frac{z}{|z|} \tag{1}$$

确定的.

假设 f 是一个将区域 Ω 映入平面内的映射，$z_0\in\Omega$ 并且 z_0 有去心邻域 $D'(z_0,r)\subset\Omega$，在这邻域里 $f(z)\neq f(z_0)$. 如果

$$\lim_{r\to 0} e^{-i\theta}A[f(z_0+re^{i\theta})-f(z_0)] \qquad (r>0) \tag{2}$$

存在并与 θ 无关，我们称 f 在 z_0 保角.

粗略地说，这个要求就是对任意两条由 z_0 发出的射线 L' 和 L''，它们的象 $f(L')$ 和 $f(L'')$ 在 $f(z_0)$ 所作成的角与 L' 和 L'' 作成的角，其大小和定向都一样.

在一个区域内每一点保角这个性质，是在这区域内导数不为零的全纯函数的特征. 这是定理 14.2 的一个推论，也是把导数不为零的全纯函数称为保形映射的理由.

14.2　定理　设 f 将一区域 Ω 映入平面内. 若对于某个 $z_0\in\Omega$，$f'(z_0)$ 存在并且 $f'(z_0)\neq 0$，则 f 在 z_0 保角. 相反，若 f 在 z_0 的微分存在并且不为零，又如果 f 在 z_0 保角，则 $f'(z_0)$ 存在并且不为零.

这里像平常一样，$f'(z_0)=\lim[f(z)-f(z_0)]/(z-z_0)$. 而 f 在 z_0 的微分是 R^2 到 R^2 的一个线性变换 L，使得记 $z_0=(x_0,y_0)$

时，

(1) $$f(x_0+x,y_0+y)=f(x_0,y_0)+L(x,y)+(x^2+y^2)^{1/2}\eta(x,y)$$

象在定义 8.22 中一样，这里当 $x\to 0$ 及 $y\to 0$ 时，$\eta(x,y)\to 0$.

证明 为简单起见，取 $z_0=f(z_0)=0$. 若 $f'(0)=a\neq 0$，则立即得到

(2) $$e^{-i\theta}A[f(re^{i\theta})]=\frac{e^{-i\theta}f(re^{i\theta})}{|f(re^{i\theta})|}\to\frac{a}{|a|}\qquad(r\to 0)$$

于是 f 在 0 保角. 反之，若 f 在 0 的微分存在且不为零，则(1)式可改写成这样的形式

(3) $$f(z)=\alpha z+\beta\bar{z}+|z|\eta(z)$$

其中当 $z\to 0$ 时，$\eta(z)\to 0$，并且 α 和 β 是不同时为零的复数. 如果 f 也在 0 保角，则

(4) $$\lim_{r\to 0}e^{-i\theta}A[f(re^{i\theta})]=\frac{\alpha+\beta e^{-2i\theta}}{|\alpha+\beta e^{-2i\theta}|}$$

存在并且与 θ 无关. 我们可以除掉那些使(4)式中分母为零的 θ；而这样的 θ 在 $[0,2\pi)$ 内最多有两个. 对所有其他的 θ，我们断定 $\alpha+\beta e^{-2i\theta}$ 位于一条通过 0 的固定的射线上，这只有当 $\beta=0$ 时才有可能. 因此 $\alpha\neq 0$，从(3)式就推出 $f'(0)=\alpha$.

注：没有全纯函数能在其导数为零的任何一点是保角的. 我们省去这个容易的证明. 但是，一个变换在微分为零的点还有可能保角. 例如，$f(z)=|z|z$，$z_0=0$.

线性分式变换

14.3 如果 a,b,c 和 d 都是复数，并且 $ad-bc\neq 0$，则映射

(1) $$z\to\frac{az+b}{cz+d}$$

称为线性分式变换. 由于考虑点∞会带来明显的方便，将(1)式看作由球面 S^2 映入球面 S^2 是合适的. 例如，当 $c\neq 0$ 时，$-d/c$ 映

为∞,而∞映为a/c. 于是容易明白每个线性分式变换是一个将S^2映到S^2上的一一映射. 而且, 每个线性分式变换可由叠加下列形式的变换得到:

(a) 平移: $z\to z+b$.

(b) 旋转: $z\to az$, $|a|=1$.

(c) 相似: $z\to rz$, $r>0$.

(d) 反演: $z\to 1/z$.

若在(1)式中, $c=0$, 这时是很明显的. 若 $c\neq 0$, 可从恒等式

$$\frac{az+b}{cz+d}=\frac{a}{c}+\frac{\lambda}{cz+d},\quad \lambda=\frac{bc-ad}{c} \tag{2}$$

得出.

易知前三个变换将直线变为直线, 圆变为圆. 而对(d)这是不成立的. 但是, 如果我们设$\mathscr{F}$是所有直线和圆组成的族, 则$\mathscr{F}$对(d)是保持的. 因此, 我们得到一个重要的结论: $\mathscr{F}$对每个线性分式变换都是保持的. [读者可能会注意到, 当把$\mathscr{F}$看成S^2的一个子集族时, 则通过球极平面射影 13.1(1), $\mathscr{F}$由S^2上的所有圆组成; 我们不打算采用$\mathscr{F}$这个性质, 并且省略其证明.]

易证经过反演变换$\mathscr{F}$仍然保持, 初等解析几何表明$\mathscr{F}$中每一元素是方程

$$\alpha z\bar{z}+\beta z+\bar{\beta}\bar{z}+\gamma=0 \tag{3}$$

的轨迹, 其中α和γ是实常数, 而β是复常数, 并且$\beta\bar{\beta}>\alpha\gamma$. 若$\alpha\neq 0$, 则(3)式确定一个圆; $\alpha=0$, 则给出直线. 用$\frac{1}{z}$换z, 则(3)式变成

$$\alpha+\beta\bar{z}+\bar{\beta}z+\gamma z\bar{z}=0 \tag{4}$$

这是相同类型的方程.

设a, b和c是不同的复数. 我们构造一个线性分式变换φ, 它将有序三元组$\{a,b,c\}$映为$\{0,1,\infty\}$, 即,

(5) $$\varphi(z)=\frac{(b-c)(z-a)}{(b-a)(z-c)}$$

这样的 φ 是唯一的. 因为若 $\varphi(a)=0$, 则必定在分子中有 $z-a$; 若 $\varphi(c)=\infty$, 则在分母中必定有 $z-c$; 若同时有 $\varphi(b)=1$, 我们就会得到(5)式. 如果 a,b,c 中有一个是∞, 类似(5)的公式是容易写出的. 如果我们在(5)之后再作一个相同类型变换的逆变换, 便得到下面的结果:

对 S^2 内的任意两个有序三元组$\{a,b,c\}$和$\{a',b',c'\}$, 都有而且仅有一个线性分式变换将 a 映射为 a', b 映射为 b', 并且将 c 映射为c'.

(当然, 假定 $a\neq b$, $a\neq c$, 和 $b\neq c$, 并且 a', b' 和 c' 也是同样假设.)

我们从这个结果可以得出结论: 每一个圆周都可以经过一个线性分式变换映到任一个圆周上. 更有兴趣的是, 每一个圆都能映到任一条直线上(若将点∞看作直线上的点), 因此, 每一开圆盘能够保形映射到任一开的半平面上.

我们来更详尽地讨论这种映射中的一个, 即

(6) $$\varphi(z)=\frac{1+z}{1-z}$$

这个 φ 映$\{-1,0,1\}$为$\{0,1,\infty\}$: 线段$(-1,1)$映射为正实轴, 单位圆周 T 经过-1和 1. 因此, $\varphi(T)$是一条经过 $\varphi(-1)=0$的直线. 因为单位圆 T 在 -1 处与实轴成一直角, 故 $\varphi(T)$在 0 与实轴成直角. 因此, $\varphi(T)$ 是虚轴. 因为 $\varphi(0)=1$, 可知 φ 是一个将开单位圆盘映到开右半平面上的一一保形映射.

线性分式变换在保形映射理论中的作用也在定理 12.6 中得到很好的说明.

14.4 线性分式变换使得有可能把关于全纯函数在接近直线

时的性态的定理代以圆弧的情况. 反射原理的非正式讨论，将足以说明这个方法.

设 Ω 是 U 内一个区域，它的边界有一部分是单位圆上的圆弧 L，f 在 $\overline{\Omega}$ 上连续，在 Ω 内全纯，并且在 L 上取实值. 函数

$$\psi(z)=\frac{z-i}{z+i} \tag{1}$$

把上半平面映到 U 上. 若 $g=f\circ\psi$，则定理 11.17 给出 g 的一个全纯开拓 G，于是 $F=G\circ\psi^{-1}$ 给出 f 的一个全纯开拓 F，满足

$$f(z^*)=\overline{F(z)} \tag{2}$$

其中 $z^*=1/\bar{z}$.

最后一个断言可由 ψ 的一个性质得出：若 $w=\psi(z)$ 和 $w_1=\psi(\bar{z})$，则 $w_1=w^*$. 这容易通过计算验证.

习题 2—5 提供这一技巧的其它应用.

正规族

Riemann 映射定理可以通过把映射函数展示为某个极值问题的解而予以证明. 这种解的存在有赖于我们现在要叙述的某些全纯函数族的一个非常有用的紧性.

14.5 定义 设对于区域 Ω，$\mathscr{F}\subset H(\Omega)$. 若族 $\mathscr{F}$ 中任一序列包含一个子序列，它在 Ω 的每一个紧子集上一致收敛，则称族 $\mathscr{F}$ 为正规族. 这个极限函数并不要求属于 $\mathscr{F}$.

（有时采用一个比较广泛的定义，只要族 $\mathscr{F}$ 中任一序列在 Ω 的每一紧子集上或者一致收敛，或者一致地趋于 ∞. 这对于讨论亚纯函数是比较合适的.）

14.6 定理 设族 $\mathscr{F}$ 在区域 Ω 任一紧子集上一致有界并且 $\mathscr{F}\subset H(\Omega)$，则族 $\mathscr{F}$ 是正规族.

证明 这个假设意味着对于每一紧集 $K\subset\Omega$，都有一个数

$M(K)<\infty$与之对应，使得对所有 $f\in\mathscr{F}$ 及 $z\in K$ 均有 $|f(z)|\leqslant M(K)$.

设$\{K_n\}$是一个紧集序列，其并等于 Ω，且 K_n 包含在 K_{n+1} 的内部. 这种序列在定理 13.3 中曾经构造过. 于是存在正数 δ_n 满足

$$D(z;2\delta_n)\subset K_{n+1} \qquad (z\in K_n) \tag{1}$$

考虑 K_n 中两点 z' 和 z''，使得 $|z'-z''|<\delta_n$. 令 γ 为圆心在 z' 半径是 $2\delta_n$ 的正向圆周，并用 Cauchy 公式估计 $|f(z')-f(z'')|$. 因为

$$\frac{1}{\zeta-z'}-\frac{1}{\zeta-z''}=\frac{z'-z''}{(\zeta-z')(\zeta-z'')}$$

我们得到

$$f(z')-f(z'')=\frac{z'-z''}{2\pi i}\int_\gamma\frac{f(\zeta)}{(\zeta-z')(\zeta-z'')}d\zeta \tag{2}$$

又因为对所有 $\zeta\in\gamma^*$，$|\zeta-z'|=2\delta_n$ 及 $|\zeta-z''|>\delta_n$，所以(2)式给出不等式

$$|f(z')-f(z'')|<\frac{M(K_{n+1})}{\delta_n}|z'-z''| \tag{3}$$

它对于所有 $f\in\mathscr{F}$，所有 z' 及 $z''\in K_n$，在假定 $|z'-z''|<\delta_n$ 时成立.

这是证明的关键性步骤：我们已经证明，对于每一 K_n，$\mathscr{F}$ 中的函数在 K_n 上的限制构成一个同等连续函数族. 根据定义，这意味着对任意 $\varepsilon>0$，存在 $\delta>0$，对于所有 $f\in\mathscr{F}$ 及所有 z' 和 $z''\in K_n$，只要 $|z'-z''|<\delta$，都有 $|f(z')-f(z'')|<\varepsilon$. 当取

$$\delta=\frac{\varepsilon\delta_n}{\varepsilon+M(K_{n+1})} \tag{4}$$

时，略为考察一下(3)式就知道，这一要求能够满足.

现在设$\{f_m\}$是在 $\mathscr{F}$ 中的一个序列. 选一个可数集 $E\subset\Omega$，使

得对于 $n=1, 2, 3, \cdots$, $E \cap K_n$ 在 K_n 内稠密. 将 E 的点排成序列 $\{w_j\}$. 因为 $\{f_m(z)\}$ 对任意 $z \in \Omega$ 都是有界的, 故 $\{f_m\}$ 有一子序列, 比如说 $\{f_{m,1}\}$, 它在 w_1 收敛. 由 $\{f_{m,1}\}$ 我们可选出一子序列, 比如说 $\{f_{m,2}\}$, 它也在 w_2 收敛. 依同样方法进行, 我们得到在 w_i 收敛的序列 $\{f_{m,i}\}$, 并且 $\{f_{m,i}\}$ 是 $\{f_{m,i-1}\}$ 的一个子序列. 因此"对角线序列" $\{f_{m,m}\}$ 在 E 的每一点收敛.

我们断言 $\{f_{m,m}\}$ 在每一个 K_n 上一致收敛(因而在 Ω 的每一个紧子集上一致收敛).

固定 K_n, 固定 $\varepsilon>0$, 并像(4)式一样选取 δ. 因为 K_n 是紧集, K_n 包含属于 E 的点: $z_1, \cdots, z_p$, 使得圆盘 $D(z_i; \delta)$ $(i=1, \cdots, p)$ 的并覆盖 K_n, 并且存在一个正整数 N, 当 $r>N, s>N$ 及 $1 \leqslant i \leqslant p$ 时,

$$|f_{r,r}(z_i)-f_{s,s}(z_i)|<\varepsilon \tag{5}$$

对每一个 $z \in K_n$, 存在有一个 z_i, 使 $1 \leqslant i \leqslant p$, $|z-z_i|<\delta$. 因此 $|f_{r,r}(z)-f_{s,s}(z)|$ 不大于

$$|f_{r,r}(z)-f_{r,r}(z_i)|+|f_{r,r}(z_i)-f_{s,s}(z_i)|+|f_{s,s}(z_i)-f_{s,s}(z)| \tag{6}$$

由我们所选取的 δ 得知, (6) 式中第一个和第三个差的模都小于 ε; 第二个差的模当 $r>N$ 及 $s>N$ 时也小于 ε. 因此

$$|f_{r,r}(z)-f_{s,s}(z)|<3\varepsilon \tag{7}$$

对每一个 $z \in K_n$, 当 $r>N$ 及 $s>N$ 时成立.

这就完成了证明.

Riemann 映射定理

14.7 保形等价性 我们称两个区域 Ω_1 和 Ω_2 是保形等价的, 如果存在一个映射 $\varphi \in H(\Omega_1)$, φ 在 Ω_1 中是一一的, 并且 $\varphi(\Omega_1)=\Omega_2$, 换句话说, 如果存在一个 Ω_1 到 Ω_2 上的一一保形映射. 在这些条件下, φ 的逆映射在 Ω_2 是全纯的(定理 10.33), 因而是一个 Ω_2

到Ω_1上的保形映射.

由此得知保形等价区域都是同胚的. 但是, 保形等价区域之间有一个更重要的关系: 设φ如上所述, 则$f\to f\circ\varphi$是$H(\Omega_2)$到$H(\Omega_1)$上的保持和与乘积的一一映射,即它是由$H(\Omega_2)$到$H(\Omega_1)$上的环同构. 如果Ω_1有一个简单的结构,那末关于$H(\Omega_2)$的问题就能转化为$H(\Omega_1)$中的问题, 并且它的解可以借助于映射函数φ返回到$H(\Omega_2)$. 这个问题最重要的场合是基于Riemann映射定理(在那里Ω_2是单位圆盘U), 对于平面上任一个单连通真子区域Ω,它把$H(\Omega)$的研究化为$H(U)$的研究. 当然,为了求出问题的显解,可能需要对于映射函数具有相当精确的知识.

14.8 定理 平面内任一(与平面本身不同的)单连通区域Ω都与开单位圆盘U保形等价.

注: 根据Liouville定理, 平面的情况显然已被排除. 因此, 平面不能保形等价于U,尽管这两个区域是同胚的.

在证明中用到单连通区域的性质仅仅在于每一个在这区域中没有零点的全纯函数都有全纯的平方根.从这里将得出定理13.11中"(i)蕴涵(a)"的结论,并将完成这个定理的证明.

证明 设Ω是平面内一个单连通区域, 并设w_0是一个复数; $w_0\notin\Omega$. 设Σ为所有$\psi\in H(\Omega)$构成的函数族,其中ψ是将Ω映入U的一一映射. 我们必须证明有$\psi\in\Sigma$将Ω映到U上.

我们先证Σ非空. 因为Ω是单连通的, 于是存在一个$\varphi\in H(\Omega)$,使得$\varphi^2(z)=z-w_0$在Ω内成立. 若$\varphi(z_1)=\varphi(z_2)$,则也有$\varphi^2(z_1)=\varphi^2(z_2)$,从而$z_1=z_2$;于是$\varphi$是一一的. 同理可证, Ω中没有两点z_1和z_2满足$\varphi(z_1)=-\varphi(z_2)$. 因为$\varphi$是一个开映射, $\varphi(\Omega)$包含一个圆盘$D(a;r)$, 其中$0<r<|a|$. 圆盘$D(-a;r)$,因此与$\varphi(\Omega)$不相交,并且,如果定义$\psi=r/(\varphi+a)$, 我们就会看出$\psi\in\Sigma$.

第二步在于证明，若$\psi\in\Sigma$，而$\psi(\Omega)$不覆盖整个U，且若$z_0\in\Omega$，则存在一个$\psi_1\in\Sigma$，满足

$$|\psi_1'(z_0)|>|\psi'(z_0)|$$

运用由

$$\varphi_a(z)=\frac{z-a}{1-\bar{a}z}$$

定义的函数φ_a将是方便的．对$a\in U$，φ_a是一个将U映射到U上的一一映射；它的逆映射是φ_{-a}(见定理 12.4)．

设$\psi\in\Sigma$，$\alpha\in U$而$\alpha\notin\psi(\Omega)$．则$\varphi_\alpha\circ\psi\in\Sigma$，且$\varphi_\alpha\circ\psi$在$\Omega$内没有零点；因此，存在一个函数$g\in H(\Omega)$，使得$g^2=\varphi_\alpha\circ\psi$．易知$g$是一个一一映射(像在证明$\Sigma\neq\varnothing$时一样)，因此$g\in\Sigma$；若$\psi_1=\varphi_\beta\circ g$，其中$\beta=g(z_0)$，便得$\psi_1\in\Sigma$．记$w^2=s(w)$，我们现在得到

$$\psi=\varphi_{-\alpha}\circ s\circ g=\varphi_{-\alpha}\circ s\circ\varphi_{-\beta}\circ\psi_1$$

因为$\psi_1(z_0)=0$，由链式法则得

$$\psi'(z_0)=F'(0)\psi_1'(z_0)$$

其中$F=\varphi_{-\alpha}\circ s\circ\varphi_{-\beta}$．易知$F(U)\subset U$，并且$F$在$U$内不是一个一一映射．于是，由 Schwarz 引理(见 12.5 节)得$|F'(0)|<1$，因而$|\psi'(z_0)|<|\psi_1'(z_0)|$．[注意，因为$\psi$在$\Omega$内是一个一一映射，故$\psi'(z_0)\neq0$．]

固定$z_0\in\Omega$，并设

$$\eta=\sup\{|\psi'(z_0)|:\psi\in\Sigma\}$$

从上面所述清楚地知道，任何函数$h\in\Sigma$，当$|h'(z_0)|=\eta$时，它将映Ω到U上．因此，只要我们证明存在这样一个函数h，定理就证明了．

由于对所有$\psi\in\Sigma$及$z\in\Omega$，$|\psi(z)|<1$，故定理 14.6 表明Σ是正规族．由η的定义知道在Σ内有序列$\{\psi_n\}$满足$|\psi_n'(z_0)|\to\eta$，并且由Σ的正规性，我们选取一个子序列(为简单起见，仍记为$\{\psi_n\}$)，

它在Ω的紧子集上一致收敛于一个极限$h\in H(\Omega)$. 由定理10.28, $|h'(0)|=\eta$. 因为$\Sigma\neq\varnothing,\eta>0$,所以$h$不是常数. 因为$\psi_n(\Omega)\subset U$, $n=1,2,3,\cdots$, 我们得到$h(\Omega)\subset\bar{U}$,但是开映射定理指出确实有$h(\Omega)\subset U$.

余下的是要证明h是一个一一映射. 取定Ω内不同的两点z_1和z_2; 令$\alpha=h(z_1)$及$\alpha_n=\psi_n(z_1)$, $n=1,2,3,\cdots$, 并设$\bar{D}$是Ω内圆心在z_2的闭圆盘, 使得$z_1\notin\bar{D}$, 并且在$\bar{D}$的边界上$h-\alpha$没有零点. 这是可能的, 因为$h-\alpha$的零点在Ω内没有极限点. 函数$\psi_n-\alpha_n$在$\bar{D}$上一致收敛于$h-\alpha$; 因为它们都是一一映射, 并且在z_1有零点, 所以它们在圆盘D内没有零点; 由 Rouche 定理得到$h-\alpha$在D内没有零点; 特别$h(z_2)\neq h(z_1)$.

于是$h\in\Sigma$, 并且定理证毕.

另一个更富有构造性的证明将略述于习题26.

14.9 评注 前面的证明也指出了$h(z_0)=0$. 因为如果$h(z_0)=\beta$而$\beta\neq0$, 则$\varphi_\beta\circ h\in\Sigma$, 并且

$$|(\varphi_\beta\circ h)'(z_0)|=|\varphi_\beta'(\beta)h'(z_0)|=\frac{|h'(z_0)|}{1-|\beta|^2}>|h'(z_0)|$$

注意到下面这一点是很有意思的. 虽然h是通过对$\psi\in\Sigma$取$|\psi'(z_0)|$的极大值得出的, 如果容许f遍取由所有Ω到U内的全纯映射(不需要是一一的)组成的类, h也是$|f'(z_0)|$的最大值. 因为, 如果f是这样的函数, 则$g=f\circ h^{-1}$映U到U内, 于是$|g'(0)|<1$, 当且仅当g是旋转时等式才成立(由 Schwarz 引理得到), 这样由链式法则得到下述结果:

若$f\in H(\Omega)$, $f(\Omega)\subset U$, 并且$z_0\in\Omega$, 则$|f'(z_0)|\leqslant|h'(z_0)|$. 当且仅当$f(z)=\lambda h(z)$, 其中$\lambda$是$|\lambda|=1$的常数时, 等式成立.

$\mathscr{S}$ 类

14.10 定义 $\mathscr{S}$ 是由在 U 中是一一的，并满足

(1) $$f(0)=0, \quad f'(0)=1$$

的全体 $f\in H(\Omega)$ 组成的类.

这样，每一个 $f\in\mathscr{S}$ 都有幂级数展开式

(2) $$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \qquad (z\in U)$$

$\mathscr{S}$ 类在加法及乘法下是不封闭的，但它有许多别的有趣性质. 我们在本节中将仅仅讨论其中一小部分. 在第二十章中证明 Mergelyan 定理时将用到定理 14.15.

14.11 例 函数

$$f(z)=\frac{z}{(1-z)^2}=\sum_{n=1}^{\infty}nz^n$$

是 $\mathscr{S}$ 类的一个元素.

因为，若 $f(z)=f(w)$，则 $(z-w)(1-zw)=0$，且当 $|z|<1$ 和 $|w|<1$ 时，第二个因式不为零.

我们留给读者对这个函数 f 求 $f(U)$.

14.12 定理 (a) 若 $f\in\mathscr{S}$，$|\alpha|=1$，并且 $g(z)=\bar{\alpha}f(\alpha z)$，则 $g\in\mathscr{S}$. (b) 若 $f\in\mathscr{S}$，则存在一个 $g\in\mathscr{S}$，使得

(1) $$g^2(z)=f(z^2) \qquad (z\in U)$$

证明 (a) 是显然的. 往证 (b)，记 $f(z)=z\varphi(z)$. 则 $\varphi\in H(U)$，$\varphi(0)=1$，并且 φ 在 U 中没有零点，因为 f 在 $U-\{0\}$ 内没有零点. 因此存在一个函数 $h\in H(U)$ 具有 $h(0)=1$，$h^2(z)=\varphi(z)$. 令

(2) $$g(z)=zh(z^2) \qquad (z\in U)$$

则 $g^2(z)=z^2h^2(z^2)=z^2\varphi(z^2)=f(z^2)$，于是得到(1)式. 显然 $g(0)$

$=0$ 和 $g'(0)=1$. 我们来证明 g 是一一映射.

设 z 和 $w\in U$, 并且 $g(z)=g(w)$. 因为 f 是一一映射, (1)蕴涵着 $z^2=w^2$, 这样, 不是 $z=w$(这就是我们要证明的)就是 $z=-w$. 在后一种情形, (2) 式给出 $g(z)=-g(w)$; 由此即得 $g(z)=g(w)=0$, 又由于 g 在 $U-\{0\}$ 内没有零点, 我们得到 $z=w=0$.

14.13 定理 如果 $F\in H(U-\{0\})$, F 在 U 内是一一的, 并且

$$F(z)=\frac{1}{z}+\sum_{n=0}^{\infty}\alpha_n z^n \qquad (z\in U) \tag{1}$$

则

$$\sum_{n=1}^{\infty}n|\alpha_n|^2\leqslant 1 \tag{2}$$

这就是通常说的面积定理, 其理由在证明中将会明白.

证明 α_0 的选取显然是无关紧要的. 因而假设 $\alpha_0=0$. 如果用 $\lambda F(\lambda z)$ ($|\lambda|=1$)代替 $F(z)$, 则既不影响假设也不影响结论. 于是我们可设 α_1 是实数.

设对于 $0<r<1$ $U_r=\{z:|z|<r\}$, $C_r=\{z:|z|=r\}$, 并且设 $V_r=\{z:r<|z|<1\}$, 则 $F(U_r)$ 是点 ∞ 的邻域 (对 $1/F$ 应用开映射定理得出); 因为 F 是一一映射, 集 $F(U_r)$, $F(C_r)$ 和 $F(V_r)$ 是不相交的. 记

$$F(z)=\frac{1}{z}+\alpha_1 z+\varphi(z) \qquad (z\in U) \tag{3}$$

$F=u+iv$, 且

$$A=\frac{1}{r}+\alpha_1 r, \quad B=\frac{1}{r}-\alpha_1 r \tag{4}$$

当 $z=re^{i\theta}$ 时, 于是我们得到

$$u=A\cos\theta+\mathrm{Re}\varphi \quad 和 \quad v=-B\sin\theta+\mathrm{Im}\varphi \tag{5}$$

在等式(5)中分别除以 A 和 B, 平方, 然后相加得到

$$\frac{u^2}{A^2}+\frac{v^2}{B^2}=1+\frac{2\cos\theta}{A}\mathrm{Re}\varphi+\left(\frac{\mathrm{Re}\varphi}{A}\right)^2-\frac{2\sin\theta}{B}\mathrm{Im}\varphi+\left(\frac{\mathrm{Im}\varphi}{B}\right)^2$$

由(3)式知道，φ 在原点至少有二阶零点. 如果我们仍用(4)式，得知存在 $\eta>0$，对所有足够小的 r，满足

$$\frac{u^2}{A^2}+\frac{v^2}{B^2}<1+\eta r^3 \qquad (z=re^{i\theta}) \tag{6}$$

这说明 $F(C_r)$ 在椭圆 E_r 的内部，E_r 的半轴是 $A\sqrt{1+\eta r^3}$ 及 $B\sqrt{1+\eta r^3}$，因此围成一块面积

$$\begin{aligned}\pi AB(1+\eta r^3)&=\pi\left(\frac{1}{r}+\alpha_1 r\right)\left(\frac{1}{r}-\alpha_1 r\right)(1+\eta r^3) \qquad (7)\\ &\leqslant\frac{\pi}{r^2}(1+\eta r^3)\end{aligned}$$

因为 $F(C_r)$ 在 E_r 的内部，我们得到 $E_r\subset F(U_r)$；因而 $F(V_r)$ 在 E_r 的内部，于是 $F(V_r)$ 的面积不大于(7). 从Cauchy-Riemann方程得出映射 $(x,y)\to(u,v)$ 的 Jacobi 行列式是 $|F'|^2$. 因此定理 8.26 给出下述结果：

$$\begin{aligned}\frac{\pi}{r^2}(1+\eta r^3)&\geqslant\iint_{V_r}|F'|^2\\ &=\int_r^1 tdt\int_0^{2\pi}\left|-t^{-2}e^{-2i\theta}+\sum_1^\infty n\alpha_n t^{n-1}e^{i(n-1)\theta}\right|^2 d\theta \qquad (8)\\ &=2\pi\int_r^1\left(t^{-3}+\sum_1^\infty n^2|\alpha_n|^2 t^{2n-1}\right)dt\\ &=\pi\left\{r^{-2}-1+\sum_1^\infty n|\alpha_n|^2(1-r^{2n})\right\}\end{aligned}$$

若在(8)式两边除以 π，然后减去 r^{-2}，我们得到

$$\sum_{n=1}^N n|\alpha_n|^2(1-r^{2n})\leqslant 1+\eta r \tag{9}$$

对所有足够小的 r 和所有正整数 N 成立. 在(9)式中令 $r\to 0$，然后令 $N\to\infty$，就得到(2)式.

系 在上述定理的同样假设下，$|\alpha_1| \leqslant 1$.

$F(z)=(1/z)+\alpha z$，$|\alpha|=1$ 这个在 U 内的一一映射说明这乃是最好的结果。

14.14 定理 如果 $f \in \mathscr{S}$，并且

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n$$

则(a) $|a_2| \leqslant 2$，及(b) $f(U) \supset D\left(0; \frac{1}{4}\right)$.

第二个结论说明 $f(U)$ 包含所有 w，这里 $|w|<\frac{1}{4}$.

证明 由定理 14.12，存在 $g \in \mathscr{S}$，使得 $g^2(z)=f(z^2)$. 如果 $G=1/g$，则将定理 14.13 应用于 G，就得到(a). 因为

$$f(z^2)=z^2(1+a_2 z^2+\cdots)$$

我们得到

$$g(z)=z\left(1+\frac{1}{2}a_2 z^2+\cdots\right)$$

因此

$$G(z)=\frac{1}{z}\left(1-\frac{1}{2}a_2 z^2+\cdots\right)=\frac{1}{z}-\frac{a_2}{2}z+\cdots$$

定理 14.13 的系指出 $|a_2|<2$.

往证(b)，假设 $w \notin f(U)$. 定义

$$h(z)=\frac{f(z)}{1-f(z)/w}$$

则 $h \in H(U)$，h 在 U 内是一一的，并且

$$h(z)=(z+a_2 z^2+\cdots)\left(1+\frac{z}{w}+\cdots\right)=z+\left(a_2+\frac{1}{w}\right)z^2+\cdots$$

因而 $h \in \mathscr{S}$. 对 h 应用(a)；我们有 $|a_2+(1/w)| \leqslant 2$. 又因为 $|a_2| \leqslant 2$，我们最后得到 $|1/w| \leqslant 4$. 所以对于每一个 $w \notin f(U)$，$|w| \geqslant \frac{1}{4}$，

这就完成了证明.

例 14.11 说明(*a*)和(*b*)都是最好结果，对于这个特别的函数 $f\in\mathscr{S}$，点$-\frac{1}{4}$不在 $f(U)$内.

14.15 定理 设 $F\in H(U-\{0\})$，F 在 U 内是一一的，在 $z=0$ 处 F 有一个一阶极点，残数是 1，并且 w_1 和 w_2 都不在 $f(U)$内，则 $|w_2-w_1|\leqslant 4$.

证明 如果 $f=1/(F-w_1)$，则 $f\in\mathscr{S}$，因此 $f(U)\supset D(0,\ \frac{1}{4})$，于是在 $F-w_1$ 的映射下，U 的象包含使 $|w|>4$ 的所有 w. 因为 w_2-w_1 不在这个象内，我们有 $|w_2-w_1|\leqslant 4$.

注意，这也是最好的结果：若 $F(z)=z^{-1}+z$，则 $F(U)$不包含点 2 和 -2. 事实上，$F(U)$在实轴上的余集正好是区间$[-2,2]$.

在边界上的连续性

在一定的条件下，每一个单连通区域 Ω 到 U 上的保形映射都可扩展为它的闭包 $\overline{\Omega}$ 到 $\overline{U}$ 上的同胚. 在这里，Ω 的边界性态起了决定性的作用.

14.16 定义 单连通平面区域 Ω 的边界点 β 称为 Ω 的简单边界点，如果 β 具有下述性质：对于 Ω 内的每一个序列 $\{\alpha_n\}$，使 $n\to\infty$ 时 $\alpha_n\to\beta$，都对应着一条曲线 γ，它具有参数区间$[0,1]$，并且有一序列 $\{t_n\}$，$0<t_1<t_2<\cdots$，$t_n\to 1$，使得 $\gamma(t_n)=\alpha_n$，$(n=1,2,3,\cdots)$. 并且当 $0\leqslant t<1$ 时，$\gamma(t)\in\Omega$.

换句话说，Ω 内有一条曲线经过所有点 α_n 且以 β 为终点.

14.17 例 因为简单边界点的例是很显然的，我们来研究一些非简单边界点.

如果 Ω 是 $U-\{x:0\leqslant x<1\}$，则 Ω 是单连通区域；并且若 $0<$

$\beta\leqslant 1$，则 β 是 Ω 的一个边界点，但不是简单边界点.

为了举出更复杂的例子，设 Ω_0 在以 $0,1,1+i$ 及 i 为顶点的正方形的内部. 在 Ω_0 内除去区间

$$\left[\frac{1}{2n},\frac{1}{2n}+\frac{n-1}{n}i\right] \text{ 和 } \left[\frac{1}{2n+1}+\frac{i}{n},\frac{1}{2n+1}+i\right]$$

这样得到的区域 Ω 是单连通区域. 假如 $0\leqslant y\leqslant 1$，则 iy 是边界点，但不是简单的.

14.18 定理 设 Ω 是平面内一个有界单连通区域，并且设 f 是 Ω 到 U 上的保形映射.

(*a*) 如果 β 是 Ω 的一个简单边界点，则 f 有一个到 $\Omega\cup\{\beta\}$ 的连续开拓. 假如 f 是这样的开拓，则 $|f(\beta)|=1$.

(*b*) 如果 β_1 和 β_2 是 Ω 的两个不同的简单边界点，并且 f 如 (*a*) 一样被连续开拓到 $\Omega\cup\{\beta_1\}\cup\{\beta_2\}$，则 $f(\beta_1)\neq f(\beta_2)$.

证明 设 g 是 f 的逆，则 $g\in H(U)$，由定理 10.33，$g(U)=\Omega$，g 是一一映射，且因为 Ω 有界，$g\in H^\infty$.

假设(*a*)不成立. 则在 Ω 内有序列 $\{\alpha_n\}$，满足 $\alpha_n\to\beta$，$f(\alpha_{2n})\to w_1$，$f(\alpha_{2n+1})\to w_2$，并且 $w_1\neq w_2$. 按定义 14.16 作 γ，并且对 $0\leqslant t<1$ 记 $\Gamma(t)=f(\gamma(t))$. 对 $0<r<1$ 令 $K_r=g(\bar{D}(0;r))$. 则 K_r 是 Ω 的紧子集. 因为当 $t\to 1$ 时，$\gamma(t)\to\beta$，于是存在一个 $t^*<1$ (依赖于 r)，使得 $t^*<t<1$ 时，$\gamma(t)\notin K_r$. 因此，当 $t^*<t<1$ 时，$|\Gamma(t)|>r$. 这说明当 $t\to 1$ 时，$|\Gamma(t)|\to 1$. 因为 $\Gamma(t_{2n})\to w_1$ 和 $\Gamma(t_{2n+1})\to w_2$，我们同时得到 $|w_1|=|w_2|=1$.

现在得知，其并是 $T-(\{w_1\}\cup\{w_2\})$ 的两段开弧 J 中有一段具有如下性质：对于 U 的每一条端点为 J 上的点的半径交 Γ 的值域于一个有一极限点在 T 的点集. 注意，当 $0\leqslant t<1$ 时，$g(\Gamma(t))=\gamma(t)$，并且因为 $g\in H^\infty$，g 在 T 上几乎处处有径向极限. 因为当 $t\to 1$ 时，$g(\Gamma(t))\to\beta$，因此

(1) $$\lim_{r\to 1} g(re^{i\theta})=\beta \qquad (\text{a·e 于 } J)$$

对 $g-\beta$ 应用定理 11.22,(1) 表明 g 是常数. 但是 g 在 U 内是一一的,于是得到矛盾. 这样,$w_1=w_2$,即(a)得证.

假设(b)不成立. 我们以绝对值为 1 的适当常数乘 f,使得到 $\beta_1\neq\beta_2$. 但有 $f(\beta_1)=f(\beta_2)=-1$.

因为 β_1 和 β_2 都是 Ω 的简单边界点,于是存在曲线 γ_i,它具有参数区间$[0,1]$,对 $i=1$ 及 2,有 $\gamma_i([0,1))\subset\Omega$,并且 $\gamma_i(1)=\beta_i$. 令 $\Gamma_i(t)=f(\gamma_i(t))$,则 $\Gamma_i([0,1))\subset U$,且 $\Gamma_1(1)=\Gamma_2(1)=-1$,因为在$[0,1)$上,$g(\Gamma_i(t))=\gamma_i(t)$,于是存在一个 $t^*<1$,使得

(2) $$|g(\Gamma_1(t_1))-g(\Gamma_2(t_2))|>\frac{1}{2}|\beta_1-\beta_2|$$
$$(t^*<t_i<1, \quad i=1,2)$$

并且存在一个正数 $\delta>0$,使得 $D(-1;\delta)$ 与 $\Gamma_1([0;t^*])$ 或 $\Gamma_2([0;t^2])$ 都不相交.

记 $A(\delta)=U\cap D(-1;\delta)$. 设 $0<r<\delta$;从(2)和 δ 的选法可以在 Γ_i 的值域上找到点 w_i 使 $|1+w_i|=r$,且

(3) $$|g(w_1)-g(w_2)|>\frac{1}{2}|\beta_1-\beta_2|$$

但 $g(w_1)-g(w_2)$ 是 g' 沿着位于 U 内的以 -1 为圆心的圆弧从 w_2 到 w_1 的积分. 因此

(4) $$\frac{1}{2}|\beta_1-\beta_2|<\int_{-\eta}^{\eta}|g'(-1+re^{i\theta})|rd\theta \qquad (0<r<\delta)$$

其中 $\eta=\eta(r)$ 是使 $-1+re^{i\theta}\in U$, $|\theta|<\eta$ 的最大数值. 于是 $\eta<\pi/2$,并且,若我们应用 Schwarz 不等式于(4),我们发现

(5) $$\frac{|\beta_1-\beta_2|^2}{4\pi r}\leqslant r\int_{-\eta}^{\eta}|g'(-1+re^{i\theta})|^2d\theta \qquad (0<r<\delta)$$

将(5)式的右边在$(0,\delta)$上对 r 积分,其结果就是 $g(A(\delta))$的

面积. 它是有限的，因为 $g(A(\delta))\subset\Omega$，而$\Omega$是有界的. 但是，(5)式左边在$(0,\delta)$上的积分是$\infty$，除非 $\beta_1=\beta_2$.

这就证明了(*b*).

14.19 定理 如果Ω是平面内的有界单连通区域，并且如果Ω的每一个边界点都是简单边界点，则每一个Ω到U上的保形映射可扩展为$\overline{\Omega}$到$\overline{U}$上的同胚.

证明 设 $f\in H(\Omega)$，$f(\Omega)=U$，且 f 是一一的. 由定理 14.18，我们可将 f 扩展为$\overline{\Omega}$到$\overline{U}$内的一个映射，使得当$\{\alpha_n\}$是Ω内的收敛于 z 的一个序列时，$f(\alpha_n)\to f(z)$. 若$\overline{\Omega}$内的序列$\{z_n\}$收敛于z，则存在点 $\alpha_n\in\Omega$，满足$|\alpha_n-z_n|<1/n$ 及$|f(\alpha_n)-f(z_n)|<1/n$. 这样，$\alpha_n\to z$，因此 $f(\alpha_n)\to f(z)$. 这就证明了 $f(z_n)\to f(z)$.

我们现在已证明了 f 的开拓在$\overline{\Omega}$上是连续的. 同样有$U\subset f(\overline{\Omega})\subset\overline{U}$. $\overline{\Omega}$的紧性蕴涵着$f(\overline{\Omega})$是紧的，因此$f(\overline{\Omega})=\overline{U}$.

定理 14.18(*b*) 证明 f 在$\overline{\Omega}$上是一一的. 因为紧集的每一个连续一一映射有一连续逆映射([26]，定理 4.17)，故定理得证.

14.20 评注

(*a*) 前述定理有一个纯拓扑学的推论：如果平面有界单连通区域Ω的每一个边界点都是简单边界点，则Ω的边界是一条 Jordan 曲线，且$\overline{\Omega}$与$\overline{U}$同胚.

(根据定义，一条 Jordan 曲线就是单位圆周的同胚像.)

逆命题也是正确的，但是我们不打算证明它：若Ω的边界是一条 Jordan 曲线，则Ω的每一个边界点都是简单边界点.

(*b*) 设 f 如在定理 14.19 中一样，a, b 和 c 是Ω的不同的边界点，并且 A, B 和 C 是 T 的不同的点. 则有一个线性分式变换φ映三元序组$\{f(a),f(b),f(c)\}$为$\{A,B,C\}$；设$\{A,B,C\}$的定向与$\{f(a),f(b),f(c)\}$相同；则$\varphi(U)=U$，且函数 $g=\varphi\circ f$是$\overline{\Omega}$到$\overline{U}$上的同胚，它在Ω内全纯，并且映$\{a,b,c\}$为原先指定的$\{A,B,C\}$. 从

14.3 节得知 g 由这些要求唯一决定.

(*c*) 定理 14.19 和上面的注(*b*)都不难推广到 Riemann 球面 S^2 内所有边界点都是简单边界点的单连通区域 Ω 上. 若假定 $S^2-\Omega$ 有非空内部,这样,通过一个线性分式变换就可以使我们回到 Ω 是平面内一个有界区域的情况. 类似地,例如,U 可以用半平面来替换.

(*d*) 更一般地,若象在定理 14.19 一样,f_1 和 f_2 分别映 Ω_1 和 Ω_2 到 U 上,则 $f=f_2^{-1}\circ f_1$ 是 $\overline{\Omega}_1$ 到 $\overline{\Omega}_2$ 上的同胚,它在 Ω_1 内是全纯的.

环域的保形映射

14.21 平面内任意两个单连通真子区域保形等价是 Riemann映射定理的一个推论,因为它们中每一个都与单位圆盘是保形等价的. 这是单连通区域的一个非常特殊的性质. 自然会问,是否可以推广到下一个最简单的情形,亦即,任意两个环域是否能保形等价? 回答是否定的.

对于 $0<r<R$,记

(1) $$A(r,R)=\{z:r<|z|<R\}$$

是具有内半径 r 和外半径 R 的环域. 若 $\lambda>0$,映射 $z\to\lambda z$ 将 $A(r,R)$ 映到 $A(\lambda r,\lambda R)$ 上. 因此,当 $R/r=R_1/r_1$ 时,$A(r,R)$ 和 $A(r_1,R_1)$ 是保形等价的. 令人惊奇的是,这个充分条件同时也是必要的;这样,在环域间对于每个大于 1 的实数都对应有一个不同的保形映射类型.

14.22 定理 $A(r_1,R_1)$ 和 $A(r_2,R_2)$ 是保形等价的,当且仅当 $R_1/r_1=R_2/r_2$.

证明 不失一般性,假定 $r_1=r_2=1$. 记

(1) $$A_1=A(1,R_1),\quad A_2=A(1,R_2)$$

并假定存在 $f\in H(A_1)$，使得 f 是一一的，且 $f(A_1)=A_2$. 设 K 是圆心在 0，半径 $r=\sqrt{R_2}$ 的圆周. 因为 f^{-1}: $A_2\to A_1$ 也是全纯的，故 $f^{-1}(K)$ 是紧的. 因此，对某个 $\varepsilon>0$ 有

$$A(1,1+\varepsilon)\cap f^{-1}(K)=\varnothing \tag{2}$$

于是，$V=f(A(1,1+\varepsilon))$ 是 A_2 的一个与 K 不相交的连通子集. 因此，$V\subset A(1,r)$ 或 $V\subset A(r,R_2)$. 在后一种情形，用 R_2/f 代替 f. 这样我们总可以假定 $V\subset A(1,r)$. 若 $1<|z_n|<1+\varepsilon$，并且 $|z_n|\to 1$，则 $f(z_n)\in V$，并且 $\{f(z_n)\}$ 在 A_2 内无极限点(因为 f^{-1} 连续)；于是 $|f(z_n)|\to 1$. 同样可知，当 $|z_n|\to R_1$ 时，$|f(z_n)|\to R_2$.

现在定义

$$\alpha=\frac{\log R_2}{\log R_1} \tag{3}$$

及

$$u(z)=2\log|f(z)|-2\alpha\log|z| \qquad (z\in A_1) \tag{4}$$

设 ∂ 为 Cauchy-Riemann 算子. 因为 $\partial\bar{f}=0$ 及 $\partial f=f'$，由链式法则得

$$\partial(2\log|f|)=\partial(\log(\bar{f}f)=f'/f \tag{5}$$

因此

$$(\partial u)(z)=\frac{f'(z)}{f(z)}-\frac{\alpha}{z} \qquad (z\in A_1) \tag{6}$$

于是 u 在 A_1 内是调和函数，由本证明的第一段，可扩展为 $\bar{A}_1$ 上的连续函数，使之在 A_1 的边界上取零值. 因为不是常数的调和函数没有局部最大值或局部最小值，我们推断出 $u=0$，于是

$$\frac{f'(z)}{f(z)}=\frac{\alpha}{z} \qquad (z\in A_1) \tag{7}$$

设 $\gamma(t)=\sqrt{R_1}e^{it}(-\pi\leqslant t\leqslant\pi)$；设 $\Gamma=f\circ\gamma$. 象定理 10.43 中的证明一样，(7)式给出

(8) $$\alpha=\frac{1}{2\pi i}\int_{\gamma}\frac{f'(z)}{f(z)}dz=\mathrm{Ind}_{\Gamma}(0)$$

于是 α 是一个整数. 由(3)式,$\alpha>0$,由(7)式, $z^{-\alpha}f(z)$ 在 A_1 的导数为 0. 于是 $f(z)=cz^{\alpha}$. 因为 f 在 A_1 内是一一的, $\alpha=1$. 因此 $R_2=R_1$.

习　题

1　求出复数 a,b,c 和 d, 使线性分式变换: $z\to(az+b)/(cz+d)$ 将上半平面映到上半平面所应当满足的充分必要条件.

2　定理 11.17 假设的简化形式是 $\Omega\subset\Pi^+$, L在实轴上, 并且当$z\to L$时, $\mathrm{Im}f(z)\to 0$. 试在下述假设之下, 利用这个定理建立类似的反射定理:

(a)　$\Omega\subset\Pi^+$, L 在实轴上, 当 $z\to L$ 时, $|f(z)|\to 1$.

(b)　$\Omega\subset U$, $L\subset T$, 当 $z\to L$ 时, $|f(z)|\to 1$.

(c)　$\Omega\subset U$, $L\subset T$, 当 $z\to L$ 时, $\mathrm{Im}f(z)\to 0$.

在(b)的情况下, 若 f 有一零点 $a\in\Omega$, 试证明 f 的开拓在$1/\bar{a}$ 有一个极点. 在(a)和(b)的情形下有什么类似结论?

3* 设 R 是一个有理函数, 使得当 $|z|=1$ 时, $|R(z)|=1$. 证明

$$R(z)=cz^m\prod_{n=1}^{k}\frac{z-a_n}{1-\bar{a}_n z}$$

其中 c 为常数, m 是整数, 并且 $a_1,\cdots,a_k$ 是满足 $a_n\neq 0$ 和 $|a_n|\neq 1$ 的复数. 注意上述每一个因式当 $|z|=1$ 时绝对值为 1.

4　试作出在 T 上取正值的有理函数的类似描述.

提示: 这个函数在 U 内必定有相同的零点数和极点数. 考虑形如

$$\frac{(z-\alpha)(1-\bar{\alpha}z)}{(z-\beta)(1-\bar{\beta}z)}$$

的因式的乘积, 其中 $|\alpha|<1$ 和 $|\beta|<1$.

5　设 f 是三角多项式

$$f(\theta)=\sum_{k=-n}^{n}a_k e^{ik\theta}$$

并且对所有实的 θ, $f(\theta)>0$. 证明存在一个多项式 $P(z)=c_0+c_1z+\cdots+c_nz^n$, 使得

$$f(\theta)=|P(e^{i\theta})|^2 \qquad (\theta \text{ 是实数})$$

提示：应用习题 4 于有理函数 $\Sigma a_k z^k$. 如果我们在假设中把 $f(\theta)>0$ 换为 $f(\theta)\geqslant 0$，结果是否仍然真确？

6 求出映射 φ_a（见定义 12.3）的不动点. 是否存在直线使 φ_a 映这条直线到它自身？

7 设映射

$$f_a(z)=\frac{z}{1+az^2}$$

求出所有复数 a，使 f_a 在 U 内是一一的. 对所有这些情况描述 $f_a(U)$.

8 设 $f(z)=z+(1/z)$，试描绘出以 0 为中心的圆和过 0 的射线在 f 之下映成的椭圆和双曲线族.

9 (a) 设 $\Omega=\{z:-1<\mathrm{Re}z<1\}$，求出一个 Ω 到 U 上的、并满足 $f(0)=0$ 和 $f'(0)>0$ 的一一保形映射 f 的明显公式. 试计算 $f'(0)$.

(b) 注意，(a) 中所构造的函数的反函数的实部在 U 中有界，而虚部无界. 试证明存在一个在 $\bar{U}$ 上连续的实函数 u，它在 U 内调和，而它的调和共轭函数 v 则在 U 内无界.（v 是使 $u+iv$ 在 U 内全纯的函数；由预先给定 $v(0)=0$，我们可以唯一地决定 v.）

(c) 设 $g\in H(U)$，在 U 内 $|\mathrm{Re}g|<1$，并且 $g(0)=0$，证明

$$|g(re^{i\theta})|\leqslant\frac{2}{\pi}\log\frac{1+r}{1-r}$$

提示：参看习题 10.

(d) 设 Ω 是象在定理 12.9 中的带域. 固定 Ω 中的一点 $\alpha+i\beta$. 设 h 是 Ω 到 Ω 上的保形一一映射，并且使 $\alpha+i\beta$ 映为 0. 证明

$$|h'(\alpha+i\beta)|=1/\cos\beta$$

10 设 f 和 g 都是 U 到 Ω 内的全纯函数，并且 f 是一一的，$f(U)=\Omega$，及 $f(0)=g(0)$. 证明

$$g(D(0;r))\subset f(D(0;r)) \qquad (0<r<1)$$

11 设 Ω 是单位圆盘 U 的上半部分. 试求保形映射 f，将 Ω 映射到 U 上，并且将 $\{-1,0,1\}$ 映为 $\{-1,-i,1\}$. 试求出 $z\in\Omega$ 使 $f(z)=0$. 并求 $f(i/2)$. 提示：$f=\phi\circ s\circ\psi$，其中 ϕ 和 ψ 是线性分式变换，而 $s(\lambda)=\lambda^2$.

12 设 Ω 是凸区域，$f\in H(\Omega)$，并且对所有 $z\in\Omega$，$\mathrm{Re}f'(z)>0$. 证明 f 在 Ω 内是一一映射. 若将假设减弱成 $\mathrm{Re}f'(z)\geqslant 0$ 时，结果是否改变？（除掉 f

为常数的平凡的情况). 试举出例子说明不能用"单连通"代替"凸".

13 设Ω是区域, $f_n\in H(\Omega)$, $n=1,2,3,\cdots$, 每一个f_n都是Ω内的一一映射, 并且在Ω的紧子集上f_n一致收敛于f. 证明f在Ω内或者是常数, 或者是一一映射. 试说明这两种情况都可以出现.

14 设$\Omega=\{x+iy:-1<y<1\}$, $f\in H(\Omega)$, $|f|<1$, 并且当$x\to\infty$时, $f(x)\to 0$. 证明

$$\lim_{x\to\infty} f(x+iy)=0 \qquad (-1<y<1)$$

并且当y限制在区间$[-a,a]$, $a<1$时, f一致地收敛于这个极限. 提示: 考虑序列$\{f_n\}$, 其中$f_n(z)=z+n$定义在正方形域: $|x|<1$, $|y|<1$内.

一个函数$g\in H^\infty$在靠近使g的径向极限存在的U的边界点时, 关于g的性态, 这个定理能告诉我们一些什么?

15 设$\mathscr{F}$是所有满足$\mathrm{Re}f>0$和$f(0)=1$的函数$f\in H(\Omega)$的族. 证明$\mathscr{F}$是正规族. 条件"$f(0)=1$"能否去掉? 能否将它改成"$f(0)\leqslant 1$"?

16 设$\mathscr{F}$是满足

$$\iint_U |f(z)|^2 dxdy\leqslant 1$$

的所有$f\in H(U)$的类, 问$\mathscr{F}$是否是正规族?

17 设Ω是区域, $f_n\in H(\Omega)$, $n=1,2,3,\cdots$, 在Ω的紧子集上f_n一致趋于f, 并且在Ω内是一一映射. 试问是否能得出对于每个紧集$K\subset\Omega$, 都对应着一个整数$N(K)$, 使当$n>N(K)$时, f_n在K上是一一映射? 给以证明或举出反例.

18 设Ω是单连通区域, $z_0\in\Omega$, 并且f和g是Ω到U上的一一保形映射, 将z_0映射为0. 问f和g之间有什么关系? 若对于某个$a\in U$, $f(z_0)=g(z_0)=a$, 试回答同样问题.

19 试求一个由U到U上的同胚, 使它不能扩展为在$\overline{U}$上的连续函数.

20 若$f\in\mathscr{S}$(定义见14.10), n是一正整数, 证明存在一个函数$g\in\mathscr{S}$, 使得对于所有$z\in U$, $g^n(z)=f(z^n)$.

21 试求所有$f\in\mathscr{S}$满足: (a) $f(U)\supset U$, (b) $f(U)\supset\overline{U}$, (c) $|a_2|=2$.

22 设f是一一保形映射, 它将Ω映射到一个以0为中心的正方形上, 并且$f(0)=0$. 证明$f(iz)=if(z)$. 若$f(z)=\sum c_n z^n$, 证明$c_n=0$, 除非$n-1$是4的倍数. 推广: 将正方形换为具有旋转对称性的单连通区域.

23 设Ω是有界区域, 它的边界由两个不相交的圆周组成. 试证存在一

个一一保形映射，它将Ω映射到环域.（对于每一个使$S^2-\Omega$恰好有两个分支的，每一分支包含多于一点的区域Ω，这结论仍然是真的，但是这个一般情形比较难于处理.）

24 试完成定理14.22的下述证明的细节. 设$1<R_2<R_1$，并且f是将$A(1, R_1)$映到$A(1, R_2)$上的一一保形映射. 定义$f_1=f$和$f_n=f\circ f_{n-1}$. 于是存在$\{f_n\}$的子序列，在$A(1, R_1)$的紧子集上一致收敛于一个函数g. 试证明g的值域不包含任何非空开集（例如用三圆定理）. 再者，证明在圆$\{z:|z|^2=R_1\}$上g不是常数. 因此，f不能存在.

25 下面是定理14.22的又一种证明. 如果f如在14.22所设. 重复运用反射原理将f扩充为整函数，使得当$|z|=1$时，$|f(z)|=1$. 这就可以推出$f(z)=\alpha z^n$，其中$|\alpha|=1$，而n是一整数，试完成这些细节.

26 重复运用定理14.8证明中的第二步，可以得出Riemann映射定理的一个证明（由Koebe作出），它在下述意义上是一种构造性的证明，即它并不要求正规族的理论，从而不依赖于某种未予确定的子序列的存在性. 对于这证法的最后一步，假设Ω有定理13.11中(h)的性质是有好处的. 因此，任何与Ω保形等价的区域都满足(h). 回忆一下，(h)显然推出(i).

根据定理14.8中的第一步，不失一般性，可以假设$0\in\Omega$，$\Omega\subset U$，并且$\Omega\neq U$. 设$\Omega=\Omega_0$. 这个证明在于构造区域$\Omega_1, \Omega_2, \Omega_3, \cdots$和函数$f_1, f_2, f_3, \cdots$，使$f_n(\Omega_{n+1})=\Omega_n$，且函数$f_n\circ f_{n-1}\circ\cdots\circ f_2\circ f_1$收敛于一个$\Omega$到$U$上的保形映射.

试完成下述提纲的各个细节.

(a)设Ω_{n-1}已造出，令r_n是满足$D(0; r_n)\subset\Omega_{n-1}$的最大数，设$\alpha_n$是$\Omega_{n-1}$的边界点，$|\alpha_n|=r_n$，选取$\beta_n$使$\beta_n^2=-\alpha_n$，并设

$$F_n=\varphi_{-\alpha_n}\circ S\circ\varphi_{-\beta_n}$$

（记号与定理14.8证明中的相同）. 试证F_n在Ω_{n-1}内有全纯的反函数G_n，并设$f_n=\lambda_n G_n$，其中$\lambda_n=|c|/c$和$c=G_n'(0)$.（这个f_n就是与Ω_{n-1}相对应的Koebe映射. 注意，f_n是初等函数，它只包含两个线性分式变换和一个平方根.）

(b)计算$f_n'(0)=(1+r_n)/2\sqrt{r_n}>1$.

(c)设$\psi_0(z)=z$和$\psi_n(z)=f_n(\psi_{n-1}(z))$. 证明$\psi_n$是将$\Omega$映射到一区域$\Omega_n\subset U$上的一一映射，$\{\psi_n'(0)\}$有界，

$$\psi_n'(0)=\prod_{k=1}^{n}\frac{1+r_k}{2\sqrt{r_k}}$$

因而当 $n\to\infty$ 时，$r_n\to 1$.

(d) 对 $z\in\Omega$，记 $\psi_n(z)=zh_n(z)$. 证明 $|h_n|<|h_{n+1}|$，应用 Harnack 定理和第十一章习题 22 于 $\{\log h_n\}$ 以证明 $\{\psi_n\}$ 在 Ω 的紧子集上一致收敛，并证明 $\lim\psi_n$ 是 Ω 到 U 上的一一映射.

27 证明 $\sum_{n=1}^{\infty}(1-r_n)^2<\infty$，其中 $\{r_n\}$ 是习题 26 已出现过的序列.

提示:

$$\frac{1+r}{2\sqrt{r}}=1+\frac{(1-\sqrt{r})^2}{2\sqrt{r}}$$

28 假若在习题 26 中我们选取 $a_n\in U-\Omega_{n-1}$，不必强调 $|a_n|=r_n$，例如，只要坚持

$$|a_n|\leqslant\frac{1+r_n}{2}$$

这样产生的序列 $\{\psi_n\}$ 是否仍收敛于所要求的映射函数？

29 设 Ω 是有界区域，$a\in\Omega$，$f\in H(\Omega)$，$f(\Omega)\subset\Omega$，并且 $f(a)=a$.

(a) 令 $f_1=f$ 及 $f_n=f\circ f_{n-1}$，试计算 $f_n'(a)$，并且推出 $|f'(a)|\leqslant 1$.

(b) 如果 $f'(a)=1$，证明对所有 $z\in\Omega$，$f(z)=z$. 提示：若

$$f(z)=z+c_m(z-a)^m+\cdots$$

计算 $f_n(z)$ 展开式中 $(z-a)^m$ 的系数.

(c) 若 $|f'(a)|=1$. 试证 f 是一一映射，并且 $f(\Omega)=\Omega$.

提示：如果 $\gamma=f'(a)$，于是存在整数 $n_k\to\infty$，使得 $\gamma^{n_k}\to 1$ 和 $f_{n_k}\to g$. 则 $g'(a)=1$，$g(\Omega)\subset U$（由第十章习题 20）. 因此由 (b) 得 $g(z)=z$. 利用 g 作出关于 f 所需要的结论.

30 设 Λ 是由所有线性分式变换组成的集.

若 $\{\alpha,\beta,\gamma,\delta\}$ 是不相同复数的四元序组，它的交比定义为

$$[\alpha,\beta,\gamma,\delta]=\frac{(\alpha-\beta)(\gamma-\delta)}{(\alpha-\delta)(\gamma-\beta)}$$

若其中有一个为 ∞，定义明显地按连续性方式修改. 同样应用于当 α 与 β（或 γ,δ）相同的情形.

(a) 若 $\varphi(z)=[z,\alpha,\beta,\gamma]$，试证 $\varphi\in\Lambda$，并且 φ 映 $\{\alpha,\beta,\gamma\}$ 为 $\{0,1,\infty\}$.

(*b*) 证明：方程$[w, a, b, c]=[z, \alpha, \beta, \gamma]$可以按$w=\varphi(z)$的形式解出，这时，$\varphi\in\Lambda$，并将$\{\alpha, \beta, \gamma\}$映为$\{a, b, c\}$.

(*c*) 若$\varphi\in\Lambda$，证明

$$[\varphi(\alpha), \varphi(\beta), \varphi(\gamma), \varphi(\delta)]=[\alpha, \beta, \gamma, \delta]$$

(*d*) 试证当且仅当这四点同在一圆周或直线上时$[\alpha, \beta, \gamma, \delta]$是实数.

(*e*) 称z和z^*两个点关于经过α, β, γ的圆周(或直线)C是对称的，如果$[z^*, \alpha, \beta, \gamma]$是$[z, \alpha, \beta, \gamma]$的复共轭数. 若$C$是单位圆周，试求$z$和$z^*$之间的简单几何关系. 同样讨论$C$是直线的情况.

(*f*) 设z和z^*关于C对称. 证明对于每一个$\varphi\in\Lambda$，$\varphi(z)$和$\varphi(z^*)$关于$\varphi(C)$对称.

31 (*a*) 证明Λ(见习题30)以复合作为群的运算时是一个群，就是说. 若$\varphi\in\Lambda$和$\psi\in\Lambda$，证明$\varphi\circ\psi\in\Lambda$，并且$\varphi$的反函数$\varphi^{-1}\in\Lambda$. 证明$\Lambda$不是交换的.

(*b*) 试证Λ中每一元素(不是恒等映射)在S^2有一个或两个不动点. [φ的不动点α就是使$\varphi(\alpha)=\alpha$的点].

(*c*) 映射φ和$\varphi_1\in\Lambda$称为共轭，如果存在一个$\psi\in\Lambda$，使$\varphi_1=\psi^{-1}\circ\varphi\circ\psi$. 证明每一个$\varphi\in\Lambda$，$\varphi$有唯一不动点，则$\varphi$与映射$z\to z+1$共轭. 证明每一个$\varphi\in\Lambda$，$\varphi$有两个不同的不动点，则$\varphi$与映射$z\to\alpha z$共轭，其中$\alpha$是复数；$\alpha$能由$\varphi$决定到何种程度？

(*d*) 设α是一复数. 证明对每一个以α为其唯一不动点的$\varphi\in\Lambda$，对应有一个β，使得

$$\frac{1}{\varphi(z)-\alpha}=\frac{1}{z-\alpha}+\beta$$

设G_α是所有这些φ加上恒等变换构成的集. 试证明G_α是Λ的子群，并且G_α与所有复数构成的加法群同构.

(*e*) 设α和β是两个不同的复数. 设$G_{\alpha,\beta}$是所有以α和β为不动点的$\varphi\in\Lambda$的集. 证明每一个$\varphi\in G_{\alpha,\beta}$由

$$\frac{\varphi(z)-\alpha}{\varphi(z)-\beta}=\gamma\cdot\frac{z-\alpha}{z-\beta}$$

给出，其中γ是复数. 并证明$G_{\alpha,\beta}$是Λ的子群，且$G_{\alpha,\beta}$与所有非零复数构成的乘法群同构.

(*f*) φ如在(*d*)(或(*e*))中所设，对哪些圆周C有$\varphi(C)=C$？这个回答应当由参数α, β及γ确定下来.

第十五章　全纯函数的零点

无穷乘积

15.1　关于区域Ω内不是常数的全纯函数f的零点集$Z(f)$，迄今我们只见到一个结果，即$Z(f)$在Ω内没有极限点．不久我们将会看到，如果没有其它条件加之于f，关于$Z(f)$所能说的也就只有这一点．因为 Weierstrass 定理(定理 15.11)断言每一个$A\subset\Omega$，当A在Ω内没有极限点时，一定有一个$f\in H(\Omega)$，使A就是$Z(f)$．如果$A=\{\alpha_n\}$，构造f的自然方式是选取函数$f_n\in H(\Omega)$，使f_n只在α_n有一个零点，并且考虑乘积

$$p_n=f_1f_2\cdots f_n$$

当$n\to\infty$时的极限．我们必须安排得使$\{p_n\}$收敛于某个$f\in H(\Omega)$，并且这个极限函数f除在指定的各点α_n外不为零．因此，从研究无穷乘积某些一般性质开始是适宜的．

15.2　**定义**　设$\{u_n\}$是一个复数序列，

(1)　　$$p_n=(1+u_1)(1+u_2)\cdots(1+u_n)$$

并且$p=\lim\limits_{n\to\infty}p_n$存在，则我们记

(2)　　$$p=\prod_{n=1}^{\infty}(1+u_n)$$

p_n是无穷乘积(2)的部分乘积．若序列$\{p_n\}$收敛，我们就说无穷乘积(2)收敛．

在研究无穷级数Σa_n时，重要的是a_n是否迅速趋于0．类似地，在研究无穷乘积时，我们感兴趣的是这些因式是否接近于1．

这说明当 u_n 接近于 0 时，上述记号：$1+u_n$ 是接近于 1 的。

15.3 引理 如果 $u_1, \cdots, u_N$ 都是复数，并且

(1) $$p_N=\prod_{n=1}^{N}(1+u_n), \qquad p_N^*=\prod_{n=1}^{N}(1+|u_n|)$$

则

(2) $$p_N^*\leqslant \exp(|u_1|+\cdots+|u_N|)$$

且

(3) $$|p_N-1|\leqslant p_N^*-1$$

证明 对于 $x\geqslant 0$，不等式 $1+x\leqslant e^x$ 可以由 e^x 的幂级数展开直接得到. 将 x 换为 $|u_1|, \cdots, |u_N|$，然后将所得的各不等式相乘，便得出不等式(2). 对于 $N=1$，(3)式是显然的. 一般情况可由归纳法得出. 对 $k=1, \cdots, N-1$，

$$p_{k+1}-1=p_k(1+u_{k+1})-1=(p_k-1)(1+u_{k+1})+u_{k+1}$$

因此，如果用 k 代替 N 时(3)式成立. 则也有

$$|p_{k+1}-1|\leqslant (p_k^*-1)(1+|u_{k+1}|)+|u_{k+1}|=p_{k+1}^*-1$$

15.4 定理 设 $\{u_n\}$ 是集 S 上的有界复函数序列，使得 $\Sigma|u_n(s)|$ 在 S 上一致收敛，则乘积

(1) $$f(s)=\prod_{n=1}^{\infty}(1+u_n(s))$$

在 S 上一致收敛，并且在某个 $s_0\in S$ 处，当且仅当对于某个 n，$u_n(s_0)=-1$ 时，$f(s_0)=0$.

再者，如果 $\{n_1, n_2, n_3, \cdots\}$ 是 $\{1, 2, 3, \cdots\}$ 的任意一个置换，则我们仍有

(2) $$f(s)=\prod_{k=1}^{\infty}(1+u_{n_k}(s)) \qquad (s\in S)$$

证明 由假设可推出 $\Sigma|u_n(s)|$ 在 S 上是有界的，并且若记 p_N

为(1)的前N项部分乘积，我们从引理 15.3 知道存在常数 $C<\infty$，对于全体N和全体 s 有$|p_N(s)|\leqslant C$.

选取 ε，$0<\varepsilon<\frac{1}{2}$. 于是存在 N_0 满足

$$\sum_{n=N_0}^{\infty}|u_n(s)|<\varepsilon \qquad (s\in S) \tag{3}$$

设$\{n_1,n_2,n_3,\cdots\}$是$\{1,2,3,\cdots\}$的一个置换. 如果 $N\geqslant N_0$，M足够大，使得

$$\{1,2,\cdots,N\}\subset\{n_1,n_2,\cdots n_M\} \tag{4}$$

并且，如果 $q_M(s)$表示无穷乘积(2)的第M个部分乘积，则

$$q_M-p_N=p_N\{\Pi(1+u_{n_k})-1\} \tag{5}$$

在(5)中出现的 n_k 是互不相同的，并且大于 N_0，因此由(3)和引理 15.3 证得

$$|q_M-p_N|\leqslant|p_N|(e^{\varepsilon}-1)\leqslant 2|p_N|\varepsilon\leqslant 2C\varepsilon \tag{6}$$

如果 $n_k=k(k=1,2,3,\cdots)$，则 $q_M=p_M$，并且(6)式表明$\{p_N\}$一致收敛于一个极限函数 f. (6)式也表明

$$|p_M-p_{N_0}|\leqslant 2|p_{N_0}|\varepsilon \qquad (M>N_0) \tag{7}$$

于是$|p_M|\geqslant(1-2\varepsilon)|p_{N_0}|$. 因此

$$|f(s)|\geqslant(1-2\varepsilon)|p_{N_0}(s)| \qquad (s\in S) \tag{8}$$

这就证明当且仅当 $p_{N_0}(s)=0$ 时 $f(s)=0$.

最后，(6)式也表明$\{q_M\}$与$\{p_N\}$收敛于相同的极限.

15.5 定理 设$0\leqslant u_n<1$，则当且仅当$\sum_{n=1}^{\infty}u_n<\infty$时，

$$\prod_{n=1}^{\infty}(1-u_n)>0$$

证明 如果$p_N=(1-u_1)\cdots(1-u_N)$，则$p_1\geqslant p_2\geqslant\cdots\geqslant p_N>0$，因此 $p=\lim p_N$ 存在. 若$\Sigma u_n<\infty$，定理15.4蕴涵着$p>0$. 另一方面，

$$p\leqslant p_N=\prod_{n=1}^{N}(1-u_n)\leqslant\exp\{-u_1-u_2-\cdots-u_N\}$$

并且，若 $\Sigma u_n=\infty$，则当 $n\to\infty$ 时，上面表达式趋于 0.

我们将经常运用定理 15.4 的下一推论.

15.6 定理 设对于 $n=1,2,3,\cdots$，$f_n\in H(\Omega)$，在 Ω 的任何分支上没有一个 f_n 恒等于 0，并且

(1) $$\sum_{n=1}^{\infty}|1-f_n(z)|$$

在 Ω 的紧子集上一致收敛，则乘积

(2) $$f(z)=\prod_{n=1}^{\infty}f_n(z)$$

在 Ω 的紧子集上一致收敛. 因此 $f\in H(\Omega)$.

再者，我们有

(3) $$m(f;z)=\sum_{n=1}^{\infty}m(f_n;z)\qquad(z\in\Omega)$$

其中 $m(f;z)$ 定义为 f 在 z 的零点的重数. [如果 $f(z)\neq 0$，则 $m(f;z)=0$.]

证明 第一部分由定理 15.4 立即得到. 对于第二部分，由(1)式可以观察到对每一个 $z\in\Omega$ 存在一个邻域 V，在 V 内至多有有限个 f_n 有零点. 首先取出这些因子. 由定理 15.4，余下的乘积在 V 内没有零点，而这个性质给出了(3)式. 附带说一句，我们也可以看出对于任何给定的 $z\in\Omega$，级数(3)至多有限项为正.

Weierstrass 因式分解定理

15.7 定义 设 $E_0(z)=1-z$，并且对于 $p=1,2,3,\cdots$，

$$E_p(z)=(1-z)\exp\left\{z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right\}$$

这些由 Weierstrass 引进的函数有时称为基本因式. 它们只在 $z=1$ 有零点. 它们所以有用是有赖于这样的事实: 虽然 $E_p(1)=0$, 但是当 $|z|<1$ 和 p 足够大时, 它们靠近 1.

15.8 引理 对于 $|z|<1$ 和 $p=0,1,2,\cdots$, 有

$$|1-E_p(z)|\leqslant|z|^{p+1}$$

证明 当 $p=0$ 时这显然成立. 当 $p\geqslant1$ 时, 直接计算得到

$$-E_p'(z)=z^p\exp\left\{z+\frac{z^2}{2}+\cdots+\frac{z^p}{p}\right\}$$

因此, $-E_p'(z)$ 在 $z=0$ 有 p 阶零点, 并且 $-E_p'$ 按 z 的幂级数展开式有非负的实系数. 因为

$$1-E_p(z)=-\int_{[0,z]}E_p'(w)dw$$

$1-E_p$ 在 $z=0$ 有 $p+1$ 阶零点, 并且如果

$$\varphi(z)=\frac{1-E_p(z)}{z^{p+1}}$$

则 $\varphi(z)=\sum a_nz^n$, 其中所有 $a_n\geqslant0$. 因此若 $|z|\leqslant1$, 则 $|\varphi(z)|\leqslant\varphi(1)=1$. 这就给出了引理的论断.

15.9 定理 设 $\{z_n\}$ 是复数序列, $z_n\neq0$, 并且当 $n\to\infty$ 时, $|z_n|\to\infty$. 如果 $\{p_n\}$ 是非负整数序列. 使

(1) $$\sum_{n=1}^{\infty}\left(\frac{r}{r_n}\right)^{1+p_n}<\infty$$

对每一个正数 r (其中 $r_n=|z_n|$) 成立, 则无穷乘积

(2) $$P(z)=\prod_{n=1}^{\infty}E_{p_n}\left(\frac{z}{z_n}\right)$$

定义一个整函数 P. P 在每个 z_n 有零点, 并且在平面内没有其它

零点.

更确切地说，如果 α 在序列 $\{z_n\}$ 中出现 m 次，则 P 在 α 有 m 阶零点.

例如，若 $p_n=n-1$，则条件(1)总是能满足的.

证明 对于每个 r，除开有限个 n 外，$r_n>2r$，因此对这些 n，$r/r_n<\frac{1}{2}$. 令 $1+p_n=n$，则(1)式成立.

现在固定 r. 如果 $|z|\leqslant r$，引理 15.8 表明

$$\left|1-E_{p_n}\left(\frac{z}{z_n}\right)\right|\leqslant\left|\frac{z}{z_n}\right|^{1+p_n}\leqslant\left(\frac{r}{r_n}\right)^{1+p_n}$$

当 $r_n\geqslant r$ 时成立. 但这对除开有限个 n 之外是满足的. 现在从(1)式可以得到级数

$$\sum_{n=1}^{\infty}\left|1-E_{p_n}\left(\frac{z}{z_n}\right)\right|$$

在平面内紧集上一致收敛，并由定理 15.6 得出所求的结论.

注：对于某些序列 $\{r_n\}$，(1) 式当 $\{p_n\}$ 是常数序列时也能成立. 使这个常数取得尽可能小是有趣的：所得到的函数(2)因此被称为对应于 $\{z_n\}$ 的典范积. 例如，如果 $\Sigma(1/r_n)<\infty$，我们取 $p_n=0$，即典范积仅仅是

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)$$

如果 $\Sigma 1/r_n=\infty$，但 $\Sigma 1/r_n^2<\infty$，则典范积是

$$\prod_{n=1}^{\infty}\left(1-\frac{z}{z_n}\right)e^{z/z_n}$$

典范积在研究有限阶整函数时是很有趣的.（定义见习题2）.

我们现在叙述 Weierstrass 因式分解定理.

15.10 定理 设 f 是整函数，$f(0)\neq 0$，并且设 $z_1, z_2, z_3, \cdots$ 是 f 的零点，并按照它们的重数列出. 则存在一个整函数 g 和一个非负整数序列 $\{p_n\}$，使

$$f(z)=e^{g(z)}\prod_{n=1}^{\infty}E_{p_n}\left(\frac{z}{z_n}\right) \tag{1}$$

注： (a) 如果 f 在 $z=0$ 有 k 阶零点，那么前述结论可应用于 $f(z)/z^k$. (b) 因式分解 (1) 并不是唯一的. 一个唯一的因式分解可以对应于其零点满足使典范积收敛所需条件的那些 f.

证明 设 P 是定理 15.9 中由 f 的零点构成的乘积，则 f/P 在平面内只有可去奇点，因此是(或扩展为)整函数. 同样，f/P 没有零点，又因为平面是单连通区域，故 $f/P=e^g$，g 是整函数.

定理 15.9 的证明容易应用于任意的开集：

15.11 定理 设 Ω 是 S^2 内的开集，$\Omega\neq S^2$. 假设 $A\subset\Omega$，并且 A 在 Ω 内无极限点，对于每个 $\alpha\in A$ 对应有一个正整数 $m(\alpha)$. 则存在一个函数 $f\in H(\Omega)$，其所有的零点都在 A 内，并且在每个 $\alpha\in A$，f 有 $m(\alpha)$ 阶零点.

证明 为了使讨论简化同时又不失一般性，假设 $\infty\in\Omega$，但 $\infty\notin A$. (若非如此，通过一个线性分式变换就可化为这种情况). 则 $S^2-\Omega$ 是平面的一个非空紧子集，并且 ∞ 不是 A 的极限点.

如果 A 是有限的，则我们可取 f 为有理函数.

如果 A 是无限的，则 A 是可数的(否则 A 在 Ω 内有极限点). 设 $\{\alpha_n\}$ 是 A 内的序列，并且在序列中每个 $\alpha\in A$ 都刚好出现 $m(\alpha)$ 次. 每个 α_n 对应一点 $\beta_n\in S^2-\Omega$ 使得 $|\beta_n-\alpha_n|\leqslant|\beta-\alpha_n|$ 对所有 $\beta\in S^2-\Omega$ 成立. 因为 $S^2-\Omega$ 是紧集，这是可能的. 于是当 $n\to\infty$ 时，

$$|\beta_n-\alpha_n|\to 0$$

否则 A 就会有一个极限点在 Ω 内. 我们断言

$$f(z)=\prod_{n=1}^{\infty}E_n\left(\frac{\alpha_n-\beta_n}{z-\beta_n}\right)$$

具有所需的性质.

记 $r_n=2|\alpha_n-\beta_n|$. 设K是Ω的紧子集. 因为 $r_n\to 0$, 于是存在N, 对所有 $z\in K$ 和所有 $n\geqslant N$, $|z-\beta_n|>r_n$. 因此,

$$\left|\frac{\alpha_n-\beta_n}{z-\beta_n}\right|\leqslant\frac{1}{2}$$

由引理 15.8, 它蕴涵着

$$\left|1-E_n\left(\frac{\alpha_n-\beta_n}{z-\beta_n}\right)\right|\leqslant\left(\frac{1}{2}\right)^{n+1}\qquad(z\in K, n\geqslant N)$$

并由定理 15.6, 这就再次完成了证明.

作为一个推论, 我们现在得到亚纯函数的一个特征 (见定义 10.4):

15.12 定理 每一个定义在开集Ω内的亚纯函数是Ω内两个全纯函数的商.

逆定理是很明显的: 如果 $g\in H(\Omega)$, $h\in H(\Omega)$, 且 h 在Ω的任何分支内不恒等于 0, 则 g/h 在Ω内是亚纯的.

证明 假设 f 在Ω内亚纯; 设 A 是 f 在Ω内所有极点的集; 并且对每个 $\alpha\in A$, 设 $m(\alpha)$是 f 在极点 α 的阶. 由定理 15.11, 存在一个$h\in H(\Omega)$, 使 h 在每一个 $\alpha\in A$ 有 $m(\alpha)$重零点, 并且 h 没有其它零点. 记 $g=fh$. g 在 A 的奇点是可去奇点, 因此我们可以扩展 g, 使 $g\in H(\Omega)$. 显然, 在$\Omega-A$ 内 $f=g/h$.

一个插值问题

Mittag-Leffler 定理和 Weierstrass 定理 15.11 结合起来可以解决下列问题: 我们是否可以任意取一个在Ω内没有极限点的集 $A\subset\Omega$, 并且找到一个函数 $f\in H(\Omega)$, 在 A 的每一点取给定的值? 回答是肯定的. 事实上, 我们可以作出更好的结果, 还可以在

A 的每一点取给定的有限阶导数：

15.13 定理 假设 Ω 是平面内的一个开集，$A\subset\Omega$，A 在 Ω 内没有极限点，并且对于每一个 $\alpha\in A$ 对应有一个非负整数 $m(\alpha)$ 和复数 $w_{n,\alpha}$，$0\leqslant n\leqslant m(\alpha)$，则存在一个函数 $f\in H(\Omega)$，满足

$$f^{(n)}(\alpha)=n!w_{n,\alpha}\quad(\alpha\in A,0\leqslant n\leqslant m(\alpha)) \tag{1}$$

证明 由定理 15.11，存在一个函数 $g\in H(\Omega)$，g 只在 A 内有零点，并且在每一个 $\alpha\in A$，g 有 $m(\alpha)+1$ 阶零点. 我们断定对于每一个 $\alpha\in A$，对应有一个形为

$$P_\alpha(z)=\sum_{j=1}^{1+m(\alpha)}c_{j,\alpha}(z-\alpha)^{-j} \tag{2}$$

的函数 P_α，使 gP_α 在以 α 为圆心的某个圆盘内有幂级数展开式

$$g(z)P_\alpha(z)=w_{0,\alpha}+w_{1,\alpha}(z-\alpha)+\cdots+w_{m(\alpha),\alpha}(z-\alpha)^{m(\alpha)}+\cdots \tag{3}$$

为了书写简单，取 $\alpha=0$ 及 $m(\alpha)=m$，并略去下标 α. 对于靠近 0 的 z，我们有

$$g(z)=b_1z^{m+1}+b_2z^{m+2}+\cdots \tag{4}$$

其中 $b_1\neq0$. 如果

$$P(z)=c_1z^{-1}+\cdots+c_{m+1}z^{-m-1} \tag{5}$$

则

$$g(z)P(z)=(C_{m+1}+C_mz+\cdots+c_1z^m)(b_1+b_2z+b_3z^2+\cdots) \tag{6}$$

这些 b 已经给定了，我们希望选取那些 c，使得

$$g(z)P(z)=w_0+w_1z+\cdots+w_mz^m+\cdots \tag{7}$$

如果我们比较(6)和(7)中 $1,z,\cdots,z^m$ 的系数，因为 $b_1\neq0$，我们可以依次在得出的方程中解出 $c_{m+1},c_m,\cdots,c_1$.

按此种方式我们求得所需的各个 P_α. 现在 Mittag-Leffler 定理给出一个在 Ω 内的亚纯函数 h，它的主要部分是这些 P_α. 并且如果我们令 $f=gh$，便可得到具有所需性质的函数.

插值问题的解可用来决定环 $H(\Omega)$ 内的全体有限生成理想.

15.14 定义 由属于 $H(\Omega)$ 内的函数 $g_1, \cdots, g_n$ 生成的理想 $[g_1, \cdots, g_n]$ 就是所有形如 $\Sigma f_i g_i$ 的函数构成的集. 其中 $f_i \in H(\Omega)$, $i=1, \cdots, n$. 一个主理想是由单独一个函数生成的理想. 注意 $[1]=H(\Omega)$.

如果 $f \in H(\Omega)$, $\alpha \in \Omega$, 并且在 α 的某个邻域内 f 不恒等于 0, f 在 α 的零点的重数记为 $m(f;\ \alpha)$. 像在定理 15.6 中一样, 若 $f(\alpha) \neq 0$, 则 $m(f;\alpha)=0$.

15.15 定理 每一个 $H(\Omega)$ 内的有限生成理想都是主理想.

说得更明白些: 如果 $g_1, \cdots, g_n \in H(\Omega)$, 则存在函数 g, f_i, $h_i \in H(\Omega)$ 满足

$$g=\sum_{i=1}^{n} f_i g_i \quad 及 \quad g_i = h_i g \qquad (1 \leqslant i \leqslant n)$$

证明 我们将假设 Ω 是一个区域. 这样做是为了避免那样一些函数所造成的问题, 它们在 Ω 的某些分支内恒等于 0, 但不在所有各分支内恒等于 0. 一旦证明了定理对于区域成立, 这个结果就可以应用于任一开集 Ω 的每一个分支, 并且整个定理也就能推出来. 我们将推证的细节留作习题.

设 $P(n)$ 是下述命题:

如果 $g_1, \cdots, g_n \in H(\Omega)$. 若没有一个 g_i 恒等于 0, 并且如果 Ω 内没有一个点是 g_i 的零点, 则 $[g_1, \cdots, g_n]=[1]$.

$P(1)$ 是不足道的. 假设 $n>1$, 并且 $P(n-1)$ 已经成立. 取 $g_1, \cdots, g_n \in H(\Omega)$, 它们没有公共零点. 由 Weierstrass 定理 15.11, 存在 $\varphi \in H(\Omega)$ 满足

(1) $$m(\varphi;\alpha)=\min\{m(g_i;\alpha): 1 \leqslant i \leqslant n-1\} \qquad (\alpha \in \Omega)$$

函数 $f_i=g_i/\varphi$ $(1 \leqslant i \leqslant n-1)$ 均属于 $H(\Omega)$ 并且在 Ω 内没有公共零点. 因为 $P(n-1)$ 已成立, $[f_1, \cdots, f_{n-1}]=[1]$, 故

(2) $$[g_1, \cdots, g_{n-1}, g_n]=[\varphi, g_n]$$

再者，我们选取的 φ 表明在集 $A=\{\alpha\in\Omega:\varphi(\alpha)=0\}$ 的每一点 $g_n(\alpha)\neq 0$，因此由定理15.13，存在 $h\in H(\Omega)$ 使

$$m(1-hg_n;\alpha)\geqslant m(\varphi;\alpha)\qquad(\alpha\in\Omega)\tag{3}$$

当 $\alpha\in A$ 和 $0\leqslant k\leqslant m(\varphi;\alpha)$ 时，这样的 h 可由适当地选择 $h^{(k)}(\alpha)$ 的预定值得到.

由(3)，$(1-hg_n)/\varphi$ 有可去奇点. 于是对某个 $f\in H(\Omega)$ 有

$$1=hg_n+f\varphi\tag{4}$$

由(2)和(4)，$1\in[g_1,\cdots,g_n]$

我们已证明 $P(n-1)$ 蕴涵 $P(n)$，因此对所有 n，$P(n)$ 成立.

最后，假设 $G_1,\cdots,G_n\in H(\Omega)$，并且没有一个 G_i 恒等于 0（这不会丧失一般性）. 再次应用定理 15.11 得出 $\varphi\in H(\Omega)$，对所有 $\alpha\in\Omega$，$m(\varphi;\alpha)=\min m(G_i;\alpha)$ 成立. 令 $g_i=G_i/\varphi$，则 $g_i\in H(\Omega)$，并且函数 $g_1,\cdots,g_n$ 在 Ω 内没有公共零点. 由 $P(n)$，$[g_1,\cdots,g_n]=[1]$. 因此 $[G_1,\cdots,G_n]=[\varphi]$. 这就完成了定理的证明.

Jensen 公式

15.16 我们从定理 15.11 看出，全纯函数在 Ω 内的零点位置除了在 Ω 内没有极限点这一条外，是不受什么限制的.

如果我们将 $H(\Omega)$ 代之以它的由某些增长条件所确定的子类，情况就大不相同了. 在那些情况下，零点分布不得不满足某些定量条件. 这些定理的大多数其基础是 Jensen 公式(定理15.18). 我们将把它应用于某些整函数类和 $H(U)$ 的某些子类.

下面的引理给出了一个运用 Cauchy 定理来计算定积分的机会.

15.17 引理 $\dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\log|1-e^{i\theta}|\,d\theta=0.$

证明 设 $\Omega=\{z:\mathrm{Re}\,z<1\}$. 因为在 Ω 内，$1-z\neq 0$，并且 Ω 是

单连通区域，于是存在一函数 $h\in H(\Omega)$，使得

$$\exp\{h(z)\}=1-z$$

在 Ω 内成立．并且如果我们要求 $h(0)=0$，则 h 是唯一的．因为在 Ω 内，$\mathrm{Re}(1-z)>0$，于是我们有

$$(1)\qquad \mathrm{Re}\,h(z)=\log|1-z|,\quad |\mathrm{Im}\,h(z)|<\frac{\pi}{2}\qquad (z\in\Omega)$$

对于很小的 $\delta>0$，设 Γ 是路径

$$(2)\qquad \Gamma(t)=e^{it}\qquad (\delta\leqslant t\leqslant 2\pi-\delta)$$

并设 γ 是圆心在 1 的圆弧，它在 U 内径由 $e^{i\delta}$ 到 $e^{-i\delta}$．则

$$(3)\qquad \frac{1}{2\pi}\int_{\delta}^{2\pi-\delta}\log|1-e^{i\theta}|\,d\theta=\mathrm{Re}\left[\frac{1}{2\pi i}\int_{\Gamma}h(z)\frac{dz}{z}\right]$$

$$=\mathrm{Re}\left[\frac{1}{2\pi i}\int_{\gamma}h(z)\frac{dz}{z}\right]$$

最后一个等式依赖 Cauchy 定理；注意 $h(0)=0$.

γ 的长小于 $\pi\delta$，因此(1)式表明(3)式最后一个积分的绝对值小于 $C\delta\log(1/\delta)$，其中 C 是常数．如果在(3)中令 $\delta\to 0$，便得出结论．

15.18 定理 假设 $\Omega=D(0;R)$，$f\in H(\Omega)$，$f(0)\neq 0$，$0<r<R$，而 $\alpha_1,\cdots,\alpha_N$ 是 f 在 $\bar{D}(0;R)$ 内的零点，按照它们的重数列出．则

$$(1)\qquad |f(0)|\prod_{n=1}^{N}\frac{r}{|\alpha_n|}=\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})|\,d\theta\right\}$$

这就是所谓的 Jensen 公式．假设 $f(0)\neq 0$ 这一点对于应用并无妨碍，因为如果 f 在 0 有 k 阶零点，则公式可应用于 $f(z)/z^k$.

证明 将点 α_j 排序，使 $\alpha_1,\cdots,\alpha_m$ 均在 $D(0;r)$ 内，而 $|\alpha_{m+1}|=\cdots=|\alpha_N|=r$．（当然，我们可能有 $m=N$ 或 $m=0$.）设

$$(2)\qquad g(z)=f(z)\prod_{n=1}^{m}\frac{r^2-\bar{\alpha}z}{r(\alpha_n-z)}\prod_{n=m+1}^{N}\frac{\alpha_n}{\alpha_n-z}$$

则 $g\in H(D)$. 其中 $D=D(0;r+\varepsilon)$. 对某个 $\varepsilon>0$, g 在 D 内没有零点,因此 $\log|g|$ 在 D 内是调和函数(定理 13.12),并且

$$\log|g(0)|=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|g(re^{i\theta})|d\theta \tag{3}$$

由(2)式,

$$|g(0)|=|f(0)|\prod_{n=1}^{m}\frac{r}{|\alpha_n|} \tag{4}$$

如果 $|z|=r$,对于 $1\leqslant n\leqslant m$, (2)式中的因式绝对值为 1. 如果 $\alpha_n=\gamma e^{i\theta_n}$,对 $m<n\leqslant N$,便有

$$\log|g(re^{i\theta})|=\log|f(re^{i\theta})|-\sum_{n=m+1}^{N}\log|1-e^{i(\theta-\theta_n)}| \tag{5}$$

因此引理 15.17 表明,如果以 f 代替 g, 则(3)中的积分不变. 与(4)比较即给出(1).

Jensen 公式产生一个包含 U 内有界全纯函数的边界值的不等式(我们回忆一下,这些函数的类我们曾记作 H^∞):

15.19 定理 如果 $f\in H^\infty$, f 不恒等于 0,定义

$$\mu_r(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})|d\theta \qquad (0<r<1) \tag{1}$$

$$\mu^*(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f^*(re^{i\theta})|d\theta. \tag{2}$$

其中 f^* 是 f 的径向极限函数,如同定理 11.21 中的一样. 则

(3) 若 $0<r<s<1$, $\mu_r(f)\leqslant\mu_s(f)$

(4) 当 $r\to 0$ 时, $\mu_r(f)\to\log|f(0)|$

并且

(5) 当 $0<r<1$ 时, $\mu_r(f)\leqslant\mu^*(f)$

注意下面的推论: 可以选取 r,使当 $|z|=r$ 时, $f(z)\neq 0$. 这样 $\mu_r(f)$ 就是有限的,并且由(5), $\mu^*(f)$ 也是有限的. 于是 $\log|f^*|\in$

$L^1(T)$，并且在T的几乎每一个点有$f^*(re^{i\theta})\neq 0$.

证明 存在整数 $m\geqslant 0$，使 $f(z)=z^m g(z)$，$g\in H^\infty$，及$g(0)\neq 0$. 用 g 替换 f，应用 Jensen 公式 15.18(1). 当 r 增加时，它的左边显然不能减少. 于是当 $r<s$ 时，$\mu_r(g)\leqslant\mu_s(g)$. 因为

$$\mu_r(f)=\mu_r(g)+m\log r$$

我们便证明了(3).

不失一般性，现在假设$|f|\leqslant 1$. 记 $f_r(e^{i\theta})$以代替 $f(re^{i\theta})$，则当 $r\to 0$ 时，$f_r\to f(0)$，并且当 $r\to 1$ 时 f_r 几乎处处趋于 f^*. 因为 $\log(1/|f_r|)\geqslant 0$，两次应用 Fatou 引理并与(3)式组合，就给出(4)和(5).

15.20 整函数的零点 假设 f 是整函数，

(1) $$M(r)=\sup_{\theta}|f(re^{i\theta})| \qquad (0<r<\infty)$$

且 $n(r)$是 f 在 $\bar{D}(0;r)$的零点数. 为了简单，假定 $f(0)=1$. 如果 $\{\alpha_n\}$是 f 的零点序列，并按 $|\alpha_1|\leqslant|\alpha_2|\leqslant\cdots$排列的话，Jensen 公式就给出

$$M(2r)\geqslant\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(2re^{i\theta})|\,d\theta\right\}=\prod_{n=1}^{n(2r)}\frac{2r}{|\alpha_n|}$$

$$\geqslant\prod_{n=1}^{n(r)}\frac{2r}{|\alpha_n|}\geqslant 2^{n(r)}$$

因此

(2) $$n(r)\log 2\leqslant\log M(2r)$$

这样一来，使 $n(r)$能够按之增长的速度(换句话说，f 的零点密度)就由$M(r)$的增长率所控制. 从更特殊的情况着眼，假设对于很大的 r，有

(3) $$M(r)<\exp\{Ar^k\}$$

其中 A和 k 是给定的正数，则由(2)式导出

(4) $$\limsup_{r\to\infty}\frac{\log n(r)}{\log r}\leqslant k.$$

例如，若 k 是正整数，而

(5) $$f(z)=1-e^{z^k}$$

则 $n(r)$ 约为 $\pi^{-1}kr^k$. 因此

(6) $$\lim_{r\to\infty}\frac{\log n(r)}{\log r}=k$$

这证明了估计式(4)不能改进.

Blaschke 乘积

Jensen 公式使我们有可能决定不是常数的函数 $f\in H^\infty$ 的零点所必须满足的精确条件.

15.21 定理 如果 $\{\alpha_n\}$ 是 U 内的序列，满足 $\alpha_n\neq 0$ 和

(1) $$\sum_{n=1}^{\infty}(1-|\alpha_n|)<\infty$$

若 k 是非负整数，并且如果

(2) $$B(z)=z^k\prod_{n=1}^{\infty}\frac{\alpha_n-z}{1-\bar{\alpha}_n z}\cdot\frac{|\alpha_n|}{\alpha_n}\qquad(z\in U)$$

则 $B\in H^\infty$，且 B 在点 α_n 外没有零点（并且若 $k>0$，则原点是零点）.

我们称函数 B 为 Blaschke 乘积. 注意，某些 α_n 可能重复，此时 B 在这些点有多重零点. 也应注意(2)内每个因式在 T 上的绝对值为 1.

当只有有限个因式时，仍称为 Blaschke 乘积. 并且甚至可以没有一个因式，在这种情况下，$B(z)=1$.

证明 级数

$$\sum_{n=1}^{\infty}\left|1-\frac{\alpha_n-z}{1-\bar{\alpha}_n z}\cdot\frac{|\alpha_n|}{\alpha_n}\right|$$

的第 n 项当 $|z|\leqslant r$ 时是

$$\left|\frac{\alpha_n+|\alpha_n|z}{(1-\bar{\alpha}_n z)\alpha_n}\right|(1-|\alpha_n|)\leqslant\frac{1+r}{1-r}(1-|\alpha_n|).$$

因此定理 15.6 表明 $B\in H(U)$，并且 B 只有给定的零点. 因为在 (2) 中每一个因式在 U 内的绝对值小于 1，于是得知 $|B(z)|<1$，并且完成了证明.

15.22 上述定理指出

$$\sum_{n=1}^{\infty}(1-|\alpha_n|)<\infty \tag{1}$$

是存在一个只有预先给定的零点 $\{\alpha_n\}$ 的函数 $f\in H^\infty$ 的充分条件. 这个条件也可以证明是必要的: 如果 $f\in H^\infty$ 且 f 不恒等于 0，则 f 的零点必须满足 (1)式. 这是定理 15.23 的特款. 令人感兴趣的是(1)式对于我们现在就要论述的一个更大的函数类也是必要条件.

对任意实数 t，若 $t\geqslant 1$ 定义 $\log^+ t=\log t$，又若 $t<1$，$\log^+ t=0$. 设 N(源于 Nevanlinna)是所有 $f\in H(U)$ 并适合

$$\sup_{0<r<1}\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|d\theta<\infty \tag{2}$$

的函数类. 很明显 $H^\infty\subset N$. 注意当 $|z|\to 1$ 时，(2)对 $|f(z)|$ 的增长率作出了限制. 然而，积分

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})|d\theta \tag{3}$$

的有界性却并未施加这样的限制. 例如对任何 $g\in H(U)$，若 $f=e^g$，则(3)与 r 无关. 这一特点在于(3)能保持很小，因为 $\log|f|$ 和它能取得大的正值一样也可以取得大的负值，然而 $\log^+|f|\geqslant 0$. 在

第 17 章我们将进一步讨论类 N.

15.23 定理 假设$f\in N$, f在U内不恒等于0, 而 $\alpha_1,\alpha_2,\alpha_3,\cdots$是$f$的零点, 并按照它们的重数列出. 则

$$\sum_{n=1}^{\infty}(1-|\alpha_n|)<\infty \tag{1}$$

(我们默认f在U内有无穷多个零点. 如果只是有限个零点, 上述和式就只有有限项, 因此没有什么要证明的. 同样, $|\alpha_n|\leqslant|\alpha_{n+1}|$.)

证明 如果f在原点有m阶零点, 又$g(z)=z^{-m}f(z)$, 则$g\in N$, 并且除原点外, g与f有相同的零点. 因此, 不失一般性, 我们可假设 $f(0)\neq 0$. 设$n(r)$是f在$\bar{D}(0;r)$内的零点数. 固定k, 并取$r<1$使$n(r)>k$. 则由 Jensen 公式

$$|f(0)|\prod_{n=1}^{n(r)}\frac{r}{|\alpha_n|}=\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f(re^{i\theta})|d\theta\right\} \tag{2}$$

可推出

$$|f(0)|\prod_{n=1}^{k}\frac{r}{|\alpha_n|}\leqslant\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{+}|f(re^{i\theta})|d\theta\right\} \tag{3}$$

我们假设$f\in N$就是相当于存在一个常数$C<\infty$, 对于所有r, $0<r<1$, 它超过(3)式的右边. 这就得出

$$\prod_{n=1}^{k}|\alpha_n|\geqslant C^{-1}|f(0)|r^k \tag{4}$$

对于每个k, 当$r\to 1$时不等式仍然成立. 因此

$$\prod_{n=1}^{\infty}|\alpha_n|\geqslant C^{-1}|f(0)|>0 \tag{5}$$

由定理 15.5, (5)蕴涵(1).

系 如果$f\in H^{\infty}$(或者甚至$f\in N$), 若$\alpha_1,\alpha_2,\alpha_3,\cdots$是$f$在$U$内

的零点，并且如果 $\Sigma(1-|\alpha_n|)=\infty$，则对于所有 $z\in U$，$f(z)=0$.

例如，没有一个非常数的有界全纯函数在 U 内能以点 $(n-1)/n$ $(n=1,2,3,\cdots)$ 为零点.

我们以一个定理来结束本节. 它描述 Blaschke 乘积靠近 U 的边界时的性态. 回忆一下，作为 H^∞ 的元素，B 在 T 的几乎所有的点有径向极限 $B^*(e^{i\theta})$.

15.24 定理 如果 B 是 Blaschke 乘积，则 $|B^*(e^{i\theta})|=1$, a.e. 并且

$$\lim_{r\to 1}\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|B(re^{i\theta})|d\theta=0 \tag{1}$$

证明 极限存在是因为积分是 r 的单调函数这个事实得出的推论. 假设 $B(z)$ 如定理 15.21 一样，又记

$$B_N(z)=\prod_{n=N}^{\infty}\frac{\alpha_n-z}{1-\bar{\alpha}_n z}\cdot\frac{|\alpha_n|}{\alpha_n} \tag{2}$$

因为 $\log(|B/B_N|)$ 在一个包含 T 的开集内连续，如果用 B_N 代替 B，极限(1)不变. 如果我们应用定理 15.19 于 B_N，便会得出

$$\begin{aligned}\log|B_N(0)|&\leqslant\lim_{r\to 1}\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|B(re^{i\theta})|d\theta \\ &\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|B^*(e^{i\theta})|d\theta\leqslant 0\end{aligned} \tag{3}$$

当 $N\to\infty$ 时，(3)式的第一项趋于 0，这就得出 (1)，并且证明了 $\int\log|B^*|=0$. 因为 $\log|B^*|\leqslant 0$, a.e.，定理 1.39(a) 现在蕴涵 $\log|B^*|=0$, a.e.

Müntz-Szasz 定理

15.25 古典的 Weierstrass 定理([26]，定理 7.24)指出多项式在 $C(I)$ 中稠密. 这里，$C(I)$ 是闭区间[0，1]上所有连续复函数

的空间，赋以上确界范数．换句话说，所有函数

(1) $$1, t, t^2, t^3, \cdots$$

的有限线性组合的集在 $C(I)$ 中是稠密的．有时候用函数族(1)张成 $C(I)$ 的说法来表示这一点．

这就提出了一个问题：如果 $0<\lambda_1<\lambda_2<\lambda_3<\cdots$，在什么条件下，函数

(2) $$1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, \cdots$$

能够张成 $C(I)$？

所提的问题自然联系到在半平面（或者圆盘；这两者保形等价）内一个有界全纯函数零点分布的问题．漂亮得出奇的回答是：函数族(2)能张成 $C(I)$，当且仅当 $\Sigma 1/\lambda_n=\infty$．

实际上，它的证明甚至给出了一个更加精确的结论：

15.26 定理 假设 $0<\lambda_1<\lambda_2<\lambda_3<\cdots$，并设 X 是 $C(I)$ 内函数

$$1, t^{\lambda_1}, t^{\lambda_2}, t^{\lambda_3}, \cdots$$

全体有限线性组合的集的闭包．

(*a*) 如果 $\Sigma 1/\lambda_n=\infty$，则 $X=C(I)$；

(*b*) 如果 $\Sigma 1/\lambda_n<\infty$，并且若 $\lambda\notin\{\lambda_n\}$，$\lambda\neq 0$，则 X 不包含 t^λ．

证明 Hahn-Banach 定理（定理 5.19）的一个推论是：$\varphi\in C(I)$ 但 $\varphi\notin X$，当且仅当存在一个 $C(I)$ 上的有界线性泛函，在 φ 不为零而在整个 X 上为零．因为每个 $C(I)$ 上的有界线性泛函是由 I 上关于复 Borel 测度的积分给出，所以(*a*)是下述命题的推论：

如果 $\Sigma 1/\lambda_n=\infty$，又 μ 是 I 上的复 Borel 测度，满足

(1) $$\int_I t^{\lambda_n} d\mu(t)=0 \qquad (n=1,2,3,\cdots)$$

则同样有

(2) $$\int_I t^k d\mu(t)=0 \qquad (k=1,2,3,\cdots)$$

如果这一点得证，前面的评注表明X包含所有函数t^k；因为$1\in X$，于是所有的多项式都在X内．因此 Weierstrass 定理蕴涵$X=C(I)$．

所以假设(1)式成立．因为(1)和(2)中的被积函数在0点处为零，我们也可以假设μ集中在$(0,1]$．我们将μ对应于一个函数

$$f(z)=\int_I t^z d\mu(t) \tag{3}$$

$t>0$时，由定义，$t^z=\exp\{z\log t\}$．我们断定f在右半平面全纯．f的连续性是容易验证的，因此可应用 Morera 定理．再者，如果$z=x+iy$，又如果$x>0$，并且$0<t\leqslant 1$，则$|t^z|=t^x\leqslant 1$．于是f在右半平面有界，并且(1)说明，对于$n=1,2,3,\cdots$，$f(\lambda_n)=0$．定义

$$g(z)=f\left(\frac{1+z}{1-z}\right)\qquad(z\in U) \tag{4}$$

则$g\in H^\infty$和$g(\alpha_n)=0$．其中$\alpha_n=(\lambda_n-1)/(\lambda_n+1)$．如果$\Sigma 1/\lambda_n=\infty$，简单的计算表明$\Sigma(1-|\alpha_n|)=\infty$．因此定理 15.23 的系告诉我们对于所有的$z\in U$，$g(z)=0$，因此$f=0$．特别，对于$k=1,2,3,\cdots$，$f(k)=0$．这就是(2)式．于是定理的$(a)$部分证毕．

为了证明(b)部分，只须在I上构造一个测度μ，使(3)式定义的函数f在半平面$\mathrm{Re}z>-1$全纯(任何否定将在这里作出)且在点$0,\lambda_1,\lambda_2,\lambda_3,\cdots$为0，在该半平面内没有别的零点．于是由这个测度$\mu$导出的泛函在$X$上为0，但当$\lambda\neq 0$和$\lambda\notin\{\lambda_n\}$时对任何函数$t^\lambda$不为零．

我们从构造一个具有这些给定零点的函数f开始，并且随后将证明f能表为(3)的形式．定义

$$f(z)=\frac{z}{(2+z)^3}\prod_{n=1}^{\infty}\frac{\lambda_n-z}{2+\lambda_n-z} \tag{5}$$

因为

$$1-\frac{\lambda_n-z}{2+\lambda_n+z}=\frac{2z+2}{2+\lambda_n+z}$$

(5)中的无穷乘积在每一个不包含 $-\lambda_n-2$ 的紧集上一致收敛. 由此得出 f 是整个平面内的、有极点 -2 和 $-\lambda_n-2$ 的亚纯函数，并且在 $0,\lambda_1,\lambda_2,\lambda_3,\cdots$ 有零点. 同时，若 $\mathrm{Re}z>-1$，则无穷乘积(5)的每一个因式的绝对值小于 1. 于是当 $\mathrm{Re}z\geqslant -1$ 时，$|f(z)|\leqslant 1$. 因式 $(2+z)^3$ 保证 f 在直线 $\mathrm{Re}z=-1$ 的约束属于 L^1.

固定 z 使 $\mathrm{Re}z>-1$，并对 $f(z)$ 运用 Cauchy 公式，积分路径由圆心在 -1，半径 $R>1+|z|$ 的半圆及由 $-1-iR$ 经 $-1+R$ 到 $-1+iR$，再加上由 $-1+iR$ 到 $-1-iR$ 的区间组成. 当 $R\to\infty$ 时，沿半圆的积分趋于 0. 因此剩下

$$(6)\qquad f(z)=-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{f(-1+is)}{-1+is-z}ds \qquad (\mathrm{Re}z>-1)$$

但是

$$(7)\qquad \frac{1}{1+z-is}=\int_0^1 t^{z-is}dt \qquad (\mathrm{Re}z>-1)$$

因此可将(6)改写为

$$(8)\qquad f(z)=\int_0^1 t^z\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}f(-1+is)e^{-is\log t}ds\right\}dt$$

交换积分次序是允许的：如果(8)中被积函数取绝对值，便得到一个有限的积分.

设 $g(s)=f(-1+is)$，则(8)式内部的积分是 $\hat{g}(\log t)$，$\hat{g}$ 是 g 的Fourier 变式. 它是 $(0,1]$ 上的有界连续函数. 如果我们令 $d\mu(t)=\hat{g}(\log t)dt$，我们就得到以所需的形式(3)表示 f 的测度.

这就完成了证明.

15.27 评注 这个定理蕴涵着当 $\{1,t^{\lambda_1},t^{\lambda_2},\cdots\}$ 张成 $C(I)$ 时，则 t^{λ_i} 的某个无穷子集可以去掉而不会改变其张成性质. 特别 $C(I)$

不包含这种类型的极小张成集．这表明它可以和 Hilbert 空间中规范正交集的下述性质相比较：如果在一个规范正交集中去掉任何一个元素，它的张成度就会减小．同样，如果$\{1, t^{\lambda_1}, t^{\lambda_2}, \cdots\}$不张成$C(I)$，去掉它的任何元素也将会减小其张成度．这可由定理 15.26(b)得出．

习　题

1　假设$\{a_n\}$和$\{b_n\}$是复数序列，满足$\Sigma|a_n-b_n|<\infty$．问在什么集上乘积

$$\prod_{n=1}^{\infty}\frac{z-a_n}{z-b_n}$$

一致收敛？在什么地方它定义一个全纯函数？

2　假设f是整函数，λ是一个正数，并且不等式

$$|f(z)|<\exp(|z|^{\lambda})$$

对所有足够大的$|z|$成立．（这样的函数称为有限阶的．满足上述条件的所有λ的最大下界就是f的阶．）如果$f(z)=\Sigma a_n z^n$，证明不等式

$$|a_n|\leqslant\left(\frac{e\lambda}{n}\right)^{n/\lambda}$$

对所有足够大的n成立．考虑函数$\exp(z^k)$，$k=1, 2, 3, \cdots$以判定上述$|a_n|$的界是否是最可能接近的．

3　试求满足$\exp(\exp(z))=1$的所有复数z，将它们作为平面上的点把它们描出．证明没有一个有限阶的整函数能以每一个这种点为零点（当然，除非$f\equiv 0$）．

4　证明函数

$$\pi\cot\pi z=\pi i\frac{e^{\pi iz}+e^{-\pi iz}}{e^{\pi iz}-e^{-\pi iz}}$$

在整数点有残数为 1 的一阶极点．同样，证明对于函数

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}\frac{2z}{z^2-n^2}=\lim_{N\to\infty}\sum_{n=-N}^{N}\frac{1}{z-n}$$

上述结论也成立．证明这两个函数是周期函数[$f(z+1)=f(z)$]，它们的差

是有界整函数，因而是常数，并证明这个常数实际上为 0，因为

$$\lim_{y\to+\infty} f(iy) = -2i\int_0^\infty \frac{dt}{1+t^2} = -\pi i$$

这就给出了部分分式分解式

$$\pi\cot\pi z = \frac{1}{z} + \sum_1^\infty \frac{2z}{z^2-n^2}$$

(与第 9 章习题 12 比较)．注意，如果 $g(z)=\sin\pi z$，$\pi\cot\pi z$ 就是 $(g'/g)(z)$．试导出乘积表示式

$$\frac{\sin\pi z}{\pi z} = \prod_{n=1}^\infty \left(1-\frac{z^2}{n^2}\right)$$

5　假设 k 是正整数，$\{z_n\}$ 是满足 $\Sigma|z_n|^{-k-1}<\infty$ 的复数序列，并且

$$f(z) = \prod_{n=1}^\infty E_k\left(\frac{z}{z_n}\right)$$

(见定义 15.7)．关于

$$M(r) = \max_\theta |f(re^{i\theta})|$$

的增长率你能说些什么？

6　假设 f 是整函数，$f(0)\neq 0$，当 $|z|$ 足够大时，$|f(z)|<\exp(|z|^s)$，并且 $\{z_n\}$ 是 f 的零点序列，按它们的重数计算，证明对每个 $\varepsilon>0$，$\Sigma|z_n|^{-s-\varepsilon}<\infty$．(对照第 12.20 节．)

7　假设 f 是一个整函数，对 $n=1,2,3,\cdots$，$f(\sqrt{n})=0$，并且存在一个正数 α，对所有足够大的 $|z|$，使得 $|f(z)|<\exp(|z|^\alpha)$．若要对所有 z 有 $f(z)=0$，α 应如何？[考虑 $\sin(\pi z^2)$．]

8　设 $\{z_n\}$ 是不相同复数的序列，$z_n\neq 0$，当 $n\to\infty$ 时，$z_n\to\infty$，并且设 $\{m_n\}$ 是正整数序列．设 g 是平面内的亚纯函数，它以点 z_n 为具有残数 m_n 的一阶极点，并且它无另外的极点．如果 $z\notin\{z_n\}$，设 $\gamma(z)$ 是由 0 到 z 不经过任何 z_n 的任意一条路径，并且定义

$$f(z) = \exp\{\int_{\gamma(z)} g(\zeta)\,d\zeta\}$$

证明 $f(z)$ 与 $\gamma(z)$ 的选择无关(即使积分本身并非如此)，f 在 $\{z_n\}$ 的余集内全纯，f 在每个点 z_n 有可去奇点．并且证明 f 的开拓在 z_n 有 m_n 阶零点．

包含在定理 15.9 内的存在性定理因此可由 Mittag-Leffler 定理推得．

9　假设 $0<\alpha<1$，$0<\beta<1$，$f\in H(U)$，$f(U)\subset U$，并且 $f(0)=\alpha$，问 f 在

圆盘 $\bar{D}(0;\beta)$ 内有多少个零点？ 如果 (a) $\alpha=\frac{1}{2}$, $\beta=\frac{1}{2}$; (b) $\alpha=\frac{1}{4}$, $\beta=\frac{1}{2}$; (c) $\alpha=\frac{2}{3}$, $\beta=\frac{1}{3}$; (d) $\alpha=1/1000$; $\beta=1/10$ 时，结果如何？

10 对于 $N=1,2,3,\cdots$，定义

$$g_N(z)=\prod_{n=N}^{\infty}\left(1-\frac{z^2}{n^2}\right)$$

证明在整函数环内由 $\{g_N\}$ 产生的理想不是主理想.

11 问在什么条件下，对于实数序列 y_n，存在一个开的右半平面内的有界全纯函数，它不恒为 0，但在每个点 $1+iy_n$ 有零点？特别，当 (a) $y_n=\log n$, (b) $y_n=\sqrt{n}$, (c) $y_n=n$, (d) $y_n=n^2$ 时能否存在？

12 假设 $0<|a_n|<1$, $\Sigma(1-|a_n|)<\infty$，并且 B 是以 a_n 为零点的 Blaschke 乘积. 设 E 是全体 $1/\bar{a}_n$ 的点集，又设 Ω 是 E 的闭包的余集. 证明这个乘积确实在 Ω 的每个紧子集上一致收敛. 因此 $B\in H(\Omega)$，并且 B 在 E 的每一点有极点. (特别有趣的是当 Ω 是连通的那些场合.)

13 对于 $n=1,2,3,\cdots$，设 $a_n=1-n^{-2}$. 设 B 是以这些点 a 为零点的 Blaschke 乘积. 证明 $\lim_{r\to 1}B(r)=0$ (理解为 $0<r<1$.)

更确切地说，如果 $a_{N-1}<r<a_N$，证明估计值

$$|B(r)|<\prod_{1}^{N-1}\frac{r-a_n}{1-a_nr}<\prod_{1}^{N-1}\frac{a_N-a_n}{1-a_n}<2e^{-N/3}$$

是正确的.

14 证明存在序列 $\{a_n\}$, $0<a_n<1$，它趋于 1 如此迅速，使得以 a_n 为零点的 Blaschke 乘积满足

$$\limsup_{r\to 1}|B(r)|=1$$

因而 B 在 $z=1$ 没有径向极限.

15 设 φ 是线性分式变换，它将 U 映到 U 上. 对任何 $z\in U$，定义 z 的 φ 轨道为集 $\{\varphi_n(z)\}$，其中 $\varphi_0(z)=z$, $\varphi_n(z)=\varphi(\varphi_{n-1}(z))$, $n=1,2,3,\cdots$. 可忽略 $\varphi(z)=z$ 的情况.

(a) 对于哪些 φ，其 φ 轨道满足 Blaschke 条件 $\Sigma(1-|\varphi_n(z)|)<\infty$？[答案部分地依赖于 φ 的不动点的位置. 可能有一不动点在 U 内，或者有一不动点在 T 上，或者两个不动点在 T 上. 对于后两种情况，可以把问题化为

对上半平面来讨论，并且考虑那样一些变换，即或者它只有∞是不动点，或者以 0 和∞为不动点．这样做是很有裨益的．]

(*b*) 对哪些 φ 存在在 φ 下不变的、即对所有 $z\in U$ 满足关系式 $f(\varphi(z))=f(z)$ 的非常数函数 $f\in H^\infty$？

16 假设 $|\alpha_1|\leqslant|\alpha_2|\leqslant|\alpha_3|\leqslant\cdots<1$，并设 $n(r)$ 是序列 $\{\alpha_j\}$ 中使 $|\alpha_j|\leqslant r$ 的项数．证明

$$\int_0^1 n(r)\,dr=\sum_{j=1}^{\infty}(1-|\alpha_j|)$$

17 如果 $B(z)=\Sigma c_k z^k$ 是一个 Blaschke 乘积，它至少有一个不同于原点的零点，问能不能对 $k=0,1,2,\cdots$ 都有 $c_k\geqslant 0$？

18 假设 B 是一个 Blaschke 乘积，其全体零点均位于区间 $(0,1)$ 上，且

$$f(z)=(z-1)^2B(z)$$

证明 f 的导数在 U 内是有界的．

19 记 $f(z)=\exp[(1+z)/(1-z)]$．利用定理 15.19 的记号，证明尽管 $f\in H^\infty$，但

$$\lim_{r\to 1}\mu_r(f)<\mu^*(f)$$

注意与定理 15.24 对比．

20 在 Müntz-Szasz 定理中，假设 $\lambda_1>\lambda_2>\cdots$ 且 $\lambda_n\to 0$．在这些条件下，该定理的结论如何？

21 以 $L^2(I)$ 代替 $C(I)$ 证明 Müntz-Szasz 定理的类似定理．

22 记 $f_n(t)=t^n e^{-t}(0\leqslant t<\infty, n=0,1,2,\cdots)$，证明这些函数的全体有限线性组合的集稠密于 $L^2(0,\infty)$．提示：若 $g\in L^2(0,\infty)$ 与每个 f_n 正交，并且

$$F(z)=\int_0^\infty e^{-tz}\overline{g(t)}\,dt\qquad(\mathrm{Re}\,z>0)$$

则 F 在 $z=1$ 的所有导数都是 0．考虑 $F(1+iy)$．

第十六章　解析延拓

因为在某区域内定义且全纯的函数常常能扩展为在某个较大的区域内全纯的函数，在本章中，我们将涉及由此引起的问题．定理 10.18 表明这样的开拓由所给的函数唯一决定．这个扩展过程称为解析延拓．一种很自然的方法使我们宁可考虑定义在 Riemann 面上而不是在平面区域内的函数．这种方案使我们能够用函数来代替"多值函数"(例如平方根函数或对数函数)．然而，Riemann 面的系统论述会使我们离题太远，所以我们将限制在平面区域内讨论．

正则点和奇点

16.1　定义　令 D 为一开圆盘，假设 $f\in H(\Omega)$，并令 β 为 D 的一边界点．若存在中心在 β 的圆盘 D_1 及函数 $g\in H(D_1)$，使得对所有 $z\in D\cap D_1$，$g(z)=f(z)$，我们就称 β 为 f 的正则点．D 的任何一个边界点若不是 f 的正则点，就称为 f 的奇点．

从定义看来，显然所有 f 的正则点的集是 D 的边界的一个开子集(也可能是空集)．

在下列定理中，不失一般性，我们将取单位圆盘 U 作为 D．

16.2　定理　假设 $f\in H(U)$，且幂级数

$$f(z)=\sum_{n=0}^{\infty}a_nz^n \qquad (z\in U) \tag{1}$$

的收敛半径为 1．则 f 在单位圆周 T 上至少有一个奇点．

证明　相反地，假设 T 的每一点均为 f 的正则点．T 的紧性

蕴涵着存在开圆盘$D_1,\cdots,D_n$及函数$g_j\in H(D_j)$，使得每个D_j的中心均在T上，$T\subset D_1\cup\cdots\cup D_n$，并且对$j=1,\cdots,n$，在$D_j\cap U$内有$g_j(z)=f(z)$.

若$D_i\cap D_j\neq\varnothing$，且$V_{ij}=D_i\cap D_j\cap U$，则$V_{ij}\neq\varnothing$（因$D_j$的中心在$T$上）. 且在$V_{ij}$内$g_i=f=g_j$. 由于$D_i\cap D_j$是连通的，由定理10.18得出在$D_i\cap D_j$内$g_i=g_j$. 因此我们能用

$$h(z)=\begin{cases}f(z) & (z\in U)\\ g_i(z) & (z\in D_i)\end{cases}\tag{2}$$

定义一个在$\Omega=U\cup D_1\cup\cdots\cup D_n$内的函数$h$.

由于$\Omega\supset\bar{U}$且Ω是开集，存在$\varepsilon>0$，使得圆盘$D(0;1+\varepsilon)\subset\Omega$. 但$h\in H(\Omega)$，$h(z)$在$U$内由(1)给定. 现在定理10.16蕴涵(1)的收敛半径至少为$1+\varepsilon$. 这与我们的假设矛盾.

16.3 定义 若$f\in H(U)$且若T的每一点都是f的奇点，则T称为f的自然边界. 在这种情形下，f没有到真正包含U的任何区域的全纯开拓.

16.4 评注 很容易看出，存在$f\in H(U)$，使T是它的自然边界. 事实上，若Ω为任意区域，容易找出函数$f\in H(\Omega)$，它不能扩展为在任何较大区域内的全纯函数. 要看出这一点，令A为Ω内任意可数集，它在Ω内无极限点，但使得Ω的每个边界点都是A的极限点. 应用定理15.11便给出一函数$f\in H(\Omega)$，它在A的每一点为零，但并不恒等于零. 若$g\in H(\Omega_1)$，此处Ω_1是一真正包含Ω的区域，且若在Ω内$g=f$，则g的零点将在Ω_1内具有极限点. 我们就得出了矛盾.

由

$$f(z)=\sum_{n=0}^{\infty}z^{2^n}=z+z^2+z^4+z^8+\cdots\qquad(z\in U)\tag{1}$$

提供了一个简单明瞭的例子. f满足函数方程

(2) $$f(z^2)=f(z)-z$$

由此得出（我们留给读者详细讨论），f 在 U 的每一条端点在 $\exp(2\pi ik/2^n)$ 的半径上都是无界的. 此处 k 和 n 都是正整数. 这些点构成了 T 的稠密子集，且由于 f 的所有奇点的集是闭的，故 f 以 T 作为它的自然边界.

这个例子是一个有大量空隙（也就是有许多零系数）的幂级数这一点并非意外. 这个例子只是定理 16.6 的特殊情形. 这个定理是 Hadamard 作出的，我们将由 Ostrowski 的下述定理来推出它.

16.5 定理 假设 λ, p_k 及 q_k 是正整数，$p_1<p_2<p_3<\cdots$，且

(1) $$\lambda q_k>(\lambda+1)p_k \qquad (k=1,2,3,\cdots)$$

假设

(2) $$f(z)=\sum_{n=0}^{\infty}a_nz^n$$

的收敛半径为 1，且对某个 k，当 $p_k<n<q_k$ 时，$a_n=0$. 若 $s_p(z)$ 是 (2) 的第 p 项部分和，且若 β 为 f 在 T 上的正则点. 则序列 $\{s_{p_k}(z)\}$ 在 β 的某邻域内收敛.

注意整个序列 $\{s_{p_k}(z)\}$ 不能在 $\overline{U}$ 外任何一点收敛. 空隙条件 (1) 保证了在 β 的邻域，因而在 $\overline{U}$ 外某些点，存在收敛的子序列. 这种现象称为过度收敛.

证明 若 $g(z)=f(\beta z)$，则 g 也满足空隙条件. 因此，不失一般性，我们可以假设 $\beta=1$. 于是 f 有到包含 $U\cup\{1\}$ 的一个区域 Ω 的全纯开拓. 令

(3) $$\varphi(w)=\frac{1}{2}(w^\lambda+w^{\lambda+1})$$

并对所有使 $\varphi(w)\in\Omega$ 的 w，定义 $F(w)=f(\varphi(w))$. 若 $|w|\leqslant 1$，但 $w\neq 1$，由 $|1+w|<2$ 得 $|\varphi(w)|<1$. 同时也有 $\varphi(1)=1$. 因此，存

在 $\varepsilon>0$, 使得 $\varphi(D(0;1+\varepsilon))\subset\Omega$, 注意区域 $\varphi(D(0;1+\varepsilon))$ 包含点1. 当 $|w|<1+\varepsilon$ 时, 级数

$$F(w)=\sum_{m=0}^{\infty}b_m w^m \tag{4}$$

是收敛的.

在 $[\varphi(w)]^n$ 内 w 的最高及最低次幂具有指数 $(\lambda+1)n$ 及 λn. 因此, 由(1), 在 $[(\varphi(w)]^{p_k}$ 中的最高指数小于 $[\varphi(w)]^{q_k}$ 中的最低指数. 由于

$$F(w)=\sum_{n=0}^{\infty}a_n[\varphi(w)]^n \qquad (|w|<1) \tag{5}$$

现在由 $\{a_n\}$ 满足的空隙条件可推出

$$\sum_{n=0}^{p_k}a_n[\varphi(w)]^n=\sum_{m=0}^{(\lambda+1)p_k}b_m w^m \qquad (k=1,2,3,\cdots) \tag{6}$$

当 $k\to\infty$, $|w|<1+\varepsilon$ 时, (6)的右边收敛. 因此, $\{s_{p_k}(z)\}$ 对所有 $z\in\varphi(D(0;1+\varepsilon))$ 收敛. 这就是所求的结论.

注: 实际上, $\{s_{p_k}(z)\}$ 在 β 的某一邻域一致收敛. 我们留给读者通过更细心地考察上述证明来证实这个结果.

16.6 定理 假设 λ 是个正整数, $\{p_k\}$ 是一个正整数序列, 使

$$p_{k+1}>\left(1+\frac{1}{\lambda}\right)p_k \qquad (k=1,2,3,\cdots) \tag{1}$$

且幂级数

$$f(z)=\sum_{k=1}^{\infty}c_k z^{p_k} \tag{2}$$

的收敛半径为 1. 则 f 以 T 作为它的自然边界.

证明 定理 16.5 的序列 $\{s_{p_k}\}$ 现在与(2)的整个的部分和序列相同(除了重复的之外). 后者不能在 $\overline{U}$ 外任何一点收敛. 因

此定理 16.5 推出 T 没有一个点能是 f 的正则点.

16.7 例 当 n 是 2 的幂时，令 $a_n=1$，其余情形令 $a_n=0$. 令 $\eta_n=\exp(-\sqrt{n})$，并定义

$$f(z)=\sum_{n=0}^{\infty}a_n\eta_n z^n \tag{1}$$

由于

$$\limsup_{n\to\infty}|a_n\eta_n|^{1/n}=1 \tag{2}$$

故(1)的收敛半径为 1. 由 Hadamard 定理，f 以 T 作为它的自然边界. 然而，f 的各阶导数的幂级数

$$f^{(k)}(z)=\sum_{n=k}^{\infty}n(n-1)\cdots(n-k+1)a_n\eta_n z^{n-k} \tag{3}$$

在闭单位圆盘上一致收敛. 因此，每个 $f^{(k)}$ 均在 $\bar{U}$ 上一致连续，并且 f 在 T 上的限制作为 θ 的函数是无穷可微的，而不管 T 是 f 的自然边界这一事实.

这例子相当值得注意地证明，在定义 16.1 的意义下，奇点的出现并不意味着会出现不连续性或者(不太严格地说)出现光滑性的任何缺陷.

在这里插入一个说明连续性排除了奇点存在的定理，看来是适宜的.

16.8 定理 假设 Ω 是一区域，L 是一条直线或一段圆弧，$\Omega-L$ 是两个区域 Ω_1 及 Ω_2 的并，f 在 Ω 内连续，且 f 在 Ω_1 及 Ω_2 内均全纯，则 f 在 Ω 内全纯.

证明 如果我们对直线 L 证明了这个定理，则应用分式线性变换，一般场合也就可以得出了. 由 Morera 定理，只要证明对 Ω 内每个三角形 $\triangle$，f 沿边界 $\partial\triangle$ 的积分均为零就够了. Cauchy 定理推出，沿 $\triangle\cap\Omega_1$ 内或 $\triangle\cap\Omega_2$ 内每一闭路径 γ，f 的积分为零. f

的连续性表明，若 γ 的一部分在 L 内，这结果仍然正确，而沿 $\partial\triangle$ 的积分最多是这种形式的两个项之和.

沿曲线的延拓

16.9 定义 一个函数元素是一个序对 (f, D). 此处 D 为一开圆盘，且 $f\in H(D)$. 若下列两条件成立：$D_0\cap D_1\neq\varnothing$，且对所有 $z\in D_0\cap D_1$，$f_0(z)=f_1(z)$，则称两个函数元素 (f_0, D_0) 及 (f_1, D_1) 是彼此直接延拓的. 在这种情形，我们记为

$$(f_0, D_0)\sim(f_1, D_1) \tag{1}$$

链是圆盘的有限序列 $\mathscr{C}$，就是说 $\mathscr{C}=\{D_0, D_1, \cdots, D_n\}$，使得对 $i=1, \cdots, n$，$D_{i-1}\cap D_i\neq\varnothing$. 若已给定 (f_0, D_0)，且若存在元素 (f_i, D_i)，使得对 $i=1, \cdots, n$，$(f_{i-1}, D_{i-1})\sim(f_i, D_i)$，则 (f_n, D_n) 称为 (f_0, D_0) 沿 $\mathscr{C}$ 的解析延拓. 注意，f_n 由 f_0 及 $\mathscr{C}$（若它全都存在）唯一决定. 为了看出这一点，假设(1)成立，并假设当用 g_1 代替 f_1 时(1)也成立. 则在 $D_0\cap D_1$ 内，$g_1=f_0=f_1$；又由 D_1 是连通的，我们在 D_1 内有 $g_1=f_1$. 现由对 $\mathscr{C}$ 的项数使用归纳法，便得出 f_n 的唯一性.

若 (f_n, D_n) 为 (f_0, D_0) 沿 $\mathscr{C}$ 的延拓，且若 $D_n\cap D_0\neq\varnothing$. 则 $(f_0, D_0)\sim(f_n, D_n)$ 不一定成立；换句话说，关系 $\sim$ 并不是传递的. 最简单的例子由平方根函数给出：设 D_0, D_1 及 D_2 为中心在 $1, w$ 及 w^2，半径为 1 的圆盘，此处 $w^3=1$. 选取 $f_j\in H(D_j)$，使得 $f_j^2(z)=z$. 因此，$(f_0, D_0)\sim(f_1, D_1)$，$(f_1, D_1)\sim(f_2, D_2)$. 在 $D_0\cap D_2$ 内，我们有 $f_2=-f_0\neq f_0$.

称链 $\mathscr{C}=\{D_0, \cdots, D_n\}$ 是用参数区间 $[0, 1]$ 覆盖曲线 γ，若存在数组 $0=s_0<s_1<\cdots<s_n=1$，使得 $\gamma(0)$ 为 D_0 的中心，$\gamma(1)$ 为 D_n 的中心，且

$$\gamma([s_i, s_{i+1}])\subset D_i \qquad (i=1, \cdots, n-1) \tag{2}$$

如果 (f_0, D_0) 能沿这样的 $\mathscr{C}$ 延拓到 (f_n, D_n)，我们称 (f_n, D_n) 为 $(f_0,$

D_0)沿 γ 的解析延拓.(唯一性将在定理 16.11 证明.)(f_0, D_0)因此被称为容许一个沿 γ 的解析延拓.

虽然关系(1)并不传递,但传递性的一个局限形式还是成立的.它提供了定理 16.11 证明的关键.

16.10 命题 假设$D_0 \cap D_1 \cap D_2 \neq \varnothing$,$(D_0, f_0) \sim (D_1, f_1)$,且$(D_1, f_1) \sim (D_2, f_2)$,则$(D_0, f_0) \sim (D_2, f_2)$.

证明 由假设,在 $D_0 \cap D_1$ 内 $f_0 = f_1$,在 $D_1 \cap D_2$ 内 $f_1 = f_2$.因此,在非空开集 $D_0 \cap D_1 \cap D_2$ 内 $f_0 = f_2$.由于 f_0 及 f_2 均在 $D_0 \cap D_2$ 内全纯,且 $D_0 \cap D_2$ 为连通的,便得出在 $D_0 \cap D_2$ 内 $f_0 = f_2$.

16.11 定理 如果(f, D)是一函数元素,并且 γ 是从 D 的中心出发的一条曲线,则(f, D)沿 γ 最多容许一个解析延拓.

这定理的断言更清楚的陈述是:如果 γ 被链 $\mathscr{C}_1 = \{A_0, A_1, \cdots, A_m\}$及 $\mathscr{C}_2 = \{B_0, B_1, \cdots, B_n\}$覆盖,此处 $A_0 = B_0 = D$.若(f, D)能沿 $\mathscr{C}_1$ 解析延拓到函数元素(g_m, A_m),同时(f, D)又能沿 $\mathscr{C}_2$ 解析延拓到(h_n, B_n),则在 $A_m \cap B_n$ 内,$g_m = h_n$.

根据假设,A_m 及 B_n 都是具有相同中心 $\gamma(1)$的圆盘,这便得出 g_m 及 h_n 具有相同的 $z - \gamma(1)$的幂级数展开式,并且我们同样能够用 A_m 及 B_n 两者中较大的一个代替它们.承认了这一点,最终就得到 $g_m = h_n$.

证明 设 $\mathscr{C}_1$ 及 $\mathscr{C}_2$ 如上述.存在数组

$$0 = s_0 < s_1 < \cdots < s_m = 1 = s_{m+1}$$

及 $0 = \sigma_0 < \sigma_1 < \cdots < \sigma_n = 1 = \sigma_{n+1}$,使得

(1) $\gamma([s_i, s_{i+1}]) \subset A_i$, $\gamma([\sigma_j, \sigma_{j+1}]) \subset B_j$ $\quad (0 \leqslant i \leqslant m, 0 \leqslant j \leqslant n)$

对$0 \leqslant i \leqslant m-1$和$0 \leqslant j \leqslant n-1$,存在函数元素$(g_i, A_i) \sim (g_{i+1}, A_{i+1})$及$(h_j, B_j) \sim (h_{j+1}, B_{j+1})$,此处 $g_0 = h_0 = f$.

我们断言,如果 $0 \leqslant i \leqslant m$ 和 $0 \leqslant j \leqslant n$,并且$[s_i, s_{i+1}]$与$[\sigma_j, \sigma_{j+1}]$相交,则$(g_i, A_i) \sim (h_j, B_j)$.

假设存在数对(i, j)使这断言错误. 则它们中有一个使$i+j$最小. 很清楚$i+j>0$. 假设$s_i \geqslant \sigma_j$, 则$i \geqslant 1$, 且由$[s_i, s_{i+1}]$与$[\sigma_j, \sigma_{j+1}]$相交, 我们看到

(2) $$\gamma(s_i) \in A_{i-1} \cap A_i \cap B_j$$

$i+j$的最小性表明$(g_{i-1}, A_{i-1}) \sim (h_j, B_j)$, 又因为$(g_{i-1}, A_{i-1}) \sim (g_i, A_i)$, 由命题16.10可推出$(g_i, A_i) \sim (h_j, B_j)$. 这与我们的假设矛盾. 用相同的方法可排除$s_i \leqslant \sigma_j$的可能性.

于是我们的断言已经确立. 特别, 它对数对(m, n)成立, 并且这就是我们要证明的.

16.12 定义 假设α和β是拓扑空间X内的点, 而φ是单位正方形$I^2 = I \times I$(此处$I = [0, 1]$)到X内的连续映射, 使得对所有$t \in I$, $\varphi(0, t) = \alpha$及$\varphi(1, t) = \beta$, 则称由

(1) $$\gamma_t(s) = \varphi(s, t) \qquad (s \in I, t \in I)$$

定义的曲线γ_t组成了X内的由α到β的一个单参数曲线族$\{\gamma_t\}$.

我们现在导出解析延拓的一个十分重要的性质:

16.13 定理 假设$\{\gamma_t\}$$(0 \leqslant t \leqslant 1)$是平面内由$\alpha$到$\beta$的单参数曲线族, D为中心在α的圆盘, 且函数元素(f, D)沿每一条γ_t都容许解析延拓到一元素(g_t, D_t), 则$g_1 = g_0$.

最后的等式是像定理16.11中的那样解释的:

$$(g_1, D_1) \sim (g_0, D_0)$$

且D_0, D_1是具有相同的中心β的圆盘.

证明 固定$t \in I$. 则存在覆盖γ_t的一个链$\mathscr{C} = \{A_0, \cdots, A_n\}$, $A_0 = D$, 使得(g_t, D_t)由(f, D)沿$\mathscr{C}$延拓得到. 并存在数组$0 = s_0 < \cdots < s_n = 1$, 使得

(1) $$E_i = \gamma_t([s_i, s_{i+1}]) \subset A_i \qquad (i = 0, 1, \cdots, n-1)$$

这样就存在$\varepsilon > 0$, 它比任意紧集E_i到对应的开圆盘A_i的余集之间的距离都要小. φ在I^2上的一致连续性(见定义16.12)表明存

在 $\delta>0$，使得当 $s\in I, u\in I, |u-t|<\delta$ 时，

(2) $$|\gamma_t(s)-\gamma_u(s)|<\varepsilon$$

假设 u 满足这些条件，则(2)表明 $\mathscr{C}$ 覆盖 γ_u. 因此，定理16.11表明 g_t 及 g_u 都由 (f,D) 沿相同的链 $\mathscr{C}$ 延拓得到，所以 $g_t=g_u$.

这样，每个 $t\in I$ 均被线段 J_t 覆盖，使得对所有 $u\in I\cap J_t$，$g_u=g_t$. 因为 I 是紧的，所以 I 被有限多个 J_t 覆盖；又因为 I 是连通的，经有限步后，我们可以看出 $g_1=g_0$.

下面定理是直观上很明显的一个拓扑事实.

16.14 定理 假设 Γ_0 及 Γ_1 是拓扑空间 X 内的曲线，具有公共的起点 α 及公共的终点 β. 如果 X 是单连通的，则存在 X 内由 α 到 β 的单参数曲线族 $\{\gamma_t\}(0\leqslant t\leqslant 1)$，使得 $\gamma_0=\Gamma_0$ 及 $\gamma_1=\Gamma_1$.

证明 令 $[0,\pi]$ 为 Γ_0 及 Γ_1 的参数区间，则

(1) $$\Gamma(s)=\begin{cases}\Gamma_0(s) & (0\leqslant s\leqslant\pi)\\ \Gamma_1(2\pi-s) & (\pi\leqslant s\leqslant 2\pi)\end{cases}$$

定义了 X 内一条闭曲线. 由于 X 是单连通的，Γ 在 X 内是零伦的，因此存在连续映射 $H:[0,2\pi]\times[0,1]\to X$，使得

(2) $$H(s,0)=\Gamma(s), H(s,t)=c\in X, H(0,t)=H(2\pi,t)$$

如果 $\Phi:\overline{U}\to X$ 由

$$\Phi(re^{i\theta})=H(\theta,1-r)\qquad(0\leqslant r\leqslant 1, 0\leqslant\theta\leqslant 2\pi)$$

定义，则(2)蕴涵着 Φ 是连续的. 令

$$\gamma_t(\theta)=\Phi[(1-t)e^{i\theta}+te^{-i\theta}]\qquad(0\leqslant\theta\leqslant\pi,\ 0\leqslant t\leqslant 1)$$

由 $\Phi(e^{i\theta})=H(\theta,0)=\Gamma(\theta)$，便得出

$$\gamma_t(0)=\Phi(1)=\Gamma(0)=\alpha\qquad(0\leqslant t\leqslant 1)$$

$$\gamma_t(\pi)=\Phi(-1)=\Gamma(\pi)=\beta\qquad(0\leqslant t\leqslant 1)$$

$$\gamma_0(\theta)=\Phi(e^{i\theta})=\Gamma(\theta)=\Gamma_0(\theta)\qquad(0\leqslant\theta\leqslant\pi)$$

和

$$\gamma_1(\theta)=\Phi(e^{-i\theta})=\Phi(e^{i(2\pi-\theta)})=\Gamma(2\pi-\theta)=\Gamma_1(\theta)\ (0\leqslant\theta\leqslant\pi)$$

这就完成了证明.

单值性定理

上面的讨论实质上已经证明了下述重要定理.

16.15 定理 假设Ω是单连通区域,(f, D)是一个函数元素,$D\subset\Omega$,且(f, D)能沿Ω内起点在D的中心的每一条曲线解析延拓.则存在$g\in H(\Omega)$,使得对所有$z\in D$,$g(z)=f(z)$.

证明 设P_0和P_1是Ω内由D的中心α到某点$\beta\in\Omega$的两条曲线.由定理16.13及16.14得出,(f, D)沿Γ_0及Γ_1的解析延拓都会导致出相同的元素(g_β, D_β).这里D_β是中心在β的一个圆盘.若D_{β_1}与D_β相交,则$(g_{\beta_1}, D_{\beta_1})$可以首先将$(f, D)$延拓到$\beta$,然后沿直线由$\beta$延拓到$\beta_1$而得到.这表明在$D_{\beta_1}\cap D_\beta$内,$g_{\beta_1}=g_\beta$.

因此定义

$$g(z)=g_\beta(z)\qquad(z\in D_\beta)$$

是相容的,并且给出了所求的f的全纯开拓.

16.16 评注 设Ω为一平面区域,固定$w\notin\Omega$,令D为Ω内的一个圆盘.由于D是单连通的,存在$f\in H(D)$,使得$\exp[f(z)]=z-w$.注意在D内$f'(z)=(z-w)^{-1}$,并且后一函数在整个Ω内全纯.这就蕴涵了(f, D)能沿Ω内从D的中心α出发的每一条路径γ解析地延拓:如果γ由α到β,$D_\beta=D(\beta, \gamma)\subset\Omega$,

(1) $$\Gamma_z=\gamma+[\beta, z]\qquad(z\in D_\beta)$$

且

(2) $$g_\beta(z)=\int_{\Gamma_z}(\xi-w)^{-1}d\xi+f(\alpha)\qquad(z\in D_\beta)$$

则(g_β, D_β)是(f, D)沿γ的解析延拓.

注意在D_β内$g'_\beta(z)=(z-w)^{-1}$.

现在假设存在 $g\in H(\Omega)$，使得在 D 内 $g(z)=f(z)$. 则对所有 $z\in\Omega$, $g'(z)=(z-w)^{-1}$，如果 Γ 是 Ω 内的闭路径，便得出

$$\text{(3)}\qquad \operatorname{Ind}_\Gamma(w)=\frac{1}{2\pi i}\int_\Gamma g'(z)\,dz=0$$

我们最终得出（借助于定理 13.11），单值性定理在每一个不是单连通的平面区域内失效.

模函数的构造

16.17 模群 这是形式为

$$\text{(1)}\qquad \varphi(z)=\frac{az+b}{cz+d}$$

的所有线性分式变换的集合 G. 这里 a, b, c 及 d 是整数，且 $ad-bc=1$.

由于 a,b,c 及 d 是实数，每个 $\varphi\in G$ 映实轴到它本身上（除了 ∞）. $\varphi(i)$ 的虚部是 $(c^2+d^2)^{-1}>0$. 因此

$$\text{(2)}\qquad \varphi(\Pi^+)=\Pi^+ \qquad (\varphi\in G)$$

此处 Π^+ 是开上半平面. 若 φ 由(1)给出，则

$$\text{(3)}\qquad \varphi^{-1}(w)=\frac{dw-b}{-cw+a}$$

于是 $\varphi^{-1}\in G$. 同时，当 $\varphi\in G$ 和 $\psi\in G$ 时，$\varphi\circ\psi\in G$.

因此 G 是一个群，以复合作为群运算. 根据(2)，习惯上把 G 看作 Π^+ 上的变换群.

变换 $z\to z+1$ ($a=b=d=1$, $c=0$) 和 $z\to -1/z$ ($a=d=0$, $b=-1, c=1$) 属于 G. 事实上，它们生成 G（也就是，没有 G 的真子群包含这两个变换）. 这能用定理 16.19(c) 所采用的同样方法证明.

一个模函数是 Π^+ 上的全纯（或亚纯）函数 f，并且 f 在 G 下或至少在 G 的某个非平凡子群 Γ 下是不变的. 这意思是说，对每个

$\varphi\in\Gamma, f\circ\varphi=f.$

16.18 一个子群 我们将取由 σ 和 τ 生成的群为 Γ. 此处

$$\sigma(z)=\frac{z}{2z+1}, \qquad \tau(z)=z+2 \tag{1}$$

我们的一个目的是构造在 Γ 下不变的某个函数 λ，且由它引导出 Picard 定理的一个快速的证明. 确实，在证明中，重要的是 λ 的映射性质，而不是它的不变性，而且也能较快地给出 λ 的构造(适当地用 Riemann 定理及反射原理). 但是，用几何的语言来研究 Γ 在 Π^+ 上的作用是很有教益的，我们将沿着这条路子来处理.

设 Q 是满足下列四个条件的所有 z 的集合，这里 $z=x+iy$:

$$y>0,\ -1\leqslant x<1,\ |2z+1|\geqslant 1,\ |2z-1|>1 \tag{2}$$

Q 界于垂直线 $x=-1$ 及 $x=1$ 之间，并且以中心在 $-\frac{1}{2}$ 及 $\frac{1}{2}$，半径为 $\frac{1}{2}$ 的两个半圆周作为下界. Q 包含了位于 Π^+ 中左半的那些边界点. Q 不含实轴上的点.

我们断言 Q 是 Γ 的基本区域. 这意思是指下列定理的陈述 (a) 及 (b) 是真确的.

16.19 定理 设 Γ 和 Q 如上述.

(a) 若 φ_1 及 $\varphi_2\in\Gamma$，且 $\varphi_1\neq\varphi_2$，则 $\varphi_1(Q)\cap\varphi_2(Q)=\varnothing$.

(b) $\bigcup_{\varphi\in\Gamma}\varphi(Q)=\Pi^+$.

(c) Γ 包含形式为

$$\varphi(z)=\frac{az+b}{cz+d} \tag{1}$$

的所有变换 $\varphi\in G$，其中 a 和 d 是奇数，b 和 c 是偶数.

证明 设 Γ_1 为 (c) 中所述的所有 $\varphi\in G$ 的集. 容易验证 Γ_1 是 G 的子群. 由 $\sigma\in\Gamma_1$ 及 $\tau\in\Gamma_1$，可得出 $\Gamma\subset\Gamma_1$. 为要证明 $\Gamma=\Gamma_1$，

也就是要证明(c)，只需证明 (a′)和(b) 成立就够了. 这里 (a′) 是由 (a) 以 Γ_1 代替 Γ 而得到的命题. 因为，如果(a′)及(b)成立，很清楚，Γ 就不能是 Γ_1 的真子集.

我们将需要关系式

$$\mathrm{Im}\varphi(z)=\frac{\mathrm{Im}z}{|cz+d|^2} \tag{2}$$

此式对由(1)给出的每个 $\varphi\in G$ 都正确. (2)的证明是用直接计算的方法，且依赖于关系式 $ad-bc=1$.

我们现在证明(a′). 假设 φ_1 及 $\varphi_2\in\Gamma_1$, $\varphi_1\neq\varphi_2$, 且定义 $\varphi=\varphi_1^{-1}\circ\varphi_2$. 如果 $z\in\varphi_1(Q)\cap\varphi_2(Q)$，则 $\varphi_1^{-1}(z)\in Q\cap\varphi(Q)$. 因此只须证明，若 $\varphi\in\Gamma_1$，且 φ 不是恒等变换，则

$$Q\cap\varphi(Q)=\varnothing \tag{3}$$

(3)的证明分为三种情形.

若在(1)中 $c=0$，则 $ad=1$，由于 a 和 d 是整数，我们有 $a=d=\pm1$. 因此，对某个整数 $n\neq0$, $\varphi(z)=z+2n$，而 Q 的描述明显地使(3)成立.

若 $c=2d$，则 $c=\pm2$ 且 $d=\pm1$(由 $ad-bc=1$). 因此，$\varphi(z)=\sigma(z)+2m$，m 为一整数. 由 $\sigma(Q)\subset\bar{D}\left(\frac{1}{2};\frac{1}{2}\right)$，(3)成立.

若 $c\neq0$，且 $c\neq2d$. 我们断定，对所有 $z\in Q$，$|cz+d|>1$. 如若不然，圆盘 $\bar{D}(-d/c;1/|c|)$与 Q 相交. Q的描述表明，若$\alpha\neq-\frac{1}{2}$为一实数，且 $\bar{D}(\alpha;r)$与 Q 相交，则点 -1, 0, 1 中至少有一点含于 $D(\alpha;r)$内，因此对 $w=-1$ 或 0 或 1，$|cw+d|<1$，但对这些 w, $cw+d$ 是一奇数，它的绝对值不能小于 1. 于是 $|cz+d|>1$. 现在由(2)可得出，对每个 $z\in Q$, $\mathrm{Im}\varphi(z)<\mathrm{Im}z$. 如果对某个 $z\in Q$, $\varphi(z)\in Q$ 是真确的话，相同的论证就可应用于 φ^{-1}，并且就能证明

$$\mathrm{Im}z=\mathrm{Im}\varphi^{-1}(\varphi(z))\leqslant\mathrm{Im}\varphi(z) \tag{4}$$

这矛盾表明(3)式成立.

因此(a')得到了证明.

要证明(b),令Σ为集$\varphi(Q)$对$\varphi\in\Gamma$取的并.很清楚,$\Sigma\subset\Pi^+$.同时,对于$n=0,\pm1,\pm2,\cdots$,Σ也包含集$\tau^n(Q)$,此处$\tau^n(z)=z+2n$.由于σ映圆周$|2z+1|=1$到圆周$|2z-1|=1$上,我们看到Σ包含使所有不等式

$$|2z-(2m+1)|\geqslant 1 \qquad (m=0,\pm1,\pm2,\cdots) \tag{5}$$

都能满足的每一个$z\in\Pi^+$.

固定$w\in\Pi^+$.由$\mathrm{Im}w>0$,仅有有限对整数c及d,使得$|cw+d|$小于任意给定的界限,并且我们能选取$\varphi_0\in\Gamma$,使得$|cw+d|$取到最小.由(2),这表示

$$\mathrm{Im}\varphi(w)\leqslant\mathrm{Im}\varphi_0(w) \qquad (\varphi\in\Gamma) \tag{6}$$

令$z=\varphi_0(w)$,则(6)变为

$$\mathrm{Im}\varphi(z)\leqslant\mathrm{Im}z \qquad (\varphi\in\Gamma) \tag{7}$$

应用(7)式于$\varphi=\sigma\tau^{-n}$和$\varphi=\sigma^{-1}\tau^{-n}$.因为

$$(\sigma\tau^{-n})(z)=\frac{z-2n}{2z-4n+1},\quad (\sigma^{-1}\tau^{-n})(z)=\frac{z-2n}{-2z+4n+1} \tag{8}$$

由(2)和(7)得出

$$|2z-4n+1|\geqslant 1,\ |2z-4n-1\geqslant 1 \qquad (n=0,\pm1,\pm2,\cdots) \tag{9}$$

这样z满足(5),因此$z\in\Sigma$;又因为$w=\varphi_0^{-1}(z)$和$\varphi_0^{-1}\in\Gamma$,我们有$w\in\Sigma$.

这就完成了证明.

下列定理概括了在16.18节提到的模函数λ的某些性质.它将用于定理16.22.

16.20 定理 如果Γ和Q均如16.18节所述,则存在一个函数$\lambda\in H(\Pi^+)$,使得

(a) 对每个$\varphi\in\Gamma$,$\lambda\circ\varphi=\lambda$.

(*b*) λ 在 Q 上是一一的.

(*c*) λ 的值域(由(*a*)，它与 $\lambda(Q)$ 相同)是不同于 0 和 1 的所有复数组成的区域.

(*d*) λ 以实轴作为它的自然边界.

证明 令 Q_0 为 Q 的右边一半. 更明确地说，Q_0 由 $z\in\Pi^+$，并满足

(1) $$0<\mathrm{Re}z<1,\ |2z-1|>1$$

的所有 z 组成. 由定理 14.19(及评注 14.20)，存在 $\bar{Q}_0$ 上的连续函数 h，它在 $\bar{Q}_0$ 上是一一的，在 Q_0 内是全纯的，并使得 $h(Q_0)=\Pi^+$，$h(0)=0$，$h(1)=1$，且 $h(\infty)=\infty$. 反射原理(定理11. 17)表明，公式

(2) $$h(-x+iy)=\overline{h(x+iy)}$$

把 h 扩展为 Q 的闭包 $\bar{Q}$ 上的连续函数. 这函数是 Q 的内部到除掉非负实轴的复平面上的保形映射. 我们同时看出 h 在 Q 上是一一的，且 $h(Q)$ 是(*c*)中所描述的区域 Ω，并有

(3) $$h(-1+iy)=h(1+iy)=h(\tau(-1+iy))\qquad(0<y<\infty)$$

和

(4) $$h\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)=h\left(\frac{1}{2}+\frac{1}{2}e^{i(\pi-\theta)}\right)$$
$$=h\left(\sigma\left(-\frac{1}{2}+\frac{1}{2}e^{i\theta}\right)\right)\qquad(0<\theta<\pi)$$

由于在 Q 的边界上 h 是实的，由(2)以及 σ 和 τ 的定义便得出(3)和(4).

我们现在定义函数 λ:

(5) $$\lambda(z)=h(\varphi^{-1}(z))\qquad(z\in\varphi(Q),\varphi\in\Gamma)$$

由定理 16. 19，每个 $z\in\Pi^+$，有一个且仅有一个 $\varphi\in\Gamma$，使 z 属于 $\varphi(Q)$. 这样，对 $z\in\Pi^+$，(5)定义了 $\lambda(z)$，并且我们立刻可以看出

λ 具有性质(a)到(c)，同时 λ 在每一个集 $\varphi(Q)$ 的内部是全纯的.

由(3)及(4)得出 λ 在

$$Q \cup \tau^{-1}(Q) \cup \sigma^{-1}(Q)$$

上连续，从而在包含 Q 的一个开集 V 上连续. 定理 16.8 现在表明 λ 在 V 内是全纯的. 由于 Π^+ 被集 $\varphi(V)$, $\varphi \in \Gamma$ 的并所覆盖，且由 $\lambda \circ \varphi = \lambda$，我们最终得出 $\lambda \in H(\Pi^+)$.

最后，所有数 $\varphi(0) = b/d$ 的集在实轴上是稠密的. 若 λ 能解析延拓到一个真正包含 Π^+ 的区域，λ 的零点将在这区域内有极限点. 但由于 λ 不是常数，这样是不可能的.

Picard 定理

16.21 所谓的"小 Picard 定理"断言每一个非常数整函数可以取到每一个值，除了有一个值可能例外. 这就是下面要证明的定理. 有一个强些的说法：每一个不是多项式的整函数可以取得每一个值无限多次，但仍然除了有一个值可能例外. 这一个例外的值确能出现，可由 $f(z) = e^z$ 说明，它删除了零值. 后面的定理在一个局部的情况下确实是真确的：若 f 在点 z_0 有孤立奇点，且若 f 在 z_0 的某个邻域内删除两个值，则 z_0 是一个可去奇点或极点. 这个所谓的"大 Picard 定理"是 Weierstrass 定理(定理10.21)值得注意的加强. 后者仅仅断言，当 f 在 z_0 有本性奇异点时，z_0 的每一个邻域的象在平面内稠密. 我们将不在此处证明它.

16.22 定理 如果 f 是整函数，且存在两个不同的复数 α 和 β，它们不在 f 的值域内，则 f 为常数.

证明 不失一般性，我们假设 $\alpha = 0$ 及 $\beta = 1$；不然的话，用 $(f-\alpha)/(\beta-\alpha)$ 代替 f. 则 f 映平面到定理16.20所述的区域 Ω 内.

对每个圆盘 $D_1 \subset \Omega$，对应有一区域 $V_1 \subset \Pi^+$(其实，有无穷多个这样的 V_1，对每个 $\varphi \in \Gamma$ 都有一个)，使得 λ 在 V_1 上是一一的，

且 $\lambda(V_1)=D_1$. 每个这样的 V_1 最多与区域 $\varphi(Q)$的两个 相交. 对应于 V_1 的每一种选取，存在函数 $\psi_1\in H(D_1)$，对所有 $z\in V_1$，有 $\psi_1(\lambda(z))=z$.

若 D_2 是 Ω 内另一圆盘，且若 $D_1\cap D_2\neq\varnothing$，则我们能选取一个对应的 V_2，使得 $V_1\cap V_2\neq\varnothing$. 因此函数 元 素 (ψ_1, D_1)和(ψ_2, D_2)就将是彼此的直接解析延拓. 注意 $\psi_i(D_i)\subset\Pi^+$.

由 于 f 的 值 域 位 于 Ω 内，故存在中心在零的圆盘 A_0 使得 $f(A_0)$含于 Ω 内一圆盘 D_0 中. 选取 $\psi_0\in H(D_0)$，如上所述，对 $z\in A_0$，令 $g(z)=\psi_0(f(z))$，并令 γ 为平面内起点在零的任意曲线. $f\circ\gamma$ 的值域是 Ω 的紧子集. 因此，γ 能被一个圆盘链 $A_0,\cdots,A_n$所覆盖，使得每一个 $f(A_i)$包含 于 Ω 内的一个圆盘 D_i 中，并且我们能选取 $\psi_i\in H(D_i)$，使得对 $i=1,\cdots,n$，(ψ_i, D_i)是(ψ_{i-1}, D_{i-1})的直接解析延拓. 这样就给出了函数元素 (g, A_0) 沿链$\{A_0,\cdots,A_n\}$的一个解析延拓. 注意 $\psi_i\circ f$ 具有正的虚部.

由于(g, A_0)能沿平面内每一条曲线解析延拓，而平面是单连通的，所以由单值性定理推出 g 能扩展成一个整函数. 同时，g的值域是在 Π^+内. 因此，$(g-i)/(g+i)$是有界的，又由 Liouville 定理，因而是常数. 这也就表明了 g 是常数，并且，由于 ψ_0 在 $f(A_0)$上是一一的，而 A_0 是非空开集，所以我们最终得出 f 是常数.

习　题

1　假设 $f(z)=\Sigma a_n z^n$，$a_n\geqslant 0$，且这级数的收敛半径为1. 证明 f 在 $z=1$有奇点. 提示：把 f 展为$\left(z-\frac{1}{2}\right)$的幂级数. 若 1 为$f$的正则点，则新级数在某个 $x>1$处收敛. 对原来的级数这将意味着什么？

2　假设(f, D)及(g, D)都是函数元素，P 是两个变量的多项式，且在 D 内，$P(f, g)=0$. 假设 f 和 g 能沿一曲线 γ 解析延拓到f_1和g_1，证明$P(f_1, g_1)=0$. 试把这个结果推广到两个函数以上. 有没有关于一个比多项式类大一些的函数类 P 的这样的定理？

3 假设Ω为一单连通区域，且u为Ω内的实调和函数．证明存在函数$f\in H(\Omega)$，使得$u=\mathrm{Re}f$．证明对每一个非单连通的区域，这个结论是不成立的．

4 假设X为平面内闭单位正方形，f是X上的连续复函数，且f在X内无零点．证明存在X上的连续函数g，使得$f=e^g$．对哪一类空间X（不同于上面的正方形），这个结果也真确？

5 证明变换$z\to z+1$和$z\to -\frac{1}{z}$生成整个模群G．设R由满足$|x|<\frac{1}{2}$，$y>0$及$|z|>1$的所有$z=x+iy$再加上它们的有$x\leqslant 0$的那些极限点组成．证明R是G的基本区域．

6 证明G也由变换φ和ψ生成，此处

$$\varphi(z)=-\frac{1}{z},\quad \psi(z)=\frac{z-1}{z}$$

证明φ具有周期2，ψ具有周期3．

7 找出线性分式变换的复合与矩阵乘法之间的关系．试用这种关系作出定理16.19(c)的一个代数证明或习题5的第一部分的证明．

8 设E为实轴上的有正Lebesgue测度的紧集，Ω为E相对于平面的余集，并定义

$$f(z)=\int_E \frac{dt}{t-z}\qquad (z\in\Omega)$$

回答下列问题：

(a) f是否是常数？

(b) f能扩展为整函数吗？

(c) 当$z\to\infty$时，$\lim zf(z)$存在吗？若存在，是什么？

(d) f在Ω内有没有全纯的平方根？

(e) 在Ω内，f的实部是否有界？

(f) 在Ω内，f的虚部是否有界？

[若(e)或(f)答复是有界，则给出一个界．]

(g) 如果γ为以E为它的内部的正向圆周，则$\int_\gamma f(z)dz$是什么？

(h) 在Ω内，是否存在非常数的有界全纯函数φ？

9 针对

$$E=[-1,1]$$

的特殊场合验证你在习题 8 中的答案.

10 若在平面内的紧集 E 的余集上不存在非常数的有界全纯函数，则称集 E 为可去的.

(a) 证明每一个可数紧集是可去的.

(b) 若 E 为实轴的紧子集，且 $m(E)=0$. 证明 E 是可去的. 提示: E 能被总长度任意小的曲线所环绕. 如在第十章习题 25 那样应用 Cauchy 公式.

(c) 假设 E 是可去的，Ω 是一区域，$E\subset\Omega$，$f\in H(\Omega-E)$，且 f 有界. 证明 f 能扩展为 Ω 内的全纯函数.

(d) 对不一定在实轴上的集 E，叙述并证明与 (b) 部分相类似的结论.

(e) 证明平面上的(多于一点)紧连通子集没有一个是可去的.

11 对每一个正数 α，令 Γ_α 为具有参数区间 $(-\infty,\infty)$ 的路径，由

$$\Gamma_\alpha(t)=\begin{cases}-t-\pi i & (-\infty<t\leqslant-\alpha)\\ \alpha+\dfrac{\pi i t}{\alpha} & (-\alpha\leqslant t\leqslant\alpha)\\ t+\pi i & (\alpha\leqslant t<\infty)\end{cases}$$

定义. 设 Ω_α 为 Γ_α^* 的余集的包含原点的分支，并定义

$$f_\alpha(z)=\frac{1}{2\pi i}\int_{\Gamma_\alpha}\frac{\exp(e^w)}{w-z}dw \qquad (z\in\Omega_\alpha)$$

证明若 $\alpha<\beta$，则 f_β 是 f_α 的解析延拓. 证明因此存在一个整函数，它在 Ω_α 的限制是 f_α. 证明对每个 $e^{i\theta}\neq1$，

$$\lim_{r\to\infty}f(re^{i\theta})=0$$

(如通常一样，这里 r 为正数，θ 为实数.) 证明 f 不是常数. [提示: 观察 $f(r)$]. 如果

$$g=f\exp(-f)$$

证明对每个 $e^{i\theta}$，

$$\lim_{r\to\infty}g(re^{i\theta})=0$$

证明存在一个整函数 h，使得

$$\lim_{n\to\infty}h(nz)=\begin{cases}1 & \text{当 } z=0\\ 0 & \text{当 } z\neq0\end{cases}$$

12 假设

$$f(z)=\sum_{k=1}^{\infty}\left(\frac{z-z^2}{2}\right)^{3k}=\sum_{n=1}^{\infty}a_n z^n$$

找出一个区域，在这区域内这两个级数收敛，指出这个例子说明了定理16.5，找出最接近于原点的 f 的奇点.

13 设 $\Omega=\left\{z:\frac{1}{2}<|z|<2\right\}$. 对 $n=1,2,3,\cdots$，设 X_n 是所有的 $g\in H(\Omega)$ 的 n 阶导函数 $f\in H(\Omega)$ 的集.［换句话说，X_n 是以 $H(\Omega)$ 为定义域的微分算子 D^n 的值域.］

(*a*) 证明当且仅当 $\int_\gamma f(z)dz=0$，这里 γ 是正向单位圆周时 $f\in X_1$.

(*b*) 证明对每一个 n，当且仅当 f 能扩展为 $D(0;2)$ 内的全纯函数时 $f\in X_n$.

第十七章　H^p-空间

这一章致力于研究由一定的增长条件所确定的 $H(U)$ 的某些子空间；事实上，它们全都包含在第十五章所定义的类 N 内．这些所谓 H^p-空间（由 G. H. Hardy 命名）具有大量的与因式分解、边界值以及用 U 的边界上的测度表达的 Cauchy 型表示有关的有趣的性质．我们只打算给出一些最重要的结果，比如，对所有的 $n<0$ Fourier 系数 $\hat{\mu}(n)$ 为 0 的测度 μ 的 F. Riesz 和 M. Riesz 定理，H^2 的不变子空间的 Beurling 分类，以及关于共轭函数的 M. Riesz 定理．

解决这个问题的一个方便的方法是运用次调和函数，我们从简要地列述它们的性质开始．

次调和函数

17.1 定义　定义在平面内开集 Ω 上的一个函数 u 称为次调和函数，如果它具有下列四个性质：

(*a*)　$-\infty \leqslant u(z) < \infty$　对所有的 $z \in \Omega$ 成立．

(*b*)　u 在 Ω 内是上半连续的．

(*c*)　当 $\bar{D}(a;r) \subset \Omega$ 时，有

$$u(a) \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} u(a+re^{i\theta})\,d\theta$$

(*d*)　(*c*)中的积分没有一个是 $-\infty$．

注意(*c*)中的积分永远存在，并且不是 $+\infty$，因为(*a*)和(*b*)蕴涵着 u 在每一个紧集 $K \subset \Omega$ 上是上有界的．[证明：若 K_n 是所有

的使得 $u(z)\geqslant n$ 的 $z\in K$ 的集，则 $K\supset K_1\supset K_2\supset\cdots$，所以或者是对某一个 n，$K_n=\varnothing$，或者是 $\bigcap K_n=\varnothing$. 在后一种情况下，必有某一个 $z\in K$，$u(z)=\infty$.]因此(d)保证(c)中的被积函数属于 $L^1(T)$.

每一个实的调和函数显然是次调和函数.

17.2 定理 如果 u 是 Ω 内的次调和函数，φ 是 R^1 上的单调增加的凸函数，则 $\varphi\circ u$ 是次调和函数.

[为使 $\varphi\circ u$ 在 Ω 的所有点上有定义，我们令 $\varphi(-\infty)=\lim\varphi(x)$，即当 $x\to-\infty$ 时的极限.]

证明 首先，因为 φ 是递增且连续的，所以 $\varphi\circ u$ 是上半连续的. 其次，若 $\bar{D}(a;r)\subset\Omega$，则有

$$\varphi(u(a))\leqslant\varphi\left(\frac{1}{2\pi}\int_{-\pi}^{\pi}u(a+re^{i\theta})d\theta\right)\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}\varphi(u(a+re^{i\theta}))d\theta$$

因为 φ 是递增的而 u 是次调和的，这些不等式中的前一个成立；从 φ 的凸性根据定理 3.3 便得出后一个不等式.

17.3 定理 如果 Ω 是一个区域，$f\in H(\Omega)$，并且 f 不恒等于 0，则 $\log|f|$ 是 Ω 内的次调和函数，从而 $\log^+|f|$ 和 $|f|^p$ $(0<p<\infty)$ 同样也是次调和的.

证明 如果 $f(z)=0$ 时，我们理解 $\log|f(z)|=-\infty$. 则 $\log|f|$ 是在 Ω 内上半连续的，并且定理 15.19 蕴涵着 $\log|f|$ 是次调和的.

我们如果在定理 17.2 中，以 $\log|f|$ 代替 u，而

$$\varphi(t)=\max(0,t)\text{和}\varphi(t)=e^{pt}$$

应用这个定理，则得出另外那两个结论.

17.4 定理 假定 u 是 Ω 内的连续次调和函数，K 是 Ω 的紧子集，h 是 K 上的连续实函数，h 在 K 的内部 V 内是调和的，并且在 K 的所有边界点上 $u(z)\leqslant h(z)$ 成立. 则 $u(z)\leqslant h(z)$ 对所有的 $z\in K$ 成立.

这个定理解释了"次调和"这一名词的含义. 这里 u 的连续性不是必须的，但是我们将不需要这个一般情形，并且留它作为一个练习.

证明 令 $u_1=u-h$，为了得出矛盾，假设对某一个 $z\in V$，$u_1(z)>0$ 成立. 因为 u_1 在 K 上是连续的，u_1 在 K 上达到它的最大值 m；因为在 K 的边界上 $u_1\leqslant 0$ 成立，集 $E=\{z\in K; u_1(z)=m\}$ 是一个 V 的非空紧子集. 设 z_0 是 E 的一个边界点. 则对于某一个 $r>0$ 有 $\bar{D}(z_0;r)\subset V$，但是 $\bar{D}(z_0;r)$ 的边界的某一段子弧位于 E 的余集内. 因此

$$u_1(z_0)=m>\frac{1}{2\pi}\int_{-\pi}^{\pi}u_1(z_0+re^{i\theta})d\theta$$

这就意味着 u_1 在 V 内不是次调和的. 但是若 u 是次调和的，根据调和函数的平均值性质，$u-h$ 也是次调和的，得出矛盾.

17.5 定理 假定 u 是 U 内的连续次调和函数，并且

(1) $$m(r)=\frac{1}{2\pi}\int_{-\pi}^{\pi}u(re^{i\theta})d\theta \qquad (0\leqslant r<1)$$

如果 $r_1\leqslant r_2$，则 $m(r_1)\leqslant m(r_2)$.

证明 设 h 是 $\bar{D}(0;r_2)$ 上的连续函数，它在 $\bar{D}(0;r_2)$ 的边界上与 u 一致，而且在 $D(0;r_2)$ 内是调和的. 根据定理 17.4，在 $D(0;r_2)$ 内 $u\leqslant h$ 成立. 因此

$$m(r_1)\leqslant\frac{1}{2\pi}\int_{-\pi}^{\pi}h(r_1e^{i\theta})d\theta=h(0)=\frac{1}{2\pi}\int_{-\pi}^{\pi}h(r_2e^{i\theta})d\theta=m(r_2)$$

H^p 空间和 N 空间

17.6 定理 如果 $f\in H(U)$，且

$$M_0(f;r)=\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|d\theta\right\}$$

$$M_p(f;r)=\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^p\,d\theta\right\}^{1/p}\qquad(0<p<\infty)$$

$$M_\infty(f;r)=\sup_\theta|f(re^{i\theta})|$$

则 M_0、M_p、M_∞ 都是$[0,1)$上的 r 的单调递增函数.

证明 对于 M_0 和 M_p，这是定理 17.3 和17.5 的直接推论. 对于 M_∞，它可以从最大模定理得出.

这就提示了下列的定义

17.7 定义 对任一 $f\in H(U)$和对 $0\leqslant p\leqslant\infty$，我们令

$$\|f\|_p=\lim_{r\to1}M_p(f;r)$$

这里 $M_p(f;r)$与定理 17.6 中的一样.

对于 $0<p\leqslant\infty$，类 H^p 定义为由所有使得$\|f\|_p<\infty$ 的那些 $f\in H(U)$组成. 注意这和我们在 $p=\infty$ 情况下原先引出的术语是一致的.

类N是由所有$\|f\|_0<\infty$的那些 $f\in H(U)$组成.

若$0<s<p<\infty$，显然有 $H^\infty\subset H^p\subset H^s\subset N$.

17.8 评注 对于$1\leqslant p\leqslant\infty$，$\|f\|_p$满足三角不等式，所以$H^p$是一个赋范线性空间. 为了看出这点，对 $M_p(f;r)$应用 Minkowski 不等式:

(1) $$M_p(f+g;r)\leqslant M_p(f;r)+M_p(g;r)\qquad(0\leqslant r<1)$$

当 $r\to1$时，我们获得

(2) $$\|f+g\|_p\leqslant\|f\|_p+\|g\|_p$$

如果$1\leqslant p\leqslant\infty$，实际上，$H^p$ 是一个 Banach 空间: 为了证明完备性，假定$\{f_n\}$是 H^p 内的 Cauchy 序列，$|z|\leqslant r<R<1$，对 f_n-f_m沿着半径为R的圆周积分，应用 Cauchy 公式，这就导出不等式

$$\begin{aligned}(R-r)|f_n(z)-f_m(z)|&\leqslant M_1(f_n-f_m;R)\\&\leqslant M_p(f_n-f_m;R)\leqslant\|f_n-f_m\|_p\end{aligned}$$

于是我们得出$\{f_n\}$在U的紧子集上一致收敛于一个函数$f\in H(U)$. 给定$\varepsilon>0$,则存在一个m,使得对所有的$n>m$,$\|f_n-f_m\|_p<\varepsilon$,因此对每一个$r<1$,

$$M_p(f-f_m;r)=\lim_{n\to\infty}M_p(f_n-f_m;r)\leqslant\varepsilon$$

这就给出当$m\to\infty$时,$\|f-f_m\|\to 0$.

对于$p<1$,H^p仍然是一个向量空间,但$\|f\|_p$不再满足三角不等式.

我们曾在定理15.23看到过:任一$f\in N$的零点满足Blaschke条件$\Sigma(1-|\alpha_n|)<\infty$. 因此,对每个$H^p$,结论仍然真确. 有趣的是:任一$f\in H^p$的零点在不增加范数的情况下能够被除去.

17.9 定理 假定$f\in N$,$f\not\equiv 0$. B是由f的零点所构成的Blaschke乘积. 令$g=f/B$. 则$g\in N$,且$\|g\|_0=\|f\|_0$.

进而言之,若$f\in H^p$,则$g\in H^p$,且$\|g\|_p=\|f\|_p(0<p\leqslant\infty)$.

证明 首先注意到

$$|g(z)|\geqslant|f(z)| \qquad (z\in U) \tag{1}$$

事实上,对每一个$z\in U$,严格的不等号能成立. 除非f在U内不存在零点,在这种情况下,$B=1$和$g=f$.

若s和t是非负实数,不等式

$$\log^+(st)\leqslant\log^+s+\log^+t \tag{2}$$

成立. 因为如果$st<1$,(2)式的左边为0;如果$st>1$,(2)式的左边为$\log s+\log t$. 因为$|g|=|f|/|B|$,(2)给出

$$\log^+|g|\leqslant\log^+|f|+\log\frac{1}{|B|} \tag{3}$$

根据定理15.24,(3)蕴涵着$\|g\|_0\leqslant\|f\|_0$,并且因为(1)成立,我们实际上有$\|g\|_0=\|f\|_0$.

现在假定对某个$p>0$,$f\in H^p$. 设B_n是由f的前n个零点所构成的有限Blaschke乘积(把重根计算在内,我们依某一顺序排

列这些零点). 对每一个 n, 令 $g_n = f/B_n$, 当 $r \to 1$ 时, $|B_n(re^{i\theta})| \to 1$ 一致地成立. 因此 $\|g_n\|_p \to \|f\|_p$. 当 $n \to \infty$ 时, $|g_n|$ 增加到 $|g|$. 所以, 根据单调收敛定理, 对所有的 $r<1$, 有

$$M_p(g;r) = \lim_{n\to\infty} M_p(g_n;r) \qquad (0<r<1) \tag{4}$$

(4)的右边至多是 $\|f\|_p$, 若我们令 $r \to 1$, 便得到 $\|g\|_p \leqslant \|f\|_p$. 象前面一样, 现在从(1)得出等式.

H^2 空间

H^2 的特殊重要性是由于它是一个 Hilbert 空间, 而且很容易地把它和 $L^2(T)$ 的某一个子空间等同起来, 这里 T 是单位圆周. 我们回顾一下, 任意一个 $g \in L^2(T)$ 的范数是

$$\|g\|_2 = \left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} |g(e^{i\theta})|^2 d\theta\right\}^{1/2}$$

并且每一个 $g \in L^2(T)$ 有 Fourier 系数

$$\hat{g}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(e^{i\theta}) e^{-in\theta} d\theta \qquad (n = 0, \pm 1, \pm 2, \cdots)$$

H^2 的基本性质概括在下列定理中.

17.10 定理

(*a*) 形如

$$f(z) = \sum_0^\infty a_n z^n \qquad (z \in U) \tag{1}$$

的一个函数 $f \in H(U)$ 当且仅当 $\Sigma |a_n|^2 < \infty$ 时是 H^2 中的函数. 在这种情况下,

$$\|f\|_2 = \left\{\sum_0^\infty |a_n|^2\right\}^{1/2} \tag{2}$$

(*b*) 如果 $f \in H^2$, 则在 T 的几乎所有各点, f 具有径向极限 $f^*(e^{i\theta})$, $f^* \in L^2(T)$; 若 $n \geqslant 0$, f^* 的第 n 个 Fourier 系数为 a_n, 若 $n<0$, f^* 的

第 n 个 Fourier 系数为 0; L^2-逼近

$$\lim_{r\to 1}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f^*(e^{i\theta})-f(re^{i\theta})|^2d\theta=0 \tag{3}$$

成立; 并且 f 不但是 f^* 的 Cauchy 积分, 而且也是 f^* 的 Poisson 积分: 若 $z=re^{i\theta}$, 则

$$f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)f^*(e^{it})dt \tag{4}$$

和

$$f(z)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f^*(\zeta)}{\zeta-z}d\zeta \tag{5}$$

这里 Γ 是正向单位圆周.

(c)映射 $f\to f^*$ 是一个 H^2 到 $L^2(T)$ 的一个子空间上的等距映射, 这个子空间是由对所有的 $n<0$, $\hat{g}(n)=0$ 的那些 $g\in L^2(T)$ 所组成.

证明 直接计算(Parseval 定理)指出,

$$M_2(f;r)=\left\{\sum_0^{\infty}|a_n|^2r^{2n}\right\}^{1/2}\qquad(0\leqslant r<1) \tag{6}$$

这便证明了(a).

现在假定 $f\in H^2$, 对 $0<s<1$ 用下式在 T 上定义函数 f_s,

$$f_s(e^{i\theta})=f(se^{i\theta})=\sum_0^{\infty}a_ns^ne^{in\theta} \tag{7}$$

因为 $\Sigma|a_n|^2<\infty$, Riesz-Fischer 定理保证函数 $g\in L^2(T)$ 的存在性, 使得 $n\geqslant 0$ 时, $\hat{g}(n)=a_n$, $n<0$ 时, $\hat{g}(n)=0$. $g-f_s$ 的 Fourier 系数当 $n\geqslant 0$ 时是 $(1-s^n)a_n$. 所以再一次应用 Parseval 定理指出了

$$\|g-f_s\|_2^2=\sum_1^{\infty}(1-s_n)^2|a_n|^2 \tag{8}$$

当 $s\to 1$ 时，(8)的右边趋于 0，我们得到了

$$\lim_{s\to 1}\|g-f_s\|_2=0 \tag{9}$$

对任一固定的 $s\in(0,1)$，$f(sz)$ 在 $D\left(0;\frac{1}{s}\right)$ 内是全纯函数. 因此，如果 $z\in U$ 和 $z=re^{i\theta}$，我们有

$$f(sz)=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_r(\theta-t)f_s(e^{it})dt \tag{10}$$

和

$$f(sz)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f_s(\zeta)}{\zeta-z}d\zeta \tag{11}$$

考虑这些积分和用 g 代替 f_s 时所对应的积分之间的差. Schwarz 不等式和(9)结合在一起，便给出了用 g 代替 f_1^* 的(4)式和(5)式. 所以 f 是 g 的 Poisson 积分；并且因为 $g\in L^1(T)$，定理 11.12的推论指出 f 的径向极限存在，并且在 T 上 a.e. 趋于 g. 这就证明了(*b*).

从(3)可得出 $\|f^*\|_2=\|f\|_2$. (*b*)的证明指出，对于 $n<0$，f^* 的所有的 Fourier 系数均为 0. 为了完成(*c*)的证明，假定 $g\in L^2(T)$ 和对所有的 $n<0$，$\hat{g}(n)=0$. 并且令

$$f(z)=\sum_{0}^{\infty}\hat{g}(n)z^n \tag{12}$$

则根据(*a*)，$f\in H^2$，(*b*)的证明指出 $f^*=g$.

17.11 评注 假定对某个 $p>0$，$f\in H^p$，B 是由 f 的零点所构成的 Blaschke 乘积，$g=f/B$. 定理 17.9 指出 $g\in H^p$，甚至指出 $\|g\|_p=\|f\|_p$. 因为 g 在 U 内不存在零点，并且 U 是单连通的，因而存在 $\varphi\in H(U)$，使得 $\exp(\varphi)=g$(定理 13.11). 令 $h=\exp(p\varphi/2)$. 则 $h\in H(U)$，并且 $|h|^2=|g|^p$，因此 $h\in H^2$. 事实上 $\|h\|_2^2=\|g\|_p^p$.

于是 f 具有下列形式的因式分解

(1) $$f=B\cdot h^{2/p}$$

这里 $h\in H^2$ 并且 h 在 U 内不存在零点. 这使得在许多情况下，把 H^2 空间的结果应用于任一 H^p 空间的函数成为可能. 以后的证明用到这个技巧.

17.12 定理 如果 $f\in H^1$，则

(1) $$f^*(e^{i\theta})=\lim_{r\to 1}f(re^{i\theta})$$

在 T 的几乎所有点上存在，并且

(2) $$\lim_{r\to 1}\frac{1}{2\pi}\int_{-\pi}^{\pi}|f^*(e^{i\theta})-f(re^{i\theta})|d\theta=0$$

证明 从定理 11.19 的系得出极限(1)a.e. 存在. 若 B 是由 f 的零点所构成的 Blaschke 乘积，上面的评注指出：存在一个 $h\in H^2$，使得 $h^2=f/B$ 和 $\|h\|_2^2=\|f\|_1$. 令 $g=Bh$. 则 $g\in H^2$，$\|g\|_2=\|h\|_2$，并且 $f=gh$. 我们已经分解 f 为两个属于 H^2 的函数的乘积.

在 T 上用 $f_r(e^{i\theta})=f(re^{i\theta})$ 定义 f_r，并用同样的方式定义 g_r 和 h_r. 因为 $f^*=g^*h^*$a.e.，我们有

(3) $$f^*-f_r=g^*(h^*-h_r)+h_r(g^*-g_r)$$

根据定理 17.10，当 $r\to 1$ 时，$\|h^*-h_r\|_2\to 0$ 和 $\|g^*-g_r\|_2\to 0$.

同时

(4) $$\|g^*\|_2^2=\|g\|_2^2=\|f\|_1$$

以及

(5) $$\|h_r\|_2^2\leqslant\|h\|_2^2=\|f\|_1$$

如果我们把 Schwarz 不等式应用到(3)式右边两项乘积的每一个乘积上，我们便得出当 $r\to 1$ 时 $\|f^*-f_r\|_1\to 0$.

系 如果 $f\in H^1$，则 f 是 f^* 的 Poisson 积分和 Cauchy 积分.

证明 如果 $R<1$ 和 $g(z)=f(Rz)$，则 $g\in H(D)$，这里 $D=D\left(0;\frac{1}{R}\right)$. 因此，对 $z\in U$

$$f(Rz)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(R\zeta)}{\zeta-z}d\zeta=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{f(Re^{it})}{e^{it}-z}e^{it}dt$$

现在固定 $z\in U$，让 $R\to 1$，并运用(2)式，这就给出了 Cauchy 表示式. 同样的方式可获得 Poisson 表示式.

F. Riesz 和 M. Riesz 定理

17.13 定理 如果 μ 是单位圆周 T 上的复 Borel 测度，并且

$$\int_{T}e^{-int}d\mu(t)=0 \qquad (n=-1,-2,-3,\cdots) \tag{1}$$

则 μ 关于 Lebesgue 测度是绝对连续的.

证明 对于 $z\in U$，定义

$$f(z)=\int_{T}\frac{d\mu(t)}{1-ze^{-it}} \tag{2}$$

若 $z=re^{i\theta}$，则

$$(1-ze^{-it})^{-1}=\sum_{n=0}^{\infty}r^{n}e^{in(\theta-t)} \tag{3}$$

并且因为

$$P_{r}(\theta-t)=\sum_{-\infty}^{\infty}r^{|n|}e^{in(\theta-t)} \tag{4}$$

(1)说明：f 也被 Poisson 积分(5)所给出

$$f(z)=\int_{T}P_{r}(\theta-t)d\mu(t) \tag{5}$$

因此，

$$M_{1}(f;r)\leqslant\int_{T}d|\mu|(t)\frac{1}{2\pi}\int_{-\pi}^{\pi}P_{r}(\theta-t)d\theta=\|\mu\| \tag{6}$$

所以 $f\in H^{1}$. 然而从定理 17.12 得出

$$f(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}P_{r}(\theta-t)f^{*}(e^{it})dt \tag{7}$$

这里 $f^{*}\in L^{1}(T)$. 根据定理 11.19，比较(5)和(7)产生

(8) $$d\mu(t)=\frac{1}{2\pi}f^*(e^{it})dt$$

这便给出了所要求的结论.

这个定理的值得注意的特征是：从一个表面上无关的条件(即，它的 Fourier 系数的一半为 0)，导出了测度的绝对连续. 近年来，这个定理已经推广到各种其它场合.

因式分解定理

从定理 17.9 我们已经知道：每一个 $f\in H^p$ (除去 $f=0$)，能分解为 Blaschke 乘积和一个在 U 内没有零点的函数 $g\in H^p$. 同时也存在一个更为精巧的 g 的因式分解. 粗略地说它与 g 沿着某些半径趋于 0 的速度有关.

17.14 定义 一个内函数是指一个函数 $M\in H^\infty$，关于这个 M，$|M^*|=1$a. e. 于 T 上成立(象通常一样，M^* 表示 M 的径向极限).

如果 φ 是 T 上的正的可测函数，使得 $\log\varphi\in L^1(T)$，并且对 $z\in U$

(1) $$Q(z)=c\cdot\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log\varphi(e^{it})dt\right\}$$

则 Q 称为一个外函数，这里 c 是一个常数，$|c|=1$.

定理 15.24 指出每一个 Blaschke 乘积是一个内函数，但也存在另外一些内函数. 它们可以描述如下.

17.15 定理 假定 c 是一个常数，$|c|=1$，B 是一个 Blaschke 乘积，μ 是 T 上正的有限 Borel 测度，它关于 Lebesgue 测度是奇异的，且

(1) $$M(z)=cB(z)\exp\left\{-\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\mu(t)\right\}\qquad(z\in U)$$

则 M 是一个内函数，并且每一个内函数都是这种形式的函数.

证明 如果(1)成立且 $g=M/B$，则 $\log|g|$ 是 $-d\mu$ 的 Poisson

积分，因此 $\log|g|\leqslant 0$，所以 $g\in H^\infty$，并且 $M\in H^\infty$ 同样是正确的. 同时，因为 μ 是奇异的，$D\mu=0$ a. e. 成立(定理8. 6)，因此$\log|g|$的径向极限均是 0 a. e. (定理11. 12). 因为$|B^*|=1$ a. e.，我们看出 M是一个内函数.

反之，设 B 是由给定的内函数M的零点所构成的 Blaschke 乘积，并且令 $g=M/B$. 则 $\log|g|$在U内是调和的. 定理 15. 24 和定理 17. 9 指出：在U内$|g|\leqslant 1$，而在T上$|g^*|=1$ a. e. 于是$\log|g|<0$. 我们从定理 11. 19 得出，对T上的某个正测度 μ，$\log|g|$是$-d\mu$的 Poisson 积分. 因为，在T上 $\log|g^*|=0$ a. e.，我们有 $D\mu=0$ a. e. 于T，所以 μ 是奇异的. 最后 $\log|g|$是函数

$$h(z)=-\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}d\mu(t)$$

的实部. 这就蕴涵着对于某个常数 c，$|c|=1$，$g=c\cdot\exp(h)$成立. 于是M是形如(1)式的函数.

这就完成了证明.

不是 Blaschke 乘积的内函数的最简单的一个例子如下：取 $c=1$ 和 $B=1$，设 μ 是集中于 $t=0$ 的单位质量. 则

$$M(z)=\exp\left\{\frac{z+1}{z-1}\right\}$$

这个函数沿着端点在 $z=1$ 的半径很快地趋于 0.

17.16 定理 假定Q是一个象在定义 17. 14 里那样的与φ有关的外函数. 则

(a) $\log|Q|$是$\log\varphi$的 Poisson 积分.

(b) $\lim\limits_{r\to 1}|Q(re^{it})|=\varphi(e^{it})$ a. e. 于T.

(c) $Q\in H^p$ 当且仅当 $\varphi\in L^p(T)$，这时$\|Q\|_p=\|\varphi\|_p$.

证明 根据视察，(a)是明显的. 并且(a)蕴涵着在T上$\log|Q|$的径向极限 a. e. 等于$\log\varphi$，这就证明了(b). 若 $Q\in H^p$，Fatou 引

理蕴涵着$\|Q^*\|_p \leqslant \|Q\|_p$，所以根据(b)，$\|\varphi\|_p \leqslant \|Q\|_p$. 反之，若$\varphi \in L^p(T)$，则根据几何平均值与算术平均值之间的不等式(定理3.3)，有

$$|Q(re^{i\theta})|^p = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\log\varphi^p(e^{it})dt\right\}$$

$$\leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t)\varphi^p(e^{it})dt$$

如果将后一个不等式对θ积分，我们发现当$p<\infty$时$\|Q\|_p \leqslant \|\varphi\|_p$. 而当$p=\infty$时，结论是明显的.

17.17 定理 假定$0<p\leqslant\infty$，$f\in H^p$，并且f不恒等于0. 则$\log|f^*|\in L^1(T)$，外函数

$$(1)\qquad Q_f(z)=\exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}\log|f^*(e^{it})|dt\right\}$$

在H^p内，并且存在一个内函数M_f，使得

$$(2)\qquad f = M_f Q_f$$

进而有

$$(3)\qquad \log|f(0)| \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f^*(e^{it})|dt$$

当且仅当M_f为常数时(3)式内的等号成立.

函数M_f和Q_f分别称为f的内因式和f的外因式；Q_f仅依赖于$|f|$的边界值.

证明 我们首先假定$f\in H^1$. 如果B是由f的零点所构成的Blaschke乘积，且$g=f/B$. 则定理17.9指出$g\in H^1$；并且因为在T上$|g^*|=|f^*|$ a.e.，所以只须用g代替f来证明定理的结论就足够了.

这样，让我们假定f在U内不存在零点，并且假定$f(0)=1$. 则$\log|f|$在U内是调和的，$\log|f(0)|=0$. 又因为$\log=\log^+-\log^-$，故调和函数的平均值性质蕴涵着

(4) $$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{-}|f(re^{i\theta})|d\theta=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log^{+}|f(re^{i\theta})|d\theta$$
$$\leqslant\|f\|_0\leqslant\|f\|_1$$

对 $0<r<1$ 成立. 从 Fatou 引理现在得出 $\log^{+}|f^{*}|$ 和 $\log^{-}|f^{*}|$ 二者都在 $L^{1}(T)$ 内. 因此 $\log|f^{*}|$ 也在 $L^{1}(T)$ 内.

这便证明了定义(1)是有意义的. 根据定理 17.16, $Q_f\in H^{1}$. 同时, 因为 $\log|f^{*}|\in L^{1}(T)$, 所以 $|Q_f^{*}|=|f^{*}|\neq 0$ a.e.. 如果我们能证明

(5) $$|f(z)|\leqslant Q_f(z)\qquad(z\in U)$$

则 f/Q_f 将是一个内函数, 并且我们获得因式分解(2).

因为, $\log|Q_f|$ 是 $\log|f^{*}|$ 的 Poisson 积分, (5)式等价于不等式

(6) $$\log|f|\leqslant P[\log|f^{*}|]$$

这就是我们现在要证明的. 我们的记号如同第十一章里一样: $P[h]$ 是函数 $h\in L^{1}(T)$ 的 Poisson 积分.

对于 $|z|\leqslant 1$ 和 $0<R<1$, 令 $f_R(z)=f(Rz)$. 固定 $z\in U$, 则

(7) $$\log|f_R(z)|=P[\log^{+}|f_R|](z)-P[\log^{-}|f_R|](z)$$

因为对所有的实数 u 和 v, $|\log^{+}u-\log^{+}v|\leqslant|u-v|$ 成立, 又因为当 $R\to 1$ 时 $\|f_R-f^{*}\|_1\to 0$(定理 17.12), 当 $R\to 1$ 时, (7)式的第一个 Poisson 积分收敛于 $P[\log^{+}|f^{*}|]$. 因此 Fatou 引理给出

(8) $$P[\log^{-}|f^{*}|]\leqslant\liminf_{R\to 1}P[\log^{-}|f_R|]=P[\log^{+}|f^{*}|]-\log|f|$$

这个不等式是与(6)相同的.

我们现在已经建立了因式分解(2). 若我们在(5)式中令 $z=0$, 我们便得到(3)式; 当且仅当 $|f(0)|=|Q_f(0)|$, 即: 当且仅当 $|M_f(0)|=1$ 时在(3)式里的等号成立; 因为 $\|M_f\|_\infty=1$, 所以这只有当 M_f 是常数时才可能.

这就完成了对 $p=1$ 时的证明.

如果 $1<p\leqslant\infty$,则 $H^p\subset H^1$,因此剩下来要证明的只是 $Q_f\in H^p$. 但是如果 $f\in H^p$, 根据 Fatou 引理, 则 $|f^*|\in L^p(T)$; 因此根据定理 17.16(c), $Q_f\in H^p$.

若 $p<1$, 我们能够运用第 17.11 节所描述的技巧. 留下这些细节作为练习.

$\log|f^*|\in L^1(T)$这一事实有一个推论, 我们在证明中已经用到过, 但是它的重要性足以分开来叙述它.

17.18 定理 如果 $0<p\leqslant\infty$, $f\in H^p$, 且 f 不恒等于 0, 则在 T 的几乎所有点上, 有 $f^*(e^{i\theta})\neq 0$.

证明 若 $f^*=0$, 则 $\log|f^*|=-\infty$, 如果在一个正测度集上发生这种情况, 则

$$\int_{-\pi}^{\pi}\log|f^*(e^{it})|\,dt=-\infty$$

注意到定理 17.18 对于 $f\in H^p$ 的径向极限的零点的位置加上了一个定量的限制. 在 U 内的零点也被 Blaschke 条件定量地限制了.

象通常一样, 我们能够把上面关于零点的结果重述为一个唯一性定理:

如果 $f\in H^p$, $g\in H^p$, 且在 T 的某个 Lebesgue 测度是正的子集上 $f^*(e^{i\theta})=g^*(e^{i\theta})$, 则对所有的 $z\in U$, $f(z)=g(z)$.

17.19 让我们浏览一下类 N, 目的是看看在那里定理 17.17 和定理 17.18 还有多少东西保持正确. 如果 $f\in N$, $f\not\equiv 0$, 我们能用 Blaschke 乘积除 f, 得到一个商 g, 它在 U 内不存在零点, 并且它属于 N(定理 17.9). 这时 $\log|g|$ 是调和的, 又因为

(1) $$|\log|g||=2\log^+|g|-\log|g|$$

和

(2) $$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|g(re^{i\theta})|\,d\theta=\log|g(0)|$$

我们看出 $\log|g|$ 满足定理 11.19 的假定，因此 $\log|g|$ 是一个实测度 μ 的 Poisson 积分．于是

$$f(z)=cB(z)\exp\left\{\int_T \frac{e^{it}+z}{e^{it}-z}d\mu(t)\right\} \tag{3}$$

这里 c 是一个常数，$|c|=1$，B 是 Blaschke 乘积．

注意，$\log^+|g|$ 的积分是有界的（它是 $|g|$ 不太靠近 ∞ 的说法的定量表达）这一假定是如何蕴涵 $\log^-|g|$ 的积分的有界性（它是说 $|g|$ 不会在太多的地方太靠近 0 的）．

若 μ 是一个负测度，在(3)式中的指数因式是在 H^∞ 内．应用 Jordan 分解于 μ，这就证明：

对每一个 $f\in N$，对应有两个函数 b_1 和 $b_2\in H^\infty$，使得 b_2 在 U 内不存在零点，且 $f=b_1/b_2$．

因为 $b_2^*\neq 0$ a. e.，所以 f a. e. 具有有限的径向极限．同时 $f^*\neq 0$ a. e.，

$\log|f^*|\in L^1(T)$ 成立吗？是的，并且其证明与定理 17.17 中给出的证明是一样的．

然而，定理 17.17 的不等式(3)，不再成立．比如，若

$$f(z)=\exp\left\{\frac{1+z}{1-z}\right\} \tag{4}$$

则 $\|f\|_0=e$，$|f^*|=1$ a. e.，并且

$$\log|f(0)|=1>0=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f^*(e^{it})|dt \tag{5}$$

移位算子

17.20 不变子空间 考虑 Banach空间 X 上的有界线性算子 S；即是说，S 是 X 到 X 内的有界线性变换．如果 X 的闭子空间 Y 具有 $S(Y)\subset Y$ 的性质，我们称 Y 为 S-不变子空间．于是 X 的 S-不变子空间恰好是由 S 把自身映入自身的那些子空间．

一个算子 S 的不变子空间的知识有助于我们想象出它的作用.(这是一个非常一般的、因而也是比较含混的原则:在研究任何一类变换中,了解该变换把什么东西保持不动.)比如,若 S 是 n 维向量空间 X 上线性算子,如果 S 有 n 个线性独立的特征向量 $x_1, \cdots, x_n$,由这些 x_i 中的任何一个所张成的一维空间是 S-不变的. 如果我们取 $\{x_1, \cdots, x_n\}$ 作为 X 的一个基,我们就得到 S 的一个非常简单的描述.

我们将描述 l^2 上的称为"移位算子" S 的不变子空间. 这里 l^2 是所有复序列

(1) $$x=\{\xi_0, \xi_1, \cdots, \xi_n, \cdots\}$$

的空间,对于这些复序列

(2) $$\|x\|=\left\{\sum_{n=0}^{\infty}|\xi_n|^2\right\}^{1/2}<\infty$$

并且 S 是把(1)式中给定的元素 $x\in l^2$ 变为

(3) $$Sx=\{0, \xi_0, \xi_1, \cdots\}$$

的变换. 显然 S 是一个 l^2 上的有界线性算子,并且 $\|S\|=1$.

一些 S-不变子空间可以立即看出:如果 Y_k 是所有前 k 个坐标均为 0 的 $x\in l^2$ 的集,则 Y_k 是 S-不变的.

为了找到另外一些不变子空间,我们利用 l^2 和 H^2 之间的 Hilbert 空间的同构,把移位算子 S 化为 H^2 上的乘法算子. 问题在于乘法算子要比原来的序列空间 l^2 的场合容易分析一些(因为作为全纯函数的空间 H^2 具有比较丰富的构造).

我们对(1)式给定的每一个 $x\in l^2$ 联系一个函数

(4) $$f(z)=\sum_{n=0}^{\infty}\xi_n z^n \qquad (z\in U)$$

根据定理 17.10,它定义了一个 l^2 到 H^2 上的线性一一映射. 如果

(5) $$y=\{\eta_n\},\ g(z)=\sum_{n=0}^{\infty}\eta_n z^n$$

并且如果 H^2 中的内积定义为

(6) $$(f,g)=\frac{1}{2\pi}\int_{-\pi}^{\pi} f^*(e^{i\theta})\overline{g^*(e^{i\theta})}d\theta$$

则 Parseval 定理指出 $(f,g)=(x,y)$. 于是我们获得一个 l^2 到 H^2 上的 Hilbert 空间的同构, 并且移位算子 S 转化为 H^2 上的乘法算子(我们继续用 S 记它):

(7) $$(Sf)(z)=zf(z)\quad (f\in H^2,\ z\in U)$$

上面提及的不变子空间 Y_k 现在可看出由所有的这些 $f\in H^2$ 所组成, 这些 f 在原点至少具有 k 阶零点. 这就给出一个线索: 对于任何有限集 $\{\alpha_1,\cdots,\alpha_k\}\subset U$, 所有使得 $f(\alpha_1)=\cdots=f(\alpha_k)=0$ 的那些 $f\in H^2$ 的空间 Y 是 S-不变的. 如果 B 是以 $\alpha_1,\cdots,\alpha_k$ 为零点的有限 Blaschke 乘积, 则当且仅当 $f/B\in H^2$ 时 $f\in Y$. 于是 $Y=BH^2$.

这就联想到无限的 Blaschke 乘积也可能导致 S-不变子空间, 更加一般地, Blaschke 乘积可能用任意的内函数 φ 代替. 不难看出每一个 φH^2 是 H^2 的闭 S-不变子空间. 但是 H^2 的每一个闭的 S-不变子空间都是这种形式的空间, 则是一个比较深入的结果.

17.21 Beurling 定理

(*a*) 对每一个内函数 φ, 空间

(1) $$\varphi H^2=\{\varphi f: f\in H^2\}$$

是 H^2 的闭的 S-不变子空间.

(*b*) 若 φ_1 和 φ_2 都是内函数, 且 $\varphi_1 H^2=\varphi_2 H^2$, 则 φ_1/φ_2 是常数.

(*c*) 除 $\{0\}$ 以外, H^2 的每一个闭的 S-不变子空间 Y 包含一个内函数 φ, 使得 $Y=\varphi H^2$.

证明 H^2 是 Hilbert 空间，它的范数为

$$\|f\|_2=\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f^*(e^{i\theta})|^2\,d\theta\right\}^{1/2} \tag{2}$$

如果 φ 是一个内函数，则 $|\varphi^*|=1$ a. e.，映射 $f\to\varphi f$ 因此是 H^2 到 H^2 内的一个等距；因为是等距的，它的值域 φH^2 是 H^2 的闭子空间.［证明：如果在 H^2 内 $\varphi f_n\to g$，则 $\{\varphi f_n\}$ 是一个 Cauchy 序列，因此 $\{f_n\}$ 也是 Cauchy 序列，从而 $f_n\to f\in H^2$，所以 $g=\varphi f\in\varphi H^2$.］因为 $z\cdot\varphi f=\varphi\cdot zf$，$\varphi H^2$ 的 S-不变性也是显而易见的．因此(*a*)成立.

如果 $\varphi_1H^2=\varphi_2H^2$，则对某个 $f\in H^2$，$\varphi_1=\varphi_2 f$. 因此 $\varphi_1/\varphi_2\in H^2$. 类似地，$\varphi_2/\varphi_1\in H^2$. 令 $\varphi=\varphi_1/\varphi_2$ 和 $h=\varphi+(1/\varphi)$. 则 $h\in H^2$，因为在 T 上 $|\varphi^*|=1$ a. e.，h^* 在 T 上几乎处处是实的．因此 h 是 h^* 的 Poisson 积分．这便得出 h 在 U 内是实的．因此 h 是常数．这时 φ 必须是常数．(*b*)得证.

(*c*) 的证明将用到由 Helson 和 Lowdenslager 开创的一个方法．假定 Y 是 H^2 的一个闭的 S-不变子空间，它不是单独由 0 组成．则存在一个最小整数 k，使得 Y 包含一个下列形式的函数 f

$$f(z)=\sum_{n=k}^{\infty}c_nz^n\qquad c_k=1 \tag{3}$$

这时 $f\notin zY$，此处我们把形如 $g(z)=zf(z)$，$f\in Y$ 的所有的 g 的集记作 zY．这就得出 zY 是一个 Y 的真闭子空间.［用在(*a*)的证明中用过的论证得出闭性.］所以 Y 包含一个正交于 zY 的非零向量(定理 4.11).

所以存在一个 $\varphi\in Y$，使得 $\|\varphi\|_2=1$ 以及 $\varphi\perp zY$. 因此 $\varphi\perp z^n\varphi$，对 $n=1,2,3,\cdots$ 成立．根据 H^2 中的内积的定义［见 17.20(6)］，这意味着

$$(4)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}|\varphi^*(e^{i\theta})|^2e^{-in\theta}d\theta=0\quad(n=1,2,3,\cdots)$$

如果我们对这些方程的左边用它们的复共轭代替，即，若我们用 $-n$ 代替 n，这些方程保持不变．于是 $|\varphi^*|^2\in L^1(T)$ 的全体 Fourier 系数除了对应于 $n=0$ 的那个等于 1 以外，其余都是 0．因为 L^1-函数是由它们的 Fourier 系数所确定的（定理 5.15），这就得出在 T 上 $|\varphi^*|=1$ a.e.．但是 $\varphi\in H^2$，所以 φ 是 φ^* 的 Poisson 积分，因此 $|\varphi|\leqslant 1$．我们得出了 φ 是一个内函数．

因为 $\varphi\in Y$，并且 Y 是 S-不变的，对于所有 $n\geqslant 0$ 有 $\varphi z^n\in Y$，因此对每一个多项式 P 有 $\varphi P\in Y$．而这些多项式是在 H^2 内稠密的（根据 Parseval 定理，任一 $f\in H^2$ 的幂级数的部分和依 H^2 的范数收敛于 f），又因为 Y 是闭的，并且 $|\varphi|\leqslant 1$，这就得出 $\varphi H^2\subset Y$．我们必须证明这个包含关系不是真正的包含．因为 φH^2 是闭的，所以只须证明 $h\in Y$ 和 $h\perp\varphi H^2$ 的假定蕴涵着 $h=0$ 即可．

如果 $h\perp\varphi H^2$，则对 $n=0,1,2,\cdots$，$h\perp\varphi z^n$，或者

$$(5)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}h^*(e^{i\theta})\overline{\varphi^*(e^{i\theta})}\,e^{-in\theta}d\theta=0\qquad(n=0,1,2,\cdots)$$

如果 $h\in Y$，$n=1,2,3,\cdots$，则 $z^nh\in zY$，我们对 φ 的选取表明 $z^nh\perp\varphi$，或者

$$(6)\qquad \frac{1}{2\pi}\int_{-\pi}^{\pi}h^*(e^{i\theta})\overline{\varphi^*(e^{i\theta})}\,e^{-in\theta}d\theta=0\qquad(n=-1,-2,-3,\cdots)$$

于是 $h^*\overline{\varphi^*}$ 的所有的 Fourier 系数均为 0，因此在 T 上 $h^*\overline{\varphi^*}=0$ a.e.；又因为 $|\varphi^*|=1$ a.e. 成立，我们有 $h^*=0$ a.e.．所以 $h=0$．证明完毕．

17.22　评注　如果我们把定理 17.15 和定理 17.21 结合起来，就可以看出：H^2 的 S-不变子空间可用下列两组数据刻划：一个复数序列 $\{\alpha_n\}$（可能是有限的，甚至是空集），使得 $|\alpha_n|<1$ 和

$\Sigma(1-|\alpha_n|)<\infty$,以及$T$上的一个正的Borel测度$\mu$,它关于Lebesgue测度是奇异的(所以$D\mu=0$ a. e.). 找出用$\{\alpha_n\}$和μ的语言叙述的条件，以保证H^2的一个S-不变子空间包含另一个S-不变子空间，这是不难的(我们把它留作练习). 这样看得出所有的S-不变子空间的半序集具有极其复杂的构造，它比从l^2上的移位算子的简单定义中所能料想到的要复杂得多.

我们用定理17.21的一个浅易的推论结束这一节. 这个推论依赖于定理17.17中所描述的因式分解.

17.23 定理 假定M_f是$f\in H^2$的内因式，Y是包含f的H^2的最小的闭S-不变子空间. 则

$$Y=M_fH^2 \tag{1}$$

特别，当且仅当f是一外函数时$Y=H^2$.

证明 设$f=M_fQ_f$是分解f为它的内因式和外因式之积的分解式. 显然$f\in M_fH^2$; 又因为M_fH^2是闭的和S-不变的，我们有$Y\subset M_fH^2$.

另一方面，定理17.21指出，存在一个内函数φ，使得$Y=\varphi H^2$. 因为$f\in Y$，存在一个$h=M_hQ_h\in H^2$，使得

$$M_fQ_f=\varphi M_hQ_h \tag{2}$$

因为内函数的绝对值在T上a. e.为1，(2)蕴涵着$Q_f=Q_h$，因此$M_f=\varphi M_h\in Y$，因而Y必须包含最小的含有M_f的S-不变闭子空间. 于是$M_fH^2\subset Y$. 这就完成了证明.

用两个问题小结这些结果可能是有趣的，这些结果将提供问题的答案.

如果$f\in H^2$，什么样的函数$g\in H^2$能够用形如fP的函数依H^2的范数逼近? 这里P跑遍所有的多项式. 答案: 恰好是使$g/M_f\in H^2$的这些g.

对于什么样的$f\in H^2$，集$\{fP\}$在H^2内稠密是正确的? 答案:

恰好是那些满足

$$\log|f(0)|=\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|f^*(e^{it})|\,dt$$

的函数 f.

共轭函数

17.24 问题的叙述 在单位圆盘 U 内的每一个实的调和函数 u 是一个而且仅是一个使得 $f(0)=u(0)$ 的 $f\in H(U)$ 的实部. 如果 $f=u+iv$, $f(0)=u(0)$ 这一要求也可以表示为 $v(0)=0$ 的形式. 这个函数 v 叫做 u 的调和共轭函数或 u 的共轭函数.

象定理 17.6 一样, 定义 $M_p(u;r)$ (用 u 代替 f). 因为 $|u|$ 是次调和的, 定理 17.6 应用于 $M_p(u;r)$, 因此若 $p\geqslant 1$

(1) $$\lim_{r\to 1}M_p(u;r)=\sup_{0<r<1}M_p(u;r)$$

我们将用 $\|u\|_p$ 记这个共同的值.

现在提出了下列问题: $\|u\|_p$ 的有限性能蕴涵 $\|v\|_p$ 的有限性吗?

这个问题等价于下面的问题: 若 $\|u\|_p<\infty$, 必定有 $f\in H^p$ 吗?

这个问题还有另一个表达形式, 它包含在习题 25 中.

如果 $1<p<\infty$, 这个答案是肯定的(由 M. Riesz 给出). (对 $p=1$ 和 $p=\infty$, 这个答案是否定的; 见习题 24.) 在定理 17.26 中将给出精确的叙述.

17.25 引理 假定 $1<p\leqslant 2$, 则存在一个仅依赖于 p 的正的常数 B, 使得

(1) $$(1+B)(B\cos\beta)^p-B\cos p\beta\geqslant 1 \qquad \left(-\frac{\pi}{2}\leqslant\beta\leqslant\frac{\pi}{2}\right)$$

证明 令 $B=(\cos\delta)^{-1}$, 这里 δ 这样地选取使 $(1+p)\delta=\pi$. 则 $\delta<\pi/2$, 所以 $B>0$ 并且 $\cos p\delta=-\cos\delta$.

如果 $0\leqslant|\beta|\leqslant\delta$, 则$B\cos\beta\geqslant B\cos\delta=1$. 所以, (1)的左边至少是$(1+B)-B=1$.

如果 $\delta<|\beta|\leqslant\pi/2$, 则 $p\delta<p|\beta|<p\pi/2\leqslant\pi$, 所以

$$\cos p\beta<\cos p\delta=-1/B$$

因此, $-B\cos p\beta>1$, (1)式再一次成立.

17.26 定理 对满足 $1<p<\infty$ 的每个 p, 对应有一个常数 A_p, 使得不等式

(1) $$\|v\|_p\leqslant A_p\|u\|_p$$

对U内的每一个实的调和函数 u 成立, 其中 v 是 u 的调和共轭函数.

证明 我们首先假定 $1<p\leqslant2$, 并且令 $f=u+iv$.

引理解决了在U内 $u>0$ 的情况. 由于 f 在U内不存在零点, 又因为U是单连通的, 故存在一个 $g\in H(U)$, 使得 $f=e^g$, $g=\alpha+i\beta$(所以 α 和 β 都是调和的), 并且 $\beta(0)=0$. 因为 $u>0$, 故在 U 内有 $|\beta|<\pi/2$. 因为

(2) $$u=e^\alpha\cos\beta=|f|\cos\beta$$

引理 17.25 指出

(3) $$|f|^p\leqslant(1+B)B^pu^p-B|f|^p\cos p\beta$$

注意到$|f|^p\cos p\beta$是$\exp(pg)$的实部, 因而它也是调和的. 所以

(4) $$\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(re^{i\theta})|^p\cos[p\beta(re^{i\theta})]d\theta=|f(0)|^p>0$$

$$(0<r<1)$$

因此(3)的积分导致

(5) $$\|f\|_p\leqslant C\|u\|_p$$

这里 $C=(1+B)^{1/p}B$.

这是证明的关键. 余下的只是例行手续并且能以不同的方式处理.

若在U内$u>0$，在$\bar{U}$上u是连续的．则(5)式可写成下列形式

$$\|f\|_p \leqslant C\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} u(e^{it})^p dt\right)^{1/p} \tag{6}$$

现在假定u在$\bar{U}$上连续，在U内是实的和调和的，并且对$z\in U$定义

$$f_1(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{it}+z}{e^{it}-z}u^+(e^{it})dt=u_1(z)+iv_1(z) \tag{7}$$

用u^-代替u^+以同样的方式定义$f_2=u_2+iv_2$．然后，以f_1，u_1和f_2，u_2代替(6)式中的f，u；应用(6)式，在T上我们有$u_1=u^+\leqslant|u|$．在$\bar{U}$上，$u=u_1-u_2$．在U内$f=f_1-f_2$．这就得出一对f，u满足以$2C$代替C的(6)式．因此，假定u是在$\bar{U}$上连续，(1)式对$A_p\cdot 2C$成立．

在一般情况下，选取$R<1$，并且对$z\in U$令$f_R(z)=f(Rz)$．将上面的结果应用于f_R，因此$\|u_R\|_p\leqslant\|u\|_p$，我们得到

$$\|f_R\|_p\leqslant A_p\|u_R\|_p\leqslant A_p\|u\|_p \tag{8}$$

对每一个$R<1$成立．因为$|v|\leqslant|f|$，这就给出了(1)．

我们现在已经证明在$1<p\leqslant 2$时定理的结论成立．

为完成这个证明，假定$2\leqslant q<\infty$，并设p是共轭于q的指数．

设u和v如定理所述，令$f=u+iv$，并且设$g(z)=\alpha(z)+i\beta(z)$是一个多项式，使$g(0)$是实数．我们要证

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}u(re^{i\theta})\beta(e^{i\theta})d\theta=-\frac{1}{2\pi}\int_{-\pi}^{\pi}v(re^{i\theta})\alpha(e^{i\theta})d\theta \tag{9}$$

对$0<r<1$成立．为看出这点，注意到$u\beta+v\alpha$是调和的．它是fg的虚部．因此

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}&[u(re^{i\theta})\beta(e^{i\theta})+v(re^{i\theta})\alpha(e^{i\theta})]d\theta\\&=u(0)\beta(0)+v(0)\alpha(0)\end{aligned} \tag{10}$$

因为$\beta(0)=0$和$v(0)=0$，我们得到(9)．Hölder不等式给出

(11) $$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} v(re^{i\theta})\alpha(e^{i\theta})\,d\theta\right| \leqslant \|u\|_q\|\beta\|_p \leqslant \|u\|_q A_p\|\alpha\|_p$$

因为实的三角多项式 α 的类在 T 上的所有实的 L^p-函数空间里是稠密的. (11) 和定理 6.16 指出

$$\|v\|_q \leqslant A_p\|u\|_q$$

我们现在已经证明了对 $2\leqslant q<\infty$ 时的定理并且我们看到 $A_q\leqslant A_p$. 如果我们对 A_p 和 A_q 取最小的允许值, 最后一道计算可以反过来, 我们就会发现 $A_p=A_q$.

习　题

1　对于上半连续的次调和函数, 证明定理 17.4 和定理 17.5.

2　假设 $f\in H(\Omega)$, 证明 $\log(1+|f|)$ 是在 Ω 内次调和的.

3　假定 $0<p\leqslant\infty$ 和 $f\in H(U)$, 证明 $f\in H^p$ 当且仅当在 U 内存在一个调和函数 u, 使得对所有的 $z\in U$, $|f(z)|^p\leqslant u(z)$. 证明若存在 $|f|^p$ 的这样一个调和的优函数 u, 则存在一个最小的这样的函数, 记作 u_f. (说明显一点, $|f|^p\leqslant u_f$ 且 u_f 是调和的; 若 $|f|^p\leqslant u$ 且 u 是调和的, 则 $u_f\leqslant u$.) 证明 $\|f\|_p=u_f(0)^{1/p}$. 提示: 对 $R<1$ 考虑在 $D(0;R)$ 上的具有边界值 $|f|^p$ 的调和函数, 并且令 $R\to 1$.

4　照样地证明当且仅当 $\log^+|f|$ 在 U 内有调和的优函数时 $f\in N$.

5　假定 $f\in H^p$, $\varphi\in H(U)$, 以及 $\varphi(U)\subset U$. 能够得出 $f\circ\varphi\in H^p$ 吗? 用 N 代替 H^p, 回答同样的问题.

6　如果 $0<r<s\leqslant\infty$, 证明 H^s 是 H^r 的真子类.

7　证明 H^∞ 是 $p<\infty$ 时所有 H^p 的交的真子类.

8　如果 $f\in H^1$ 并且 $f^*\in L^p(T)$, 证明 $f\in H^p$.

9　假定 $f\in H(U)$ 和 $f(U)$ 在平面内不是稠密的. 证明 f 在 T 的几乎所有的点上具有有限的径向极限.

10　固定 $a\in U$. 证明映射 $f\to f(a)$ 是 H^2 上的有界线性泛函. 因为 H^2 是 Hilbert 空间, 这个泛函能表示为和某个 $g\in H^2$ 的内积. 找出这个 g.

11　固定 $a\in U$, 若 $\|f\|_2\leqslant 1$, $|f'(a)|$ 能够有多大? 找出极值函数. 对 $f^{(n)}(a)$ 同样地做.

12 假定 $p\geqslant 1, f\in H^p$, 并且 f^* 在 T 上几乎处处是实的，证明 f 是常数. 指出这个结果对每一个 $p<1$ 不成立.

13 假定 $f\in H(U)$ 和假定存在一个 $M<\infty$，使得 f 将每一个半径 $r<1$ 中心为 0 的圆周映到长度至多是 M 的一条曲线 γ_r 上. 证明 f 具有到 $\overline{U}$ 上的连续开拓，并且 f 在 T 上的限制是绝对连续的.

14 假定 μ 是 T 上的复 Borel 测度，使得

$$\int_T e^{int}d\mu(t)=0 \qquad (n=1, 2, 3, \cdots)$$

证明或者 $\mu=0$，或者 μ 的支集是整个 T.

15 假定 K 是单位圆周 T 的真紧子集. 证明 K 上的每一个连续函数在 K 上能够用多项式一致逼近. 提示：应用习题 14.

16 完成定理 17.17 对于 $0<p<1$ 时的证明.

17 设 φ 是非常数的内函数，它在 U 内没有零点.

(a) 证明若 $p>0$ 则 $1/\varphi\notin H^p$.

(b) 证明至少存在一个 $e^{i\theta}\in T$，使得 $\lim\limits_{r\to 1}\varphi(re^{i\theta})=0$.

提示：$\log|\varphi|$ 是一个非负的调和函数.

18 假定 φ 是一个非常数的内函数，$|\alpha|<1$ 和 $\alpha\notin\varphi(U)$. 证明

$$\lim_{r\to 1}\varphi(re^{i\theta})=\alpha$$

至少对一个 $e^{i\theta}\in T$ 成立.

19 假定 $f\in H^1$ 和 $1/f\in H^1$. 证明 f 是一个外函数.

20 假定 $f\in H^1$ 以及对所有 $z\in U$, $\mathrm{Re}[f(z)]>0$. 证明 f 是一个外函数.

21 证明 $f\in N$ 当且仅当 $f=g/h$，其中 g 和 $h\in H^\infty$，h 在 U 内不存在零点.

22 证明下面的定理 15.24 的逆定理：

如果 $f\in H(U)$ 且

$$(*)\qquad \lim_{r\to 1}\int_{-\pi}^{\pi}|\log|f(re^{i\theta})||d\theta=0$$

则 f 是一个 Blaschke 乘积. 提示：(*)蕴涵着

$$\lim_{r\to 1}\int_{-\pi}^{\pi}\log^+|f(re^{i\theta})|d\theta=0$$

因为 $\log^+|f|\geqslant 0$，从定理 17.3 和定理 17.5 得出 $\log^+|f|=0$. 所以 $|f|\leqslant 1$. 现在 $f=Bg$，g 不存在零点，$|g|\leqslant 1$. 用 $1/g$ 代替 f，(*)仍成立. 根据前面的论证，$|1/g|\leqslant 1$，因此 $|g|=1$.

23 找出在 17.22 节里所提到的条件.

24 U 到一个垂直带上的保形映射说明关于共轭函数的 M. Riesz 定理不能推广到 $p=\infty$ 的情况. 试推证出它也不能推广到 $p=1$ 的情况.

25 假定 $1<p<\infty$, 将每一个 $f\in L^p(T)$ 对应于它的 Fourier 系数

$$\hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(e^{it})e^{-int}dt \qquad (n=0,\pm1,\pm2,\cdots)$$

试从定理 17.26推导下列结论:

(a) 对于每一个 $f\in L^p(T)$, 对应有一个 $g\in L^p(T)$, 使得对 $n\geqslant 0$时 $\hat{g}(n)=\hat{f}(n)$, 但当 $n<0$ 时 $\hat{g}(n)=0$. 事实上, 存在一个仅依赖于 p 的常数 C, 使得

$$\|g\|_p\leqslant C\|f\|_p$$

于是映射 $f\to g$ 是 $L^p(T)$ 到 $L^p(T)$ 内的有界线性射影. 从 f 的 Fourier 级数中删去 $n<0$ 的那些项就获得 g 的 Fourier 级数.

(b) 如果我们删去 $n<k$ 的那些项, 这里 k 是任意给定的正整数, 证明同样的结论是正确的.

(c) 从(b)推导出: 任一$f\in L^p(T)$的 Fourier级数的部分和s_n构成$L^p(T)$内的一个有界序列. 我们进而获得: 实际上有

$$\lim_{n\to\infty}\|f-s_n\|_p=0$$

(d) 若 $f\in L^p(T)$, 且

$$F(z)=\sum_{n=0}^{\infty}\hat{f}(n)z^n$$

则 $F\in H^p$, 并且每一个 $F\in H^p$ 都能这样得出. 于是(a)里所述及的射影可以看成一个 $L^p(T)$ 到 H^p 上的映射.

26 证明当 $p=2$ 时定理 17.26 存在一个很简单的证明, 并且找出 A_2的最好值.

27 假定在 U 内 $f(z)=\sum_{0}^{\infty}a_nz^n$并且 $\Sigma|a_n|<\infty$. 证明

$$\int_0^1|f'(re^{i\theta})|dr<\infty$$

对所有的 θ 成立.

28 证明下列命题是正确的: 如果$\{n_k\}$是一个很快地趋于$+\infty$ 的正整数序列, 若设

$$f(z)=\sum_{k=1}^{\infty}\frac{z^{n_k}}{k}$$

则对所有的使得

$$1-\frac{1}{n_k}<|z|<1-\frac{1}{2n_k}$$

成立的那些 z，有 $|f'(z)|>n_k/(10k)$ 成立. 因此，虽然对几乎所有的 θ

$$\lim_{R\to 1}\int_0^R f'(re^{i\theta})\,dr$$

存在(并且是有限的). 但是对每一个 θ

$$\int_0^1 |f'(re^{i\theta})|\,dr=\infty$$

用 U 的半径在 f 之下的象的长度对此作出几何解释.

第十八章　Banach 代数的初等理论

引言

18.1　定义　复代数是一个复数域上的向量空间 A，其中定义了一个可结合和可分配的乘法运算，即

(1)　　$x(yz)=(xy)z$，$(x+y)z=xz+yz$，$x(y+z)=xy+xz$

对 $x, y, z\in A$ 成立，并且与纯量乘法的关系是对 $x, y\in A$ 和纯量 α，有

(2)　　$$\alpha(xy)=x(\alpha y)=(\alpha x)y$$

如果在 A 中定义了一个范数，使 A 成为一个赋范线性空间，并且满足乘法不等式

(3)　　$$\|xy\|\leqslant\|x\|\|y\| \qquad (x, y\in A)$$

则 A 是一个赋范的复代数．如果再加上 A 关于这一范数而言是完备度量空间，即若 A 是 Banach空间，那末我们就称 A 为 Banach 代数．

不等式(3)使乘法成为一个连续运算．这意味着：如果 $x_n\to x$ 和 $y_n\to y$，则 $x_ny_n\to xy$，这可以从(3)及恒等式

(4)　　$$x_ny_n-xy=(x_n-x)y_n+x(y_n-y)$$

得出．

注意，我们未曾要求 A 是交换的，即对所有 x 和 $y\in A$，$xy=yx$．并且除非明显地提了出来，以后也将不要求是交换的．

不过，我们将假定 A 有个单位元．这是一个元素 e，它使得

(5)　　$$xe=ex=x \qquad (x\in A)$$

成立．容易看出，至多存在一个这样的 e $(e'=e'\cdot e=e)$，并且由(3)，$\|e\|\geqslant 1$．我们将作出一个附带的假设

(6) $$\|e\|=1$$

元素$x\in A$称为可逆的，如果x在A中有一个逆元，就是说，如果存在一个元素$x^{-1}\in A$使得

(7) $$x^{-1}x=xx^{-1}=e$$

再者，容易看出没有一个$x\in A$能有多于一个逆元.

如果x和y在A中都是可逆的，则x^{-1}和xy亦然，因为$(xy)^{-1}=y^{-1}x^{-1}$. 因此，可逆元对于乘法构成一个群.

一个元素$x\in A$的谱是使$x-\lambda e$不可逆的全体复数λ组成的集. 我们用$\sigma(x)$表示x的谱.

18.2 Banach代数的理论包含着以代数性质为一方和以拓扑性质为另一方之间的大量的相互影响. 在定理9.21中我们已经看到这方面的一个例子，并且还将会看到别的例子. 在Banach代数和全纯函数之间也有着密切的关系：$\sigma(x)$决不会是空集这一基本事实的最简易的证明有赖于关于整函数的Liouville定理，而谱半径公式则自然由关于幂级数的定理得到. 这就是我们着意讨论复Banach代数的原因之一. 实Banach代数的理论(其定义应当是很明显的，从略)是不那么合适的.

可逆元

在本节中，A是具有单位元e的复Banach代数，而G是A内的全体可逆元的集.

18.3 定理 如果$x\in A$而$\|x\|<1$，则$e+x\in G$

(1) $$(e+x)^{-1}=\sum_{n=0}^{\infty}(-1)^n x^n$$

且

(2) $$\|(e+x)^{-1}-e+x\|\leqslant\frac{\|x\|^2}{1-\|x\|}$$

证明 乘法不等式18.1(3)表明$\|x^n\|\leqslant\|x\|^n$. 若

(3) $$S_N=e-x+x^2-\cdots+(-1)^Nx^N$$

便得知$\{S_n\}$是A中的Cauchy序列. 因此,(1)中的级数(关于A的范数)收敛于一个元素$y\in A$. 因为乘法运算是连续的,且

(4) $$(e+x)S_N=e+(-1)^Nx^{N+1}=S_N(e+x)$$

可见$(e+x)y=e=y(e+x)$. 这就给出了(1),而(2)可从

(5) $$\left\|\sum_{n=2}^{\infty}(-1)^nx^n\right\|\leqslant\sum_{n=2}^{\infty}\|x^n\|\leqslant\sum_{n=2}^{\infty}\|x\|^n=\frac{\|x\|^2}{1-\|x\|}$$

得出.

18.4 定理 假设$x\in G,\|x^{-1}\|=1/\alpha,h\in A$,且$\|h\|=\beta<\alpha$. 则$x+h\in G$,且

(1) $$\|(x+h)^{-1}-x^{-1}+x^{-1}hx^{-1}\|\leqslant\frac{\beta^2}{\alpha^2(\alpha-\beta)}$$

证明 $\|x^{-1}h\|\leqslant\beta/\alpha<1$,由定理18.3,$e+x^{-1}h\in G$;又由于$x+h=x(e+x^{-1}h)$,有$x+h\in G$,且

(2) $$(x+h)^{-1}=(e+x^{-1}h)^{-1}x^{-1}$$

这样,

(3) $$(x+h)^{-1}-x^{-1}+x^{-1}hx^{-1}=[(e+x^{-1}h)^{-1}-e+x^{-1}h]x^{-1}$$

在定理18.3中以$x^{-1}h$代替x便得不等式(1).

系1 G是开集,并且映射$x\to x^{-1}$是G到G上的同胚.

因为,若$x\in G$且$\|h\|\to0$,则(1)蕴涵着$\|(x+h)^{-1}-x^{-1}\|\to0$. 因此$x\to x^{-1}$是连续的;显然它映$G$到$G$上. 并且由于它本身就是它的逆映射,所以它是一个同胚.

系2 映射$x\to x^{-1}$是可微的,它在任一$x\in G$的微分是将$h\in A$对应于$-x^{-1}hx^{-1}$的线性算子.

这也可以从(1)看出来. 注意变换的微分这一概念并不象定义8.22那样仅在R^k中,而是在任意赋范线性空间中都有意义. 如

果 A 是交换的，上述微分将 h 变为 $-x^{-2}h$，这和全纯函数 z^{-1} 的导数是 $-z^{-2}$ 这一事实是相一致的.

系 3 对每个 $x\in A$, $\sigma(x)$ 是紧的. 并且当 $\lambda\in\sigma(x)$ 时，$|\lambda|\leqslant\|x\|$.

因为，若 $|\lambda|>\|x\|$，由定理18.3，$e-\lambda^{-1}x\in G$. 对于 $x-\lambda e=-\lambda(e-\lambda^{-1}x)$ 结论也同样真确；所以 $\lambda\notin\sigma(x)$. 欲证 $\sigma(x)$ 是闭的，注意，(a) $\lambda\in\sigma(x)$ 当且仅当 $x-\lambda e\notin G$；(b) 由系 1，G 的余集是 A 的闭子集；(c) 映射 $\lambda\to x-\lambda e$ 是复平面到 A 内的连续映射.

18.5 定理 设 Φ 是 A 上的有界线性泛函，固定 $x\in A$，并定义

(1) $$f(\lambda)=\Phi[(x-\lambda e)^{-1}] \qquad (\lambda\notin\sigma(x))$$

则 f 在 $\sigma(x)$ 的余集上是全纯的，且 $\lambda\to\infty$ 时，$f(\lambda)\to 0$.

证明 固定 $\lambda\notin\sigma(x)$，在定理 8.14 中用 $x-\lambda e$ 代替 x，$(\lambda-\mu)e$ 代替 h，可以看出存在一个依赖于 x 和 λ 的常数 C，使得

(2) $$\|(x-\mu e)^{-1}-(x-\lambda e)^{-1}+(\lambda-\mu)(x-\lambda e)^{-2}\|\leqslant C|\mu-\lambda|^2$$

对所有充分接近 λ 的 μ 成立. 因此当 $\mu\to\lambda$ 时

(3) $$\frac{(x-\mu e)^{-1}-(x-\lambda e)^{-1}}{\mu-\lambda}\longrightarrow(x-\lambda e)^{-2}$$

并且，若我们将 Φ 作用于(3)两边，则 Φ 的连续性和线性表明

(4) $$\frac{f(\mu)-f(\lambda)}{\mu-\lambda}\longrightarrow\Phi[(x-\lambda e)^{-2}]$$

于是 f 是可微的，从而在 $\sigma(x)$ 之外是全纯的. 最后，当 $\lambda\to\infty$ 时，由 G 中逆映射的连续性，有

(5) $$\lambda f(\lambda)=\Phi[\lambda(x-\lambda e)^{-1}]=\Phi\left[\left(\frac{x}{\lambda}-e\right)^{-1}\right]\to\Phi(-e)$$

18.6 定理 对每个 $x\in A$, $\sigma(x)$ 是紧的和不空的.

证明 我们已经知道了 $\sigma(x)$ 是紧的. 固定 $x\in A$，并固定 $\lambda_0\notin\sigma(x)$. 则 $(x-\lambda_0 e)^{-1}\neq 0$，Hahn-Banach 定理蕴涵着存在一个 A 上的有界线性泛函 Φ，使得 $f(\lambda_0)\neq 0$，此处 f 如定理18.5中所定

义. 如果$\sigma(x)$是空的，由定理18.5就会推出f是一个在∞处趋于0的整函数. 因此，由Liouville定理，对每个λ，$f(\lambda)=0$，这和$f(\lambda_0)\neq 0$矛盾. 所以，$\sigma(x)$是不空的.

18.7 定理(Gelfand-Mazur) 如果A是具有单位元的复Banach代数，其中每个非零元都是可逆的，则A就是(等距同构于)复数域.

每个非零元都是可逆的代数称为可除代数. 注意，A的交换性并不是假设的一部分；它是结论的一部分.

证明 若$x\in A$且$\lambda_1\neq\lambda_2$，则$x-\lambda_1 e$和$x-\lambda_2 e$必定有一个是可逆的，因为它们不能同时为零. 现在由定理18.6得知：对每个$x\in A$，$\sigma(x)$刚好由一个点组成，记它为$\lambda(x)$. 由于$x-\lambda(x)e$不可逆，它一定是0，因此$x=\lambda(x)e$. 映射$x\to\lambda(x)$因此是A到复数域上的同构. 因为对所有$x\in A$，$|\lambda(x)|=\|\lambda(x)e\|=\|x\|$，所以它也是一个等距.

18.8 定义 对任意$x\in A$，x的谱半径$\rho(x)$是包含$\sigma(x)$的、中心在原点的最小闭圆盘的半径(有时也称为x的谱范数；参看习题14)：

$$\rho(x)=\sup\{|\lambda|:\lambda\in\sigma(x)\}$$

18.9 定理(谱半径公式) 对每个$x\in A$，

(1) $$\lim_{n\to\infty}\|x^n\|^{1/n}=\rho(x)$$

(极限的存在性是结论的一部分.)

证明 固定$x\in A$，设n是一个正整数. λ是复数. 并假定$\lambda^n\notin\sigma(x^n)$. 我们有

(2) $$(x^n-\lambda^n e)=(x-\lambda e)(x^{n-1}+\lambda x^{n-2}+\cdots+\lambda^{n-1}e)$$

以$(x^n-\lambda^n e)^{-1}$乘(2)两边，就证明了$x-\lambda e$是可逆的，因此$\lambda\notin\sigma(x)$.

这样一来，若$\lambda\in\sigma(x)$，则$\lambda^n\in\sigma(x^n)$对$n=1,2,3,\cdots$成立. 定

理 18.4 的系 3 指出了 $|\lambda^n|\leqslant\|x^n\|$，因此 $|\lambda|\leqslant\|x^n\|^{1/n}$．这就给出

(3) $$\rho(x)\leqslant\liminf_{n\to\infty}\|x^n\|^{1/n}$$

现在，若 $|\lambda|>\|x\|$，容易验证

(4) $$(\lambda e-x)\sum_{n=0}^{\infty}\lambda^{-n-1}x^n=e$$

上述级数因此就是 $-(x-\lambda e)^{-1}$．设 Φ 是 A 上的有界线性泛函并如定理 18.5 中一样定义 f．由(4)，展开式

(5) $$f(\lambda)=-\sum_{n=0}^{\infty}\Phi(x^n)\lambda^{-n-1}$$

对所有使得 $|\lambda|>\|x\|$ 的 λ 都成立．由定理 18.5，f 在 $\sigma(x)$ 外，因此也在集 $\{\lambda:|\lambda|>\rho(x)\}$ 中是全纯的．于是得出若 $|x|>\rho(x)$，则幂级数(5)收敛．特别对 A 上的每一个有界线性泛函 Φ，

(6) $$\sup_{n}|\Phi(\lambda^{-n}x^n)|<\infty \qquad (|\lambda|>\rho(x))$$

A 中任意一个元素的范数与它作为 A 的对偶空间上的线性泛函的范数相同，这一点是 Hahn-Banach 定理（第 5.21 节）的推论．因为(6)对每个 Φ 成立，现在我们可以应用 Banach-Steinhaus 定理并作出结论：对每个使 $|\lambda|>\rho(x)$ 的 λ 都对应有一个实数 $C(\lambda)$ 使

(7) $$\|\lambda^{-n}x^n\|\leqslant C(\lambda) \qquad (n=1,2,3,\cdots)$$

以 $|\lambda|^n$ 乘(7)并取 n 次根，若 $|\lambda|>\rho(x)$，则可给出

(8) $$\|x^n\|^{1/n}\leqslant|\lambda|[C(\lambda)]^{1/n} \qquad (n=1,2,3,\cdots)$$

因此

(9) $$\limsup_{n\to\infty}\|x^n\|^{1/n}\leqslant\rho(x)$$

从(3)和(9)即得本定理．

18.10 评注

(a) A 的一个元素是否在 A 中可逆，纯粹是一种代数性质．

所以 x 的谱及诸如谱半径之类是用 A 的代数结构来定义的，没有涉及到任何度量(或拓扑)方面的考虑．另一方面，定理 18.9 的陈述中的极限则依赖于 A 的度量性质．这是该定理引人注意的特色之一：它断言了完全由不同的途径提出来的两个数量的相等．

(*b*) 我们所考虑的代数也可能是一个较大的 Banach 代数 B 的子代数 (一个例子见后)，因此很可能出现这样的情况，某个 $x\in A$，在 A 中不是可逆的，但在 B 中是可逆的．因此 x 的谱与代数有关；利用很明显的记号，有 $\sigma_A(x)\supset\sigma_B(x)$，并且可以是真正的包含关系．但是，$x$ 的谱半径却不受此影响，因为定理 18.9 指出它可以用 x 的幂的度量性质表示出来，而它们却与 A 之外所发生的任何事情无关．

18.11 例 设 $C(T)$ 是单位圆周 T 上的全体复连续函数(关于点态加法、乘法及上确界范数)的代数，A 是所有 $f\in C(T)$ 的集，而这些 f 可以开拓为单位圆盘 U 的闭包上的连续函数 F，并使 F 在 U 中是全纯的．容易看出 A 是 $C(T)$ 的子代数．如果 $f_n\in A$ 并且 $\{f_n\}$ 在 T 上一致收敛，那末最大模定理保证了相应的序列 $\{F_n\}$ 在 U 的闭包上一致收敛．这表明 A 是 $C(T)$ 的闭子代数，从而 A 本身也是 Banach 代数．

由 $f_0(e^{i\theta})=e^{i\theta}$ 定义 f_0．则 $F_0(z)=z$．f_0 作为 A 的元素，它的谱由闭单位圆盘组成．对于 $C(T)$，f_0 的谱仅由单位圆组成．根据定理 18.9，这两个谱半径是相同的．

理想与同态

从现在起我们只讨论交换代数．

18.12定义 交换的复代数 A 的子集 I 称为理想，如果：(*a*) I 是 A 的子空间(在向量空间的意义下)，并且(*b*)当 $x\in A$，$y\in I$ 时，$xy\in I$．若 $I\neq A$，I 就称为真理想．极大理想是一个不能被任一更

大的真理想所包含的真理想. 注意, 没有一个真理想能包含可逆元.

如果B是另一个复代数, A到B的映射φ称为同态, 若它是一个线性映射, 同时也保持乘法运算: 对所有$x, y\in A$, $\varphi(x)\varphi(y)=\varphi(xy)$. φ的核(或零空间)是全体使$\varphi(x)=0$的$x\in A$的集. 验证一个同态核是理想是极容易的. 其逆命题请参看18.14节.

18.13 定理 如果A是具有单位元的交换复代数, 则A的每一个真理想都包含在一个极大理想之内. 如果再加上A是Banach代数, 则A的每一个极大理想都是闭的.

证明 第一部分几乎是Hausdorff极大原理的一个直接推论. (并且在任何一个具有单位元的交换环中都成立.) 将所有包含I的A的真理想组成的族$\mathscr{P}$半序化(根据集的包含关系). 设M是$\mathscr{P}$的某个极大线性序子族$\mathscr{Q}$中诸理想之并, 因为$\mathscr{P}$的任何成员都不包含单位元, 故M是一个理想 (因为它是理想的线性序族之并), $I\subset M$且$M\neq A$. $\mathscr{Q}$的极大性可推出M是A的极大理想.

若A是Banach代数, 则M的闭包$\overline{M}$仍是一个理想 (这一命题的证明细节留给读者). 因为M不包含A的可逆元, 并且全体可逆元的集是开的, 故$\overline{M}\neq A$. 于是M的极大性指出$\overline{M}=M$.

18.14 商空间和商代数 假定J是向量空间A的子空间, 并且对每个$x\in A$, 对应一个陪集

$$\varphi(x)=x+J=\{x+y: y\in J\} \tag{1}$$

若$x_1-x_2\in J$, 则$\varphi(x_1)=\varphi(x_2)$. 若$x_1-x_2\notin J$, 则$\varphi(x_1)\cap\varphi(x_2)=\varnothing$. J的陪集的全体用A/J表示. 如果我们对任意x, $y\in A$和纯量λ定义

$$\varphi(x)+\varphi(y)=\varphi(x+y),\quad \lambda\varphi(x)=\varphi(\lambda x) \tag{2}$$

则A/J是一个向量空间. 由于J是向量空间, 运算(2)是完全确定的. 这就是说, 若$\varphi(x)=\varphi(x')$和$\varphi(y)=\varphi(y')$, 则

恩

(3) $\varphi(x)+\varphi(y)=\varphi(x')+\varphi(y')$, $\lambda\varphi(x)=\lambda\varphi(x')$

显然，φ 也是一个由 A 到 A/J 上的线性映射；A/J 的零元素就是 $\varphi(0)=J$.

下一步假设 A 不仅是一个向量空间而且是一个交换代数. J 是 A 的一个真理想. 若 $x'-x\in J$ 且 $y'-y\in J$，则恒等式

(4) $$x'y'-xy=(x'-x)y'+x(y'-y)$$

表明 $x'y'-xy\in J$. 因此在 A/J 中的乘法可以用一种协调的方式予以定义：

(5) $$\varphi(x)\varphi(y)=\varphi(xy) \qquad (x \text{ 和 } y\in A)$$

容易验证 A/J 是一个代数，并且 φ 是 A 到 A/J 上的同态，其核是 J.

若 A 有单位元 e，则 $\varphi(e)$ 是 A/J 的单位元. 而且当且仅当 J 是极大理想时，A/J 是一个域.

为看出这点，假设 $x\in A$ 而 $x\notin J$，并令

(6) $$I=\{ax+y: a\in A, y\in J\}$$

则 I 是 A 的一个理想. 因为 $x\in I$，故 I 真正包含 J. 当 J 是极大时，$I=A$，因此对某个 $a\in A$ 和 $y\in J$ 有 $ax+y=e$. 因此 $\varphi(a)\varphi(x)=\varphi(e)$；而这就是说，$A/J$ 的每个非零元都是可逆的，于是 A/J 是域. 若 J 不是极大的，我们可以象上述一样选择 x，使 $I\neq A$. 因此 $e\notin I$，从而 $\varphi(x)$ 在 A/J 中不是可逆的.

18.15 商范数 假设 A 是一个赋范线性空间，J 是 A 的闭子空间，且如上所述，$\varphi(x)=x+J$. 定义

(1) $$\|\varphi(x)\|=\inf\{\|x+y\|: y\in J\}$$

注意 $\|\varphi(x)\|$ 是陪集 $\varphi(x)$ 中诸元素的范数的最大下界；这就与从 x 到 J 的距离相同. 称 A/J 中由(1)定义的范数为 A/J 的商范数. 它有下列性质：

(a) A/J 是赋范线性空间.

(*b*) 若A是Banach空间，则A/J亦然.

(*c*) 若A是交换Banach代数而J是一个真闭理想，则A/J是交换Banach代数.

这些都是容易验证的:

若$x\in J$，则$\|\varphi(x)\|=0$. 若$x\notin J$，则J闭这一事实蕴涵着$\|\varphi(x)\|>0$. 显然$\|\lambda\varphi(x)\|=|\lambda|\,\|\varphi(x)\|$. 若$x_1$和$x_2\in A$且$e>0$，则存在$y_1$和$y_2\in J$使得

$$\|x_i+y_i\|<\|\varphi(x_i)\|+e \qquad (i=1,2) \tag{2}$$

因此

$$\begin{aligned}\|\varphi(x_1+x_2)\|&\leqslant\|x_1+x_2+y_1+y_2\|\\&<\|\varphi(x_1)\|+\|\varphi(x_2)\|+2e\end{aligned} \tag{3}$$

它给出了三角不等式并证明了(*a*).

假如A是完备的且$\{\varphi(x_n)\}$是A/J中的Cauchy序列，则存在子序列使得

$$\|\varphi(x_{n_i})-\varphi(x_{n_{i+1}})\|<2^{-i} \qquad (i=1,2,3,\cdots) \tag{4}$$

并存在元素z_i使$z_i-x_{n_i}\in J$，且$\|z_i-z_{i+1}\|<2^{-i}$. 这样，$\{z_i\}$是A内的一个Cauchy序列；由于A是完备的，存在$z\in A$使$\|z_i-z\|\to 0$. 由此得出在A/J中，$\varphi(x_{n_i})$收敛于$\varphi(z)$. 但是一个Cauchy序列若有收敛子序列则整个序列收敛. 因此A/J是完备的，并且证明了(*b*).

为证明(*c*)，选择x_1和$x_2\in A$以及$e>0$，并选择$y_1, y_2\in J$使(2)成立. 注意$(x_1+y_1)(x_2+y_2)\in x_1x_2+J$，所以

$$\|\varphi(x_1x_2)\|\leqslant\|(x_1+y_1)(x_2+y_2)\|\leqslant\|x_1+y_1\|\,\|x_2+y_2\| \tag{5}$$

现在(2)蕴涵着

$$\|\varphi(x_1x_2)\|\leqslant\|\varphi(x_1)\|\,\|\varphi(x_2)\| \tag{6}$$

最后，如果e是A的单位元，在(6)中取$x_1\notin J$及$x_2=e$，就给出$\|\varphi(e)\|\geqslant 1$但$e\in\varphi(e)$，商范数的定义表明$\|\varphi(e)\|\leqslant\|e\|=1$. 所以

$\|\varphi(e)\|=1$，证明完毕.

18.16 在论述了这些预备知识后，现在我们就能够导出有关交换Banach代数的一些关键性事实了.

如前一样，假设A是一个有单位元的交换的复Banach代数. 对于A，我们让它对应于A的全体复同态的集Δ；这些同态是A到复数域上的同态，或者用不同的术语，是A上的不恒等于零的乘法线性泛函. 如前一样，$\sigma(x)$表示元素$x\in A$的谱而$\rho(x)$是x的谱半径.

那末，下述关系成立：

18.17 定理

(*a*) A的每个极大理想都是某个$h\in\Delta$的核.

(*b*) $\lambda\in\sigma(x)$当且仅当$h(x)=\lambda$对某个$h\in\Delta$成立.

(*c*) x在A中是可逆的当且仅当对每个$h\in\Delta$, $h(x)\neq 0$.

(*d*) 对每个$x\in A$和$h\in\Delta$, $h(x)\in\sigma(x)$.

(*e*) 对每个$x\in A$和$h\in\Delta$, $|h(x)|\leqslant\rho(x)\leqslant\|x\|$.

证明 如果M是A的极大理想，则A/M是域；因为M是闭的(定理18.13)，A/M是Banach代数. 由定理18.7，存在A/M到复数域上的同构j. 若$h=j\circ\varphi$，这里φ是A到A/M上的同态，其核为M，则$h\in\Delta$而h的核是M. (*a*)就得证.

如$\lambda\in\sigma(x)$，则$x-\lambda e$是不可逆的；因此，对所有$y\in A$，元素$(x-\lambda e)y$的集是A的一个真理想. (由定理18.13)它位于一个极大理想之内. (*a*) 表明存在$h\in\Delta$使$h(x-\lambda e)=0$. 因为$h(e)=1$，即给出$h(x)=\lambda$.

另一方面，如果$\lambda\notin\sigma(x)$，则对某个$y\in A$有$(x-\lambda e)y=e$，由此得知对每个$h\in\Delta$，$h(x-\lambda e)h(y)=1$，所以$h(x-\lambda e)\neq 0$或者$h(x)\neq\lambda$. (*b*)式得证.

因为当且仅当$0\notin\sigma(x)$时，x是可逆的，故由(*b*)可得(*c*).

最后，(*d*)和(*e*)都是(*b*)的直接推论.

注意，(e)蕴涵着 h 作为线性泛函其范数至多是 1．特别每个 $h\in\Delta$是连续的．这已经在早些时候证明过了(定理9.21)．

应用

现在给出一些定理的例子，这些定理的陈述并不包含代数的概念，然而却可以通过 Banach 代数的技巧来证明．

18.18 定理 设 $A(U)$是开单位圆盘 U 的闭包 $\bar{U}$ 上全体连续函数的集，这些函数在 U 上的限制是全纯的．假设 f_1，…，f_n是 $A(U)$的元素，使得

(1) $$|f_1(z)|+\cdots+|f_n(z)|>0$$

对每个 $z\in\bar{U}$ 成立．则存在 $g_1,\cdots,g_n\in A(U)$使得

(2) $$\sum_{i=1}^{n}f_i(z)g_i(z)=1 \qquad (z\in\bar{U})$$

成立．

证明 由于全纯函数的和、积以及一致极限是全纯的，故 $A(U)$赋以上确界范数时是 Banach 代数．对$A(U)$的任意元素 g_i，所有形如 Σf_ig_i的函数组成的集 J 是$A(U)$的一个理想．我们必须证明 J 包含 $A(U)$的单位元 1．由定理 18.13，当且仅当 J 不包含于 $A(U)$的最大理想时才会出现这情况．由定理 18.17(a)，只需证明不存在$A(U)$到复数域上的同态 h 使得 $h(f_i)=0$对每个i $(1\leqslant i\leqslant n)$成立就够了．

在确定这些同态之前，先要注意到多项式构成 $A(U)$的稠密子集．为看出这一点，设 $f\in A(U)$和$\varepsilon>0$，因为 f 在 $\bar{U}$ 上一致连续，存在$r<1$使$|f(z)-f(rz)|<\varepsilon$对所有的 $z\in\bar{U}$ 成立．当$|rz|<1$时，$f(rz)$ 关于 z 的幂级数展开式是收敛的，因此它对于 $z\in\bar{U}$ 一致收敛于$f(rz)$．这就给出了所需的逼近．

现在设 h 是 $A(U)$的一个复同态．令 $f_0(z)=z$，则 $f_0\in A(U)$．

很明显 $\sigma(f_0)=\overline{U}$. 由定理18.17(d), 存在 $\alpha\in\overline{U}$ 使得 $h(f_0)=\alpha$. 因此 $h(f_0^n)=\alpha^n=f_0^n(\alpha)$, 对 $n=1,2,3,\cdots$ 成立. 于是, 对每个多项式 P, $h(P)=P(\alpha)$. 因为 h 连续而多项式稠密于 $A(U)$, 由此得知对每个 $f\in A(U)$ 有 $h(f)=f(\alpha)$.

我们的假设 (1) 蕴涵着至少有一个指标 $i(1\leqslant i\leqslant n)$ 使得 $|f_i(\alpha)|>0$. 因此 $h(f_i)\neq 0$.

我们已经证明对每个 $h\in\Delta$ 都至少对应有一个给定的函数 f_i, 使得 $h(f_i)\neq 0$. 而这一点, 正如上面曾注意到的一样, 足以证明本定理.

注: 根据前述证明, 我们已经确定了 $A(U)$ 所有的极大理想. 因为其中每一个都是某个 $h\in\Delta$ 的核: 如果 $\alpha\in\overline{U}$ 且 M_α 是所有使得 $f(\alpha)=0$ 的 $f\in A(U)$ 的集, 则 M_α 就是 $A(U)$ 的一个极大理想, 而且 $A(U)$ 的全体极大理想都能由这个途径得出.

$A(U)$ 通常被称为圆盘代数.

18.19 $A(U)$ 的元素在单位圆周 T 上的限制构成了一个 $C(T)$ 的闭子代数. 这就是在例 18.11 中讨论过的代数 A. 事实上 A 是 $C(T)$ 的一个极大子代数. 说得更明确些, 若 $A\subset B\subset C(T)$, 且 B 是 $C(T)$ 的 (关于上确界范数而言的) 闭子代数, 则 $B=A$ 或 $B=C(T)$.

从定理 11.21 得知 A 恰好由使得

$$\text{(1)}\qquad \hat{f}(n)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(e^{i\theta})e^{-in\theta}d\theta=0$$

$$(n=-1,-2,-3,\cdots)$$

成立的那些 $f\in C(T)$ 组成. 因此上面提到的极大性定理可以作为一个逼近定理来叙述:

18.20 定理 假设 $g\in C(T)$, 而对某个 $n<0$, $\hat{g}(n)\neq 0$. 则对每个 $f\in C(T)$ 和 $\varepsilon>0$, 都对应一个多项式

(1) $$P_n(e^{i\theta})=\sum_{k=0}^{m(n)}a_{n,k}e^{ik\theta} \qquad (n=0,\cdots,N)$$

使得

(2) $$|f(e^{i\theta})-\sum_{n=0}^{N}P_n(e^{i\theta})g^n(e^{i\theta})|<\varepsilon \qquad (e^{i\theta}\in T)$$

证明 设 B 是所有形如

(3) $$\sum_{n=0}^{N}P_ng^n$$

的函数的集在 $C=C(T)$ 中的闭包. 这时定理断言 $B=C$. 现在假定 $B\neq C$.

所有形如(3)的函数的集（注意 N 不是固定的）构成一个复代数. 它的闭包是 Banach 代数. 它包含了函数 f_0, 这里 $f_0(e^{i\theta})=e^{i\theta}$. $B\neq C$ 的假定蕴涵着 $1/f_0\notin B$. 如若不然, 对全体整数 n, B 就会包含 f_0^n. 因而全体三角多项式都属于 B, 但由于三角多项式在 C 中稠密(定理4.25)就应当有 $B=C$.

于是 f_0 在 B 中是不可逆的. 由定理 18.17, 存在 B 的复同态 h, 使 $h(f_0)=0$. 每个到复数域上的同态都满足 $h(1)=1$; 而因为 $h(f_0)=0$, 故也有

(4) $$h(f_0^n)=[h(f_0)]^n=0 \qquad (n=1,2,3,\cdots)$$

我们知道 h 是 B 上的线性泛函, 其范数至多是1. Hahn-Banach 定理可将 h 开拓为 C 上具有相同范数的线性泛函(仍以 h 记之). 因为 $h(1)=1$ 而 $\|h\|\leqslant 1$, 5.22节所用的论证表明 h 是 C 上的正线性泛函. 特别, 对于实的 f, $h(f)$ 是实的, 因此 $h(\bar{f})=\overline{h(f)}$. 由于 f_0^{-n} 是 f_0^n 的共轭复数, 得知(4)对 $n=-1,-2,-3,\cdots$ 也成立. 这样

(5) $$h(f_0^n)=\begin{cases}1 & \text{若 } n=0\\ 0 & \text{若 } n\neq 0\end{cases}$$

由于三角多项式稠于C，故C上只有一个满足(5)的有界线性泛函. 因此h由公式

$$h(f)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(e^{i\theta})d\theta \qquad (f\in C) \tag{6}$$

给出.

现在当n是正整数时，$gf_0^n\in B$; 由于h在B上是可乘的以及根据(5)式，(6)给出

$$\begin{aligned}\hat{g}(-n)&=\frac{1}{2\pi}\int_{-\pi}^{\pi}g(e^{i\theta})e^{in\theta}d\theta=h(gf_0^n)\\&=h(g)\cdot h(f_0^n)=0.\end{aligned} \tag{7}$$

这就与定理的假设矛盾.

我们用Wiener所作出的一个定理结束.

18.21 定理 假设

$$f(e^{i\theta})=\sum_{-\infty}^{\infty}c_ne^{in\theta},\quad \sum_{-\infty}^{\infty}|c_n|<\infty \tag{1}$$

且对每一个实的θ，$f(e^{i\theta})\neq 0$. 则

$$\frac{1}{f(e^{i\theta})}=\sum_{-\infty}^{\infty}\gamma_ne^{in\theta},\ \text{且有}\ \sum_{-\infty}^{\infty}|\gamma_n|<\infty \tag{2}$$

证明 设A为所有在单位圆周上满足(1)的复函数的空间，赋以范数

$$\|f\|=\sum_{-\infty}^{\infty}|c_n| \tag{3}$$

显然A是一个Banach空间. 事实上，A等距同构于l^1，即整数集上的关于计数测度可积的全体复函数的空间. 但A对于点态乘法而言也是一个交换Banach代数. 因为，若$g\in A$而$g(e^{i\theta})=\Sigma b_ne^{in\theta}$，则

(4) $$f(e^{is})g(e^{is})=\sum_n\Big(\sum_k c_{n-k}b_k\Big)e^{ins}$$

并因之有

(5) $$\|fg\|=\sum_n|\sum_k c_{n-k}b_k|\leqslant\sum_k|b_k|\sum_n|c_{n-k}|$$

$$=\|f\|\cdot\|g\|$$

同样，函数 1 是 A 的单位元，且 $\|1\|=1$.

令 $f_0(e^{i\theta})=e^{i\theta}$ 如前，则 $f_0\in A$, $1/f_0\in A$, 且对 $n=0,\pm1,\pm2,\cdots$ $\|f_0^n\|=1$. 若 h 是 A 的任意一个复同态且 $h(f_0)=\lambda$, 则 $\|h\|\leqslant1$ 的事实蕴涵着

(6) $$|\lambda^n|=|h(f_0^n)|\leqslant\|f_0^n\|=1\qquad(n=0,\pm1,\pm2,\cdots)$$

因此 $|\lambda|=1$. 换言之，对每个 h 都对应有一个点 $e^{i\alpha}\in T$, 使 $h(f_0)=e^{i\alpha}$, 于是

(7) $$h(f_0^n)=e^{in\alpha}=f_0^n(e^{i\alpha})\qquad(n=0,\pm1,\pm2,\cdots)$$

如果 f 由(1)给出，则 $f=\Sigma c_nf_0^n$. 这个级数在 A 中收敛；由于 h 是 A 上的连续线性泛函，从(7)可得出结论

(8) $$h(f)=f(e^{i\alpha})\qquad(f\in A)$$

因此 T 上没有一个点能使 f 变为 0 这一假设说明了 f 不能属于 A 的任何一个复同态的核. 现在，定理 18.17 蕴涵着 f 在 A 中是可逆的. 而这恰好是本定理所断言的.

习　题

1　假设 $B(X)$ 是 Banach 空间 X 上全体有界线性算子的代数，当 A, A_1 和 $A_2\in B(X)$ 时，有

$$(A_1+A_2)(x)=A_1x+A_2x,\ (A_1A_2)(x)=A_1(A_2x),$$

$$\|A\|=\sup\frac{\|Ax\|}{\|x\|}$$

证明 $B(X)$ 是 Banach 代数.

2 设 n 是一个正整数，X 是全体 n 元复数序组的空间(用任一种方式赋范，只要能满足赋范线性空间的公理)，而设 $B(X)$ 如习题 1. 证明 $B(X)$ 中每一个元素的谱至多由 n 个复数组成. 它们是些什么?

3 取 $X=L^2(-\infty,\infty)$，假设 $\varphi\in L^2(-\infty,\infty)$，并设 M 是一个乘法算子，它将 $f\in L^2$ 变为 φf. 证明 M 是 L^2 上有界线性算子并且 M 的谱等于 φ 的本性值域(第三章习题 19).

4 l^2 上的移位算子的谱是什么?(关于定义请参看定理 17.20.)

5 证明 Banach 代数中的理想的闭包仍是一个理想.

6 若 X 是紧 Hausdorff 空间，找出 $C(X)$ 中的全体极大理想.

7 假设 A 是一个具有单位元的交换 Banach 代数，它由单个元素 x 生成. 这就是说 x 的多项式在 A 中稠密. 证明 $\sigma(x)$ 的余集是平面上的一个连通子集. 提示: 若 $\lambda\notin\sigma(x)$，则存在多项式 P_n 使得在 A 中 $P_n(x)\to(x-\lambda e)^{-1}$. 证明对于 $z\in\sigma(x)$，$P_n(z)\to(z-\lambda e)^{-1}$ 一致地成立.

8 假设 $\sum_0^\infty|c_n|<\infty$，$f(z)=\sum_0^\infty c_nz^n$，对每个 $z\in\overline{U}$，$|f(z)|>0$，且 $1/f(z)=\sum_0^\infty a_nz^n$，证明 $\sum_0^\infty|a_n|<\infty$.

9 证明 Banach 代数 $L^1(R^1)$(见 9.19 节)的闭线性子空间当且仅当它是一个理想时，是平移不变的.

10 证明 $L^1(T)$ 的乘法按

$$(f*g)(t)=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(t-s)g(s)\,ds$$

定义时是一个(没有单位元的)交换 Banach 代数. 如定理 9.23 一样，找出 $L^1(T)$ 的全体复同态. 如果 E 是一个整数集且 I_E 是对所有的 $n\in E$，$f(n)=0$ 的全体 $f\in L^1(T)$ 的集，证明 I_E 是 $L^1(T)$ 的闭理想，并证明 $L^1(T)$ 的每个闭理想都可按这一方式得出.

11 具有单位元的 Banach 代数中的元素 x 的预解式 $R(\lambda,x)$ 定义为

$$R(\lambda,x)=(\lambda e-x)^{-1}$$

它对所有使这个逆元素存在的复数 λ 定义. 证明恒等式

$$R(\lambda,x)-R(\mu,x)=(\mu-\lambda)R(\lambda,x)R(\mu,x)$$

并用它给出定理 18.5 的另一种证明.

12 设 A 是具有单位元的交换 Banach 代数. A 的根式定义为 A 的全体极大理想之交. 证明关于元素 $x\in A$ 的下述三个命题是等价的.

(a) x 属于 A 的根式.

(b) $\lim_{n\to\infty}\|x^n\|^{1/n}=0$.

(c) 对于 A 的每个复同态 h, $h(x)=0$.

13 在 Banach 代数 A 中, 找出一个元素 x (例如, Hilbert 空间上的一个有界线性算子)使得对所有 $n>0$, $x^n\neq 0$, 但 $\lim_{n\to\infty}\|x^n\|^{1/n}=0$.

14 如 18.16 节一样, 假设 A 是具有单位元的交换 Banach 代数, 而 Δ 是 A 的全体复同态的集. 对每个 $x\in A$, 由公式

$$\hat{x}(h)=h(x)\qquad(h\in\Delta)$$

对应有 Δ 上的函数 $\hat{x}$, $\hat{x}$ 称为 x 的 Gelfand 变式.

证明映射 $x\to\hat{x}$ 是 A 到 Δ 上的复函数所组成的关于点态乘法运算的代数 $\hat{A}$ 上的一个同态. 在关于 A 的什么条件下, 这个同态是一个同构? (参看习题 12.)

证明谱半径 $\rho(x)$ 等于

$$\|\hat{x}\|_\infty=\sup\{|\hat{x}(h)|:h\in\Delta\}$$

证明 $\hat{x}$ 的值域恰好是谱 $\sigma(x)$.

15 如果 A 是一个没有单位元的交换 Banach 代数, 设 A_1 是所有序对 (x,λ) (这里 $x\in A$, λ 是复数)的代数, 其加法和乘法按"明显的"方式定义, 且 $\|(x,\lambda)\|=\|x\|+|\lambda|$. 证明 A_1 是具有单位元的交换 Banach 代数, 并且映射 $x\to(x,0)$ 是 A 到 A_1 的一个极大理想上的等距同构. 这是一个没有单位元的代数到一个具有单位元的代数的标准的嵌入.

16 证明 H^∞ 对于上确界范数及点态的加法和乘法是一个交换 Banach 代数. 映射 $f\to f(\alpha)$ 当 $|\alpha|<1$ 时是 H^∞ 的一个复同态. 证明一定还有其他的复同态.

17 证明所有函数 $(z-1)^2f$ 的集是 H^∞ 内的一个非闭的理想, 这里 $f\in H^\infty$. 提示: 当 $|z|<1$, $\varepsilon>0$ 时

$$|(1-z)^2(1+\varepsilon-z)^{-1}-(1-z)|<\varepsilon$$

18 假设 φ 是一个内函数, 证明 $\{\varphi f:f\in H^\infty\}$ 是 H^∞ 的闭理想. 换言之, 证明如果 H^∞ 的序列 $\{f_n\}$, 使得 $\varphi f_n\to g$ 在 U 上一致成立, 则 $g/\varphi\in H^\infty$.

第十九章　全纯 Fourier 变式

引言

19.1　在第九章，R^1上的函数 f 的 Fourier 变式定义为 R^1 上的一个函数 $\hat{f}$. 通常 $\hat{f}$ 能开拓为在平面上某区域内全纯的函数. 例如，若 $f(t)=e^{-|t|}$，则 $\hat{f}(x)=(1+x^2)^{-1}$，是一个有理函数. 这并不会太奇怪. 因为对每个实数 t，核 e^{itz} 是 z 的整函数，于是人们自然期望在 f 满足某些条件下，$\hat{f}$ 将在某个区域内全纯.

我们将描述用这种方法引出的两类全纯函数.

对于第一类，设 F 是 $L^2(-\infty,\infty)$ 的任意一个函数，它在 $(-\infty,0)$ 上为零. [也就是，取 $F\in L^2(0,\infty)$.]并定义

$$f(z)=\int_0^\infty F(t)e^{itz}dt \qquad (z\in\Pi^+) \tag{1}$$

此处 Π^+ 为 $y>0$ 的、所有 $z=x+iy$ 的集. 若 $z\in\Pi^+$，则 $|e^{itz}|=e^{-ty}$，这表明积分(1)作为 Lebesgue 积分存在.

如果 $\mathrm{Im}z>\delta>0$，$\mathrm{Im}z_n>\delta$，且 $z_n\to z$. 则控制收敛定理表明

$$\lim_{n\to\infty}\int_0^\infty |\exp(itz_n)-\exp(itz)|^2dt=0$$

因为被积函数以 L^1-函数 $4\exp(-2\delta t)$ 为界，并且对每个 $t>0$，它是趋于 0 的. 因此由 Schwarz 不等式推出 f 在 Π^+ 内连续. Fubini 定理和 Cauchy 定理表明，对 Π^+ 内每一条闭路径 γ，$\int_\gamma f(z)dz=0$. 由 Morera 定理，$f\in H(\Pi^+)$.

让我们改写(1)为如下形式

$$f(x+iy)=\int_0^\infty F(t)e^{-ty}e^{itx}dt \tag{2}$$

把 y 看作固定的，并应用 Plancherel 定理．对每个 $y>0$，我们得到

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}|f(x+iy)|^2dx$$

$$=\int_0^{\infty}|F(t)|^2e^{-2ty}dt\leqslant\int_0^{\infty}|F(t)|^2dt \tag{3}$$

[注意，我们现在的符号与第九章的不同．那里基本的测度是 Lebesgue 测度除以 $\sqrt{2\pi}$．此处我们正好是用的 Lebesgue 测度，这说明了(3)内出现的因子 $\frac{1}{2\pi}$．]这就表明

(*a*) 若 f 属于形式(1)，则 f 在 Π^+ 内全纯，且它们在 Π^+ 的水平线上的限制构成了 $L^2(-\infty,\infty)$ 内一个有界集．

第二类由所有形如

$$f(z)=\int_{-A}^{A}F(t)e^{izt}dt \tag{4}$$

的 f 组成．此处 $0<A<\infty$ 且 $F\in L^2(-A,A)$．这些函数 f 是整函数(证明同上)，并且它们满足增长条件：

$$|f(z)|\leqslant\int_{-A}^{A}|F(t)|e^{-ty}dt\leqslant e^{A|y|}\int_{-A}^{A}|F(t)|dt \tag{5}$$

如果 C 是最后一个积分，则 $C<\infty$，且由(5)可推出

$$|f(z)|\leqslant Ce^{A|z|} \tag{6}$$

[满足(6)的整函数称为指数型的．]因而：

(*b*) 每个属于(4)型的 f 都是满足(6)的整函数，且它们在实轴上的限制属于 L^2(由 Plancherel 定理)．

值得注意的是(*a*)和(*b*)的逆仍真．这是定理 19.2 及 19.3 的内容．

Paley 和 Wiener 的两个定理

19.2 定理 假设 $f\in H(\Pi^+)$ 及

(1) $$\sup_{0<y<\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty}|f(x+iy)|^2dx=C<\infty$$

则存在 $F\in L^2(0,\infty)$，使得

(2) $$f(z)=\int_0^{\infty}F(t)e^{izt}dt \qquad (z\in\Pi^+)$$

且

(3) $$\int_0^{\infty}|F(t)|^2dt=C$$

注　我们寻求的函数 F 具有这样的性质，它使 $f(x+iy)$ 是 $F(t)e^{-yt}$ 的Fourier 变式(我们把 y 视为一个正的常数)，应用反演公式(这样做正确与否不是主要的，我们是在尝试引导出下列的证明)，所求的 F 将具有形式

(4) $$F(t)=e^{ty}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x+iy)e^{-itx}dx=\frac{1}{2\pi}\int f(z)e^{-itz}dz$$

最后的积分是沿 Π^+ 内一条水平直线的，并且当这一推理完全正确时，这个积分将不依赖于我们偶然选取的特殊直线．这暗示应当求助于 Cauchy 定理．

证明　固定 y，$0<y<\infty$．对每个 $\alpha>0$，令 Γ_α 为顶点在 $\pm\alpha+i$ 及 $\pm\alpha+iy$ 的长方形路径．由 Cauchy 定理

(5) $$\int_{\Gamma\alpha}f(z)e^{-itz}dz=0$$

我们仅考虑 t 的实数值．令 $\Phi(\beta)$ 为 $f(z)e^{-itz}$ 沿 $\beta+i$ 到 $\beta+iy$ (β 为实数)的直线区间的积分．若 $y<1$，令 $I=[y,1]$，若 $1<y$，令 $I=[1,y]$，则

(6) $$|\Phi(\beta)|^2=\left|\int_I f(\beta+iu)e^{-it(\beta+iu)}du\right|^2$$
$$\leqslant\int_I|f(\beta+iu)|^2du\int_I e^{2tu}du$$

令

(7) $$A(\beta)=\int_I |f(\beta+iu)|^2 du$$

由 Fubini 定理，则(1)表明，有

(8) $$\frac{1}{2\pi}\int_{-\infty}^{\infty} A(\beta)d\beta \leqslant Cm(I)$$

因此存在序列$\{\alpha_j\}$，使得$\alpha_j\to\infty$，并且

(9) $$A(\alpha_j)+A(-\alpha_j)\to 0 \qquad (j\to\infty)$$

由(6)，这蕴涵着当 $j\to\infty$ 时，

(10) $$\Phi(\alpha_j)\to 0,\ \Phi(-\alpha_j)\to 0$$

注意，对每个 t 这个式子都成立，且序列$\{\alpha_j\}$不依赖于 t。

让我们定义

(11) $$g_j(y,t)=\frac{1}{2\pi}\int_{-\alpha_j}^{\alpha_j} f(x+iy)e^{-itx}dx$$

则我们由(5)和(10)推导出

(12) $$\lim_{j\to\infty}[e^{ty}g_j(y,t)-e^t g_j(1,t)]=0 \qquad (-\infty<t<\infty)$$

记 $f(x+iy)$为 $f_y(x)$. 由假设 $f_y\in L^2(-\infty,\infty)$，且Plancherel 定理断言

(13) $$\lim_{j\to\infty}\int_{-\infty}^{\infty} |\hat{f}_y(t)-g_j(y,t)|^2 dt=0$$

此处 $\hat{f}_y$ 是 f_y 的 Fourier 变式. 因此$\{g_j(y,\ t)\}$有一个子序列，对几乎所有的 t，点态收敛于 $\hat{f}_y(t)$(定理 3.12). 如果我们定义

(14) $$F(t)=e^t\hat{f}_1(t)$$

由(12)即得

(15) $$F(t)=e^{ty}\hat{f}_y(t)$$

注意(14)不包含 y，且(15)对每个 $y\in(0,\infty)$成立. Plancherel定理能应用于(15)：

(16) $$\int_{-\infty}^{\infty} e^{-2ty}|F(t)|^2 dt=\int_{-\infty}^{+\infty}|\hat{f}_y(t)|^2 dt$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}|f_y(x)|^2dx\leqslant C$$

令 $y\to\infty$, (16)表明在$(-\infty,0)$内几乎处处有 $F(t)=0$.

令 $y\to 0$, (16)表明

$$\int_0^{\infty}|F(t)|^2dt\leqslant C \tag{17}$$

现由(15)得出, $y>0$时, $\hat{f}_y\in L^1$. 因此定理 9.14 给出

$$f_y(x)=\int_{-\infty}^{\infty}\hat{f}_y(t)e^{ixt}dt \tag{18}$$

或

$$f(z)=\int_0^{\infty}F(t)\,e^{-yt}e^{ixt}dt=\int_0^{\infty}F(t)e^{izt}dt \qquad (z\in\Pi^+) \tag{19}$$

这就是(2). 现在由(17)及公式 19.1(3)得出(3).

19.3 定理 假设 A 和 C 是正的常数, f 是一个整函数, 使得对所有 z

$$|f(z)|\leqslant Ce^{A|z|} \tag{1}$$

且

$$\int_{-\infty}^{\infty}|f(x)|^2dx<\infty \tag{2}$$

则存在一个 $F\in L^2(-A,A)$, 使得

$$f(z)=\int_{-A}^{A}F(t)e^{izt}dt \tag{3}$$

证明 对 $\varepsilon>0$ 及实数 x, 令 $f_\varepsilon(x)=f(x)e^{-\varepsilon|x|}$, 我们将证明

$$\lim_{\varepsilon\to 0}\int_{-\infty}^{\infty}f_\varepsilon(x)e^{-ixt}dx=0 \qquad (t\text{ 是实数},\ |t|>A) \tag{4}$$

因为当 $\varepsilon\to 0$ 时, $\|f_\varepsilon-f\|_2\to 0$, Plancherel 定理蕴涵着在 L^2 内 f_ε 的 Fourier 变式收敛于 f 的 Fourier 变式 F. (更确切一点, 是 f 在实轴上的限制.)因此, 由(4)将推出 F 在$[-A,A]$外为零. 这样, 定理 9.14 表明(3)对几乎每一个实数 z 都成立. 由于(3)的

每一边都是整函数，便得出(3)对每一个复数 z 成立。

这样，(4)蕴涵着本定理.

对每一实数 α，设 Γ_α 是由

$$\Gamma_\alpha(s)=se^{i\alpha} \qquad (0\leqslant s<\infty) \tag{5}$$

定义的路径，令

$$\Pi_\alpha=\{w:\mathrm{Re}(we^{i\alpha})>A\} \tag{6}$$

并且，当 $w\in\Pi_\alpha$ 时，定义

$$\Phi_\alpha(w)=\int_{\Gamma_\alpha} f(z)\,e^{-wz}dz=e^{i\alpha}\int_0^\infty f(se^{i\alpha})\exp(-wse^{i\alpha})ds \tag{7}$$

由(1)和(5)，被积函数的绝对值最多是

$$C\exp\{-[\mathrm{Re}(we^{i\alpha})-A]s\}$$

由此得出(如在 19.1 节一样)，Φ_α 在半平面 Π_2 内全纯.

然而，进一步当 $\alpha=0$ 及 $\alpha=\pi$ 时也是正确的：我们有

$$\Phi_0(w)=\int_0^\infty f(x)e^{-wx}dx \qquad (\mathrm{Re}w>0) \tag{8}$$

$$\Phi_\pi(w)=-\int_{-\infty}^0 f(x)e^{-wx}dx \qquad (\mathrm{Re}w<0) \tag{9}$$

因为(2)，Φ_0 及 Φ_π 在所指定的半平面是全纯的.

函数 Φ_α 对于(4)的意义，在于下述容易验证的关系：

$$\int_{-\infty}^\infty f_\varepsilon(x)e^{-itx}dx=\Phi_0(\varepsilon+it)-\Phi_\pi(-\varepsilon+it) \qquad (t\text{为实数}) \tag{10}$$

因此我们要证明，若 $t>A$ 和 $t<-A$，则当 $\varepsilon\to0$ 时，(10)的右边趋于零.

我们的证明，将通过证明任意两个函数 Φ_α 在它们的定义域交集内是相同的，也就是说，它们能彼此解析延拓来实现. 一旦做了这一点，我们就能在 $t<-A$ 时用 $\Phi_{\frac{\pi}{2}}$，而 $t>A$ 时，用 $\Phi_{-\frac{\pi}{2}}$ 来代替(10)中的 Φ_0 和 Φ_π，因此当 $\varepsilon\to0$ 时，差趋于 0 就是很明显的了.

于是假设 $0<\beta-\alpha<\pi$. 令

$$\gamma=\frac{\alpha+\beta}{2},\ \eta=\cos\frac{\beta-\alpha}{2}>0 \tag{11}$$

如果 $w=|w|e^{-i\gamma}$, 则

$$\mathrm{Re}(we^{i\alpha})=\eta|w|=\mathrm{Re}(we^{i\beta}) \tag{12}$$

于是一旦 $|w|>A/\eta$, 则 $w\in\Pi_\alpha\cap\Pi_\beta$. 考虑沿由 $\Gamma(t)=re^{it}$, $\alpha\leqslant t\leqslant\beta$ 的圆弧 Γ 的积分

$$\int_\Gamma f(z)e^{-wz}dz \tag{13}$$

由于

$$\mathrm{Re}(-wz)=-|w|r\cos(t-\gamma)\leqslant-|w|r\eta \tag{14}$$

(13)内的被积函数的绝对值不会超过

$$C\exp\{(A-|w|\eta)r\}$$

如果 $|w|>A/\eta$, 便得出当 $r\to\infty$ 时, (13)趋于 0.

我们现在应用 Cauchy 定理. $f(z)e^{-wz}$ 沿区间 $[0,\ re^{i\beta}]$ 的积分等于(13)与沿 $[0, re^{i\alpha}]$ 的积分之和. 由于当 $r\to\infty$ 时, (13)趋于 0, 我们最终得出, 如果 $w=|w|e^{-i\gamma}$ 和 $|w|>A/\eta$. 则有 $\Phi_\alpha(w)=\Phi_\beta(w)$. 这样, 定理 10.18 表明 Φ_α 及 Φ_β 在它们原来定义的半平面的交集内重合.

这就完成了证明.

19.4 评注 上述两个证明每一个都依靠 Cauchy 定理的一个典型的应用. 在定理 19.2, 我们用沿另一水平线的积分代替沿一水平线的积分来证明 19.2(15)不依赖于 y; 在定理 19.3 用另一条射线代替一条射线用以构成解析延拓. 这结果确实表明这些函数 Φ_α 都是一个在 $[-A_1,\ A_1]$ 的余集内全纯的函数 Φ 的限制.

在定理 19.2 中所描述的函数类是在第十七章讨论的 H^2 类的半平面类似物. 定理 19.3 将用于 Denjoy-Carleman 定理 (定理 19.11)的证明.

拟解析类

19.5 如果 Ω 是一个区域且 $z_0\in\Omega$，每一个 $f\in H(\Omega)$ 是由数 $f(z_0)$, $f'(z_0)$, $f''(z_0)$ …唯一确定的. 另一方面，存在 R^1 上的无限次可微函数，它不恒等于零，但在某个区间上为零. 这样，我们在此得到全纯函数具有的而在 C^∞（在 R^1 所有无限次可微的复函数类）不成立的唯一性性质.

如果 $f\in H(\Omega)$，序列 $\{|f^n(z_0)|\}$ 的增长，受到定理 10.26 限制. 因此有理由问，上述的唯一性性质，是否对 C^∞ 中导数的增长遵守某些限制的函数组成适当的子类仍然成立？这就是作出下列定义的动机. 问题的答案由定理 19.11 给出.

19.6 **$C\{M_n\}$ 类** 如果 M_0, M_1, M_2, …是正数，我们令 $C\{M_n\}$ 为所有 $f\in C^\infty$，并满足下列不等式的类

$$\|D^n f\|_\infty\leqslant\beta_f B_f^n M_n \qquad (n=0,1,2,\cdots) \tag{1}$$

此处 $D^0 f=f$，$n\geqslant 1$ 时，$D^n f$ 是 f 的 n 阶导数. 范数是在 R^1 上的上确界范数，而 β_f 和 B_f 是正的常数（依赖于 f，但不依赖 n）.

如果 f 满足(1)，则

$$\limsup_{n\to\infty}\left\{\frac{\|D^n f\|_\infty}{M_n}\right\}^{1/n}\leqslant B_f \tag{2}$$

这表明 B_f 是比 β_f 更有意义的量. 然而，若在(1)内省略掉 β_f，则 $n=0$ 的场合将推出 $\|f\|_\infty\leqslant M_0$. 这是一个不希望有的限制. 而包括 β_f 在内就使 $C\{M_n\}$ 成为一向量空间.

每一个 $C\{M_n\}$ 在仿射变换下是不变的. 更清楚地说，假设 $f\in C\{M_n\}$ 及 $g(x)=f(ax+b)$，则 g 满足(1)，并有 $\beta_g=\beta_f$ 及 $B_g=aB_f$.

在研究时，我们将对序列 $\{M_n\}$ 作出两个补充假设：

$$M_0=1 \tag{3}$$

(4) $$M_n^2 \leqslant M_{n-1}M_{n+1} \qquad (n=1,2,3,\cdots)$$

假设(4)也可以表示为这样的形式：$\{\log M_n\}$是一个凸序列.

这些假设将简化我们某些工作，并且也不会丧失一般性.［能够证明，每一类$C\{M_n\}$是等于某一类$C\{\bar{M}_n\}$，其中$\{\bar{M}_n\}$满足(3)和(4)，不过我们不打算这样做.］

下列结果说明(3)及(4)的作用.

19.7 定理 每个$C\{M_n\}$关于点态乘法是一个代数.

证明 假设f和$g\in C\{M_n\}$，且β_f, B_f, β_g及B_g为对应的常数. 微分乘积法则表明

(1) $$D^n(fg)=\sum_{j=0}^{n}\binom{n}{j}(D^j f)(D^{n-j}g)$$

因此

(2) $$|D^n(fg)| \leqslant \beta_f\beta_g\sum_{j=0}^{n}\binom{n}{j}B_f^j B_g^{n-j}M_jM_{n-j}$$

$\{\log M_n\}$的凸性与$M_0=1$联系起来，表明对于$0\leqslant j\leqslant n$，就有$M_jM_{n-j}\leqslant M_n$. 因此，由(2)，二项式定理引导出

(3) $$\|D^n(fg)\|_\infty \leqslant \beta_f\beta_g(B_f+B_g)^n M_n \qquad (n=0,1,2,\cdots)$$

于是$fg\in C\{M_n\}$.

19.8 定义 如果条件

(1) $$f\in C\{M_n\},\ (D^n f)(0)=0 \qquad (n=0,1,2,\cdots)$$

能推出$f(x)=0$对所有$x\in R^1$成立，则称类$C\{M_n\}$为拟解析的.

$(D^n f)(0)$用$(D^n f)(x_0)$代替时，对于任何给定的点x_0，这个定义的内容当然不会改变.

因此拟解析类是具有我们在19.6节陈述的唯一性性质的函数类的一种. 这些类中之一是密切地关系到全纯函数的.

19.9 定理 类$C\{n_l\}$由所有这样的f组成，f对应有一个

$\delta>0$，使得 f 在由 $|\mathrm{Im}(z)|<\delta$ 定义的带形域内，能开拓为有界全纯函数.

因此，$C\{n_l\}$ 是一个拟解析类.

证明 假设 $f\in H(\Omega)$，且对所有 $z\in\Omega$，$|f(z)|<\beta$. 这里 Ω 由具有 $|y|<\delta$ 的所有 $z=x+iy$ 组成. 由定理 10.26 得出，对所有实数 x

$$|(D^n f)(x)|\leqslant\beta\delta^{-n}n_l \qquad (n=0,1,2,\cdots) \tag{1}$$

因此，f 在实轴上的限制属于 $C\{n_l\}$.

相反地，假设 f 定义在实轴上，且 $f\in C\{n_l\}$，换句话说

$$\|D^n f\|_\infty\leqslant\beta B^n n_l \qquad (n=0,1,2,\cdots) \tag{2}$$

我们断言表示式

$$f(x)=\sum_{n=0}^{\infty}\frac{(D^n f)(a)}{n_l}(x-a)^n \tag{3}$$

当 $a-B^{-1}<x<a+B^{-1}$ 时，对所有 $a\in R^1$ 是正确的. 由 Taylor 公式重复进行分部积分得出

$$f(x)=\sum_{j=0}^{n}\frac{(D^j f)(a)}{j_l}(x-a)^j+\frac{1}{(n-1)_l}\int_a^x(x-t)^{n-1}(D^n f)(t)dt \tag{4}$$

由(2)式知(4)中最后一项(余项)由

$$n\beta B^n\left|\int_a^x(x-t)^{n-1}dt\right|=\beta|B(x-a)|^n \tag{5}$$

所控制. 如果 $|B(x-a)|<1$，当 $n\to\infty$ 时，它就趋于 0，从而得出(3)式.

我们现在能在(3)式内用满足 $|z-a|<1/B$ 的任何一个复数 z 来代替 x. 这就在中心为 a，半径为 $1/B$ 的圆盘内定义了一个全纯函数 F_a，并且当 x 是实数，而 $|x-a|<1/B$ 时，$F_a(x)=F(x)$. 因

此不同的函数 F_a 都是彼此的解析延拓，它们在带形域 $|y|<1/B$ 内构成 f 的全纯开拓.

如果 $0<\delta<1/B$, 且 $|z|=a+iy$, $|y|<\delta$, 则

$$|F(z)|=|F_a(z)|=\left|\sum_{n=0}^{\infty}\frac{(D^nf)(a)}{n!}(iy)^n\right|$$

$$\leqslant\beta\sum_{n=0}^{\infty}(B\delta)^n=\frac{\beta}{1-B\delta}$$

这表明 F 在带形域 $|y|<\delta$ 内有界，并完成了证明.

19.10 定理 类 $C\{M_n\}$ 是拟解析的，当且仅当 $C\{M_n\}$ 不包含具有紧支集的非平凡函数.

证明 如果 $C\{M_n\}$ 是拟解析的，$f\in C\{M_n\}$，且 f 有紧支集，则显然 f 和它的所有导数在某点为零. 因此对所有 x, $f(x)=0$.

假设 $C\{M_n\}$ 不是拟解析的，则存在一个 $f\in C\{M_n\}$，使得对 $n=0,1,2,\cdots$, $(D^nf)(0)=0$, 但对某个 x_0, $f(x_0)\neq0$. 我们可以假定 $x_0>0$. 若对 $x\geqslant0$, 令 $g(x)=f(x)$; 对 $x<0$, 令 $g(x)=0$, 则 $g\in C\{M_n\}$. 令 $h(x)=g(x)g(2x_0-x)$. 由定理 19.7, $h\in C\{M_n\}$. 同样，若 $x<0$ 和 $x>2x_0$, 则 $h(x)=0$. 但 $h(x_0)=f^2(x_0)\neq0$. 因此，h 是 $C\{M_n\}$ 的一个具有紧支集的非平凡元素.

我们现在完成了关于拟解析类基本定理的预备工作.

Denjoy Carleman 定理

19.11 定理 假设 $M_0=1$, 对 $n=1,2,3,\cdots$, $M_n^2\leqslant M_{n-1}M_{n+1}$, 且对 $x>0$

$$Q(x)=\sum_{n=0}^{\infty}\frac{x^n}{M_n},\quad q(x)=\sup_{n\geqslant0}\frac{x^n}{M_n}$$

则下列五个条件中任何一个可推出其余四个:

(a) $C\{M_n\}$不是拟解析的.

(b) $\int_0^\infty \log Q(x)\frac{dx}{1+x^2}<\infty$.

(c) $\int_0^\infty \log q(x)\frac{dx}{1+x^2}<\infty$.

(d) $\sum_{n=1}^\infty \left(\frac{1}{M_n}\right)^{1/n}<\infty$.

(e) $\sum_{n=1}^\infty \frac{M_{n-1}}{M_n}<\infty$.

注: 如果当 $n\to\infty$时, $M_n\to\infty$十分迅速, 则当 $x\to\infty$时, $Q(x)$慢慢地趋向无穷. 这五个条件, 以它们自己的方式说明, $M_n\to\infty$是很迅速的. 同时注意 $Q(x)\geqslant 1$和 $q(x)\geqslant 1$. 这样, (b)、(c)内的积分恒有定义. 可能对某些 $x<\infty$出现 $Q(x)=\infty$. 在这种情况下, 积分(b)是$+\infty$, 并且定理断言 $C\{M_n\}$是拟解析的.

若 $M_n=n!$, 则$M_{n-1}/M_n=1/n$, 因此(e)被破坏了. 这时定理就断言, $C\{n!\}$是拟解析的, 与定理 19.9 一致.

证明 (a)推出(b) 假设 $C\{M_n\}$ 不是拟解析的, 则 $C\{M_n\}$包含一个具有紧支集的非平凡函数(定理 19.10). 通过对变量的一个仿射变换给出一个函数 $F\in C\{M_n\}$, 在某区间$[0,A]$内具有支集, 使得

(1) $$\|D^nF\|_\infty\leqslant 2^{-n}M_n \qquad (n=0,1,2,\cdots)$$

并使得 F不恒等于零. 定义

(2) $$f(z)=\int_0^A F(t)e^{izt}dt$$

和

(3) $$g(w)=f\left(\frac{i-iw}{1+w}\right)$$

则f是整函数. 若$\mathrm{Im}z>0$, (2)内被积函数的绝对值最多是$|F(t)|$.

于是，f 在上半平面有界；因此 g 在 U 内有界．同时，g 在 $\bar{U}$ 上，除点 $w=-1$ 外连续．因为 f 不恒等于 0（由 Fourier 变式的唯一性定理），g 也不恒等于 0．定理 15.19 表明

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\log|g(e^{i\theta})|d\theta>-\infty \tag{4}$$

如果 $x=i(1-e^{i\theta})/(1+e^{i\theta})=\operatorname{tg}(\theta/2)$，则 $d\theta=2(1+x^2)^{-1}dx$，于是(4)与

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\log|f(x)|\frac{dx}{1+x^2}>-\infty \tag{5}$$

相同．另一方面，由于 F 和它的各阶导数在 0 及 A 为零，(2)的分部积分给出

$$f(z)=(iz)^{-n}\int_0^A(D^nF)(t)e^{izt}dt \qquad (z\neq 0) \tag{6}$$

现在由(1)和(6)得出

$$|x^nf(x)|\leqslant 2^{-n}AM_n \qquad (x\text{ 为实数},n=0,1,2,\cdots) \tag{7}$$

因此

$$Q(x)|f(x)|=\sum_{n=0}^{\infty}\frac{x^n|f(x)|}{M_n}\leqslant 2A \qquad (x\geqslant 0) \tag{8}$$

且(5)及(8)推出(*b*)成立．

证明(*b*)推出(*c*)　$q(x)\leqslant Q(x)$．

证明（*c*）推出（*d*）　令 $a_n=M_n^{1/n}$，由 $M_0=1$ 和 $M_n^2\leqslant M_{n-1}M_{n+1}$，容易验证对 $n>0$，$a_n\leqslant a_{n+1}$．若 $x\geqslant ea_n$，则 $x^n/M_n\geqslant e^n$，于是

$$\log q(x)\geqslant\log\frac{x^n}{M_n}\geqslant\log e^n=n \tag{9}$$

因此，对每一个 N，

(10) $$e\int_{ea_1}^{\infty}\log q(x)\frac{dx}{x^2}\geqslant e\sum_{n=1}^{N}n\int_{ea_n}^{ea_{n+1}}x^{-2}dx+e\int_{ea_{N+1}}^{\infty}(N+1)x^{-2}dx$$

$$=\sum_{n=1}^{N}n\left(\frac{1}{a_n}-\frac{1}{a_{n+1}}\right)+\frac{N+1}{a_{N+1}}=\sum_{n=1}^{N+1}\frac{1}{a_n}$$

这表明(c)推出(d).

证明(d)推出(e)　令

(11) $$\lambda_n=\frac{M_{n-1}}{M}\qquad(n=1,2,3,\cdots)$$

则 $\lambda_1\geqslant\lambda_2\geqslant\lambda_3\geqslant\cdots$，并且若如上述，$a_n=M_n^{1/n}$，我们便有

(12) $$(a_n\lambda_n)^n\leqslant M_n\cdot\lambda_1\lambda_2\cdots\lambda_n=1$$

这样，$\lambda_n\leqslant 1/a_n$和$\Sigma 1/a_n$的收敛性推出$\Sigma\lambda_n$收敛.

证明(e)推出(a)　现在假设$\Sigma\lambda_n<\infty$，其中 λ_n 由(11)给出. 我们断言函数

(13) $$f(z)=\left(\frac{\sin z}{z}\right)^2\prod_{n=1}^{\infty}\frac{\sin\lambda_n z}{\lambda_n z}$$

是一个不恒等于 0 的，指数型整函数，满足不等式

(14) $$|x^k f(x)|\leqslant M_k\left(\frac{\sin x}{x}\right)^2\qquad(x\text{ 为实数},k=0,1,2,\cdots)$$

首先注意 $1-z^{-1}\sin z$ 在原点具有零点. 因此，存在一个常数 B，使得

(15) $$\left|1-\frac{\sin z}{z}\right|\leqslant B|z|\qquad(|z|\leqslant 1)$$

于是得出

(16) $$\left|1-\frac{\sin\lambda_n z}{\lambda_n z}\right|\leqslant B\lambda_n|z|\qquad\left(|z|\leqslant\frac{1}{\lambda_n}\right)$$

于是级数

(17) $$\sum_{n=1}^{\infty}\left|1-\frac{\sin\lambda_n z}{\lambda_n z}\right|$$

在紧集上一致收敛.(注意,因为$\Sigma\lambda_n<\infty$,所以当$n\to\infty$时,$1/\lambda_n\to\infty$.)因此无穷乘积(13)定义一个不恒等于零的整函数f.

其次,恒等式

$$\frac{\sin z}{z}=\frac{1}{2}\int_{-1}^{1}e^{izt}dt \tag{18}$$

表明,若$z=x+iy$,则$|z^{-1}\sin z|\leqslant e^{|y|}$. 因此

$$|f(z)|\leqslant e^{A|z|},\text{其中}A=2+\sum_{n=1}^{\infty}\lambda_n \tag{19}$$

对实数x,我们有$|\sin x|\leqslant|x|$,及$|\sin x|\leqslant 1$. 因此

$$\begin{aligned}|x^kf(x)|&\leqslant|x^k|\left(\frac{\sin x}{x}\right)^2\prod_{n=1}^{k}\left|\frac{\sin\lambda_nx}{\lambda_nx}\right| \\ &\leqslant\left(\frac{\sin x}{x}\right)^2(\lambda_1\cdots\lambda_k)^{-1}=M_k\left(\frac{\sin x}{x}\right)^2\end{aligned} \tag{20}$$

这就给出了(14). 如果我们积分(14),便得到

$$\frac{1}{\pi}\int_{-\infty}^{\infty}|x^kf(x)|dx\leqslant M_k\qquad(k=0,1,2,\cdots) \tag{21}$$

我们已证明出f满足定理19.3的假设. f的Fourier变式

$$F(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)e^{-itx}dx\qquad(t\text{为实数}) \tag{22}$$

因此是不恒等于零的,具有紧支集的函数. 由(21)知$F\in C^\infty$,且由重复应用定理9.2(f),有

$$(D^kF)(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}(-ix)^kf(x)e^{-itx}dx \tag{23}$$

因此,由(21),$\|D^kF\|_\infty\leqslant M_k$. 它表明$F\in C\{M_n\}$.

因此$C\{M_n\}$不是拟解析的,证明就完成了.

习　题

1　假使f是指数型的整函数,且

$$\varphi(y)=\int_{-\infty}^{\infty}|f(x+iy)|^2dx$$

证明或者对所有实数 y, $\varphi(y)=\infty$, 或者对所有实数 y, $\varphi(y)<\infty$. 证明如果 φ 是有界函数, 则 $f=0$.

2 假设 f 是指数型的整函数, 使得 f 在两条非平行直线上的限制属于 L^2. 证明 $f=0$.

3 假设 f 是指数型的整函数, 它在两条非平行直线上的限制是有界的. 证明 f 是常数. (应用第十二章, 习题 9.)

4 假设 f 是整函数, $|f(z)|<C\exp(A|z|)$, 且 $f(z)=\sum a_nz^n$, 令

$$\Phi(w)=\sum_{n=0}^{\infty}\frac{n!a_n}{w^{n+1}}$$

证明: 若 $|w|>A$, 则级数收敛, 若 $\Gamma(t)=(A+\varepsilon)e^{it}$, $0\leqslant t\leqslant 2\pi$, 则

$$f(z)=\frac{1}{2\pi i}\int_{\Gamma}\Phi(w)e^{wz}dw$$

并且 Φ 就是定理 19.3 的证明中出现的函数(也可参看第 19.4 节).

5 假设 f 满足定理 19.2 的假设, 证明 Cauchy 公式

$$(*)\qquad f(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f(\xi+i\varepsilon)}{\xi-i\varepsilon-z}d\xi\qquad(0<\varepsilon<y)$$

成立, 此处 $z=x+iy$. 证明

$$f^*(x)=\lim_{y\to 0}f(x+iy)$$

对几乎所有 x 存在. f^* 与定理 19.2 出现的函数 F 之间有什么关系? 若在(*)中令 $\varepsilon=0$, 且在被积函数中用 f^* 代替 f, (*)是否仍然真确?

6 假设 $\varphi\in L^2(-\infty,\infty)$ 且 $\varphi>0$. 证明, 当且仅当

$$\int_{-\infty}^{\infty}\log\varphi(x)\frac{dx}{1+x^2}>-\infty$$

时, 存在一个 f, $|f|=\varphi$, 使得 f 的 Fourier 变式在半直线上为零. 提示: 如习题 5 一样, 考虑 f^*, 此处 $f=\exp(u+iv)$, 且

$$u(z)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(x-t)^2+y^2}\log\varphi(t)dt$$

7 设 f 是平面内闭集 E 上的复函数. 证明下列关于 f 的两个条件等价:

(a) 存在开集 $\Omega\supset E$ 和一个函数 $F\in H(\Omega)$, 使得对 $z\in E$, $F(z)=f(z)$.

(*b*) 对每一个 $\alpha\in E$, 对应有 α 的邻域 V_α 和一个函数 $F_\alpha\in H(V_\alpha)$, 使得在 $V_\alpha\cap E$ 内, $F_\alpha(z)=f(z)$. (它的一种特殊情形, 已在定理 19.9 证明.)

8 证明 $C\{n!\}=C\{n^n\}$.

9 证明存在比 $C\{n!\}$ 大的拟解析类.

10 如在定理 19.11 证明中一样, 令 $\lambda_n=M_{n-1}/M_n$. 取 $g_0\in C_c(R^1)$, 且定义

$$g_n(x)=(2\lambda_n)^{-1}\int_{-\lambda_n}^{\lambda_n}g_{n-1}(x-t)\,dt \qquad (n=1,2,3,\cdots)$$

直接证明 (不用 Fourier 变式或全纯函数) $g=\lim g_n$ 是一个函数, 它证明了定理 19.11 中的 (*e*) 推出 (*a*) (可以任意选择 g_0 这一点是方便的).

11 找出函数 $\varphi\in C^\infty$ 的一个明显的公式, 这个函数的支集在 $[-2,2]$ 内, 并且 $-1<x<1$ 时, $\varphi(x)=1$.

12 证明对于每一个复数序列 $\{a_n\}$, 都对应有一个函数 $f\in C^\infty$, 使得当 $n=0,1,2,\cdots$ 时, $(D^nf)(0)=a_n$. 提示: 设 φ 为习题 11 中所述. 如果 $\beta_n=a_n/n!$, $g_n(x)=\beta_nx^n\varphi(x)$, 且

$$f_n(x)=\lambda_n^{-n}g_n(\lambda_nx)=\beta_nx^n\varphi(\lambda_nx)$$

则对于 $k=0,\cdots,n-1$, 当 λ_n 充分大时, $|D^kf_n|<2^{-n}$. 取 $f=\Sigma f_n$.

13 构造一个函数 $f\in C^\infty$, 使幂级数

$$\sum_{n=0}^{\infty}\frac{(D^nf)(a)}{n!}(x-a)^n$$

对每个 a 都有 0 收敛半径. 提示: 令

$$f(z)=\sum_{k=1}^{\infty}c_ke^{i\lambda_kx}$$

其中 $\{c_k\}$ 和 $\{\lambda_k\}$ 是正数序列, 选择得使 $\Sigma c_k\lambda_k^n<\infty$ 对 $n=0,1,2,\cdots$ 都成立, 并使 $c_n\lambda_n^n$ 增大得非常迅速, 要比级数 $\Sigma c_k\lambda_k^n$ 中其余所有各项之和大得多.

例如, 令 $c_k=\lambda_k^{1-k}$, 并选 $\{\lambda_k\}$ 使

$$\lambda_k>2\sum_{j=1}^{k-1}c_j\lambda_j^k \text{ 和 } \lambda_k>k^{2k}$$

14 假设 $C\{M_n\}$ 是拟解析的, $f\in C\{M_n\}$, 并且有无穷多个 $x\in[0,1]$, 使 $f(x)=0$. 这能推出什么结论?

15 设 X 是满足下列条件的整函数 f 组成的向量空间: 存在 $C<\infty$ 使

$|f(z)| \leqslant Ce^{\tau|z|}$，并且它在实轴上的限制属于 L^2. 将每个 $f \in X$ 对应于它在整数集上的限制. 证明 $f \to \{f(n)\}$ 是 X 到 l^2 上的一一线性映射.

16 假定 f 是 $(-\infty, \infty)$ 上的可测函数，对所有 x，$|f(x)| < e^{-|x|}$. 证明它的 Fourier 变式 $\hat{f}$ 不能有紧支集，除非 $f(x) = 0$ a. e.

第二十章　用多项式一致逼近

引言

20.1　设 K^0 是复平面内紧集 K 的内部.（依照定义，K^0 是所有 K 的开圆盘子集的并；当然，即使 K 是非空的，K^0 也可能是空集.）设 $P(K)$ 表示所有在 K 上是 z 的多项式的一致极限的函数集.

什么函数属于 $P(K)$？

有两个必要条件立即引起注意：如果 $f\in P(K)$，则 $f\in C(K)$ 和 $f\in H(K^0)$.

问题提出来了，这些必要条件是否也是充分条件？当 K 分离平面时（换句话说，当 K 关于平面的余集是不连通时），回答是否定的. 我们在 13.8 节看到过. 另一方面，如果 K 是实轴上的区间（在这种情况 $K^0=\varnothing$），则 Weierstrass 逼近定理断言

$$P(K)=C(K)$$

如果 K 是区间，那末回答是肯定的. Runge 定理也指出这个方向，因为它说明，对于不分离平面的紧集 K，$P(K)$ 至少包含所有能够全纯扩展到某个开集 $\Omega\supset K$ 的 $f\in C(K)$.

在本章中，我们将证明 Mergelyan 定理，它说明假若 K 不分离平面，那末，不需要再加任何多余的假设，上面提到的必要条件同样也是充分的.

构成证明的主要组份是：Tietze 开拓定理，一个包含卷积的光滑化过程，Runge 定理和引理 20.2，其证明依赖第十四章介绍的 $\mathscr{S}$ 类的性质.

一些引理

20.2 引理 假设D是半径$r>0$的开圆盘，$E\subset D$，E是紧和连通的，$\Omega=S^2-E$是连通的，并且E的直径至少是r. 则存在函数$g\in H(\Omega)$和常数b，具有下述性质：如果

(1) $$Q(\zeta,z)=g(z)+(\zeta-b)g^2(z)$$

则不等式

(2) $$|Q(\zeta,z)|<\frac{100}{r}$$

(3) $$\left|Q(\zeta,z)-\frac{1}{z-\zeta}\right|<\frac{1000r^2}{|z-\zeta|^3}$$

对于所有$z\in\Omega$和所有$\zeta\in D$成立.

我们回顾一下，S^2是Riemann球面，而E的直径是数$|z_1-z_2|$的上确界，其中$z_1\in E$和$z_2\in E$.

证明 不失一般性，假设D的中心在原点. 于是$D=D(0;r)$.

定理13.11的蕴涵式$(d)\to(b)$说明了Ω是单连通的.（注意，$\infty\in\Omega$.）由Riemann映射定理，因而有一个U到Ω上的保形映射F，使$F(0)=\infty$. F有形如

(4) $$F(w)=\frac{a}{w}+\sum_{n=0}^{\infty}c_nw^n \qquad (w\in U)$$

的展开式. 我们定义

(5) $$g(z)=\frac{1}{a}F^{-1}(z) \qquad (z\in\Omega)$$

其中F^{-1}是映Ω到U上的F的逆映射，并且我们令

(6) $$b=\frac{1}{2\pi i}\int_\Gamma zg(z)\,dz$$

其中Γ为圆心在0和半径为r的正向圆周.

依照(4)式，定理14.15可用于F/a. 它断定$(F/a)(U)$的余

集的直径至多是 4，所以 $\operatorname{diam}E\leqslant 4|a|$. 因为 $\operatorname{diam}E\geqslant r$，结果是

$$|a|\geqslant\frac{r}{4} \tag{7}$$

因为 g 是 Ω 到 $D(0;1/|a|)$ 上的保形映射，故(7)式表明

$$|g(z)|<\frac{4}{r}\qquad(z\in\Omega) \tag{8}$$

又因为 Γ 是 Ω 内的路径，长为 $2\pi r$，所以(6)给出

$$|b|<4r \tag{9}$$

如果 $\zeta\in D$，则 $|\zeta|<r$，于是(1)，(8)和(9)式蕴涵着

$$|Q|\leqslant\frac{4}{r}+5r\left(\frac{16}{r^2}\right)<\frac{100}{r} \tag{10}$$

这就证明了(2)式.

固定 $\zeta\in D$.

如果 $z=F(w)$，则 $zg(z)=wF(w)/a$；又因为当 $w\to 0$ 时，$wF(w)\to a$，所以当 $z\to\infty$ 时，$zg(z)\to 1$. 因此 g 有展开式

$$g(z)=\frac{1}{z-\zeta}+\frac{\lambda_2(\zeta)}{(z-\zeta)^2}+\frac{\lambda_3(\zeta)}{(z-\zeta)^3}+\cdots \tag{11}$$

$$(|z-\zeta|>2r)$$

设 Γ_0 是中心在 0 的一个大圆周；(11)（依照 Cauchy 定理）给出

$$\lambda_2(\zeta)=\frac{1}{2\pi i}\int_{\Gamma_0}(z-\zeta)g(z)dz=b-\zeta \tag{12}$$

将 $\lambda_2(\zeta)$ 的值代入(11)，则(1)表明函数

$$\varphi(z)=\left[Q(\zeta,z)-\frac{1}{z-\zeta}\right](z-\zeta)^3 \tag{13}$$

当 $z\to\infty$ 时有界. 因此 φ 在 ∞ 有可去奇点. 如果 $z\in\Omega\cap D$，则 $|z-\zeta|<2r$，于是(2)和(3)给出

$$|\varphi(z)|<8r^3|Q(\zeta,z)|+4r^2<1000r^2 \tag{14}$$

由最大模定理，(14)对于所有 $z\in\Omega$ 成立. 这便证明了(3).

20.3 引理 假设 $f\in C_c'(R^2)$, $C_c'(R^2)$ 是所有具有紧支集的在全平面内连续可微的函数的空间. 令

$$\bar{\partial}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) \tag{1}$$

则下述"Cauchy 公式"成立:

$$f(z)=-\frac{1}{\pi}\iint_{R^2}\frac{(\bar{\partial}f)(\xi)}{\xi-z}d\xi d\eta \qquad (\xi=\xi+i\eta) \tag{2}$$

证明 这可由 Green 定理推得. 但是, 下面却是一个简单的直接证明:

设 $\varphi(r,\theta)=f(z+re^{i\theta})$, $r>0$, θ 是实数. 如果 $\xi=z+re^{i\theta}$, 链式法则给出

$$(\bar{\partial}f)(\xi)=\frac{1}{2}e^{i\theta}\left[\frac{\partial}{\partial r}+\frac{i}{r}\frac{\partial}{\partial\theta}\right]\varphi(r,\theta) \tag{3}$$

因此, 当 $\varepsilon\to 0$ 时, (2)式右边等于极限

$$-\frac{1}{2\pi}\int_{\varepsilon}^{\infty}\int_{0}^{2\pi}\left(\frac{\partial\varphi}{\partial r}+\frac{i}{r}\frac{\partial\varphi}{\partial\theta}\right)d\theta dr \tag{4}$$

对于每一个 $r>0$, φ 是 θ 的周期函数, 周期为 2π. 于是 $\partial\varphi/\partial\theta$ 的积分为0, 并且(4)变成

$$-\frac{1}{2\pi}\int_{0}^{2\pi}d\theta\int_{\varepsilon}^{\infty}\frac{\partial\varphi}{\partial r}dr=\frac{1}{2\pi}\int_{0}^{2\pi}\varphi(\varepsilon,\theta)d\theta \tag{5}$$

当 $\varepsilon\to 0$ 时, $\varphi(\varepsilon,\theta)$ 一致地趋于 $f(z)$. 这就给出了(2).

我们将使用在证明 Urysohn 引理时所用的同样处置建立 Tietze 开拓定理, 因为它完全是这个引理的直接推论.

20.4 Tietze 开拓定理 假设 K 是局部紧 Hausdorff 空间 X 的紧子集, 并且 $f\in C(K)$, 则存在一个函数 $F\in C_c(X)$, 对于所有 $x\in K$, 满足 $F(x)=f(x)$.

(象在 Lusin定理中一样, 我们也可以安排得使 $\|F\|_X=\|f\|_K$.)

证明 假设 f 是实的, $-1\leqslant f\leqslant 1$. 设 W 是具有紧闭包的开

集，使得 $K\subset W$. 设

(1) $$K^+=\{x\in K: f(x)\geqslant\frac{1}{3}\},\ K^-=\{x\in K: f(x)\leqslant-\frac{1}{3}\}$$

则 K^+和 K^-是 W 的不相交的紧子集.

作为 Urysohn 引理的推论，存在函数 $f_1\in C_c(X)$ 在 K^+ 满足 $f_1(x)=\frac{1}{3}$，在 K^-上 $f_1(x)=-\frac{1}{3}$，对所有 $x\in X$，$-\frac{1}{3}\leqslant f_1(x)\leqslant\frac{1}{3}$，并且 f_1 的支集在 W 内. 因此，

(2) $$\text{在 } K \text{ 上，} |f-f_1|\leqslant\frac{2}{3};\ \text{在 } X \text{ 上，} |f_1|\leqslant\frac{1}{3}$$

用 $f-f_1$ 代替 f 重复这一构造过程：存在 $f_2\in C_c(X)$，其支集在 W 内，使得

(3) $$\text{在 } K \text{ 上，} |f-f_1-f_2|\leqslant\left(\frac{2}{3}\right)^2;\ \text{在 } X \text{ 上，} |f_2|\leqslant\frac{1}{3}\cdot\frac{2}{3}$$

按同样方法，我们得到 $f_n\in C_c(X)$，其支集在 W 内，使得

(4) $$\text{在 } K \text{ 上，} |f-f_1-\cdots-f_n|\leqslant\left(\frac{2}{3}\right)^n;\ \text{在 } X \text{ 上，}$$

$$|f_n|\leqslant\frac{1}{3}\cdot\left(\frac{2}{3}\right)^{n-1}$$

令 $F=f_1+f_2+f_3+\cdots$. 依照(4)式，这个级数在 K 上收敛于 f，并且它在 X 上一致收敛. 因此，F 是连续函数. 同样，F 的支集在 $\overline{W}$ 内.

Mergelyan 定理

20.5 定理 如果 K 是平面内的紧集，它的余集是连通的，若 f 是在 K 上连续，在 K 的内部全纯的复函数，并且如果 $\varepsilon>0$，则存在多项式 P，对于所有 $z\in K$，满足 $|f(z)-P(z)|<\varepsilon$.

如果 K 的内部是空集，则假设部分无所谓满足的问题，因而结论对于所有 $f\in C(K)$ 成立. 注意 K 不必是连通的.

证明 由 Tietze 定理，f 能开拓为一个在全平面内具有紧支集的连续函数. 我们固定这样的一个开拓，并且仍记为 f.

对于任意 $\delta>0$，设 $\omega(\delta)$ 是

$$|f(z_2)-f(z_1)|$$

的上确界，其中 z_1 和 z_2 都满足条件 $|z_2-z_1|\leqslant\delta$. 因为 f 一致连续，我们有

(1) $$\lim_{\delta\to 0}\omega(\delta)=0$$

从现在起固定 δ. 我们将证明存在多项式 P，满足

(2) $$|f(z)-P(z)|<10000\omega(\delta) \qquad (z\in K)$$

由(1)，这就证明了本定理.

我们的第一个目标是构造一个函数 $\Phi\in C_c'(R^2)$，对于所有 z，满足

(3) $$|f(z)-\Phi(z)|\leqslant\omega(\delta)$$

(4) $$|(\bar\partial\Phi)(z)|\leqslant\frac{2\omega(\delta)}{\delta}$$

并且

(5) $$\Phi(z)=-\frac{1}{\pi}\iint_X\frac{(\bar\partial\Phi)(\zeta)}{\zeta-z}d\xi d\eta \qquad (\zeta=\xi+i\eta)$$

其中 X 是所有在 Φ 的支集中与 K 的余集的距离不超过 δ 的点的集. (因此，X 不包含"深入 K 中"的点.)

我们构造 Φ 作为 f 和一个光滑函数 A 的卷积，当 $r>\delta$ 时，令 $a(r)=0$，当 $0\leqslant r\leqslant\delta$ 时，令

(6) $$a(r)=\frac{3}{\pi\delta^2}\left(1-\frac{r^2}{\delta^2}\right)^2$$

并且对于所有复数 z，定义

(7) $$A(z)=a(|z|)$$

很明显 $A\in C_c'(R^2)$. 我们断言

(8) $$\iint_{R^2} A = 1$$

(9) $$\iint_{R^2} \bar{\partial} A = 0$$

(10) $$\iint_{R^2} |\bar{\partial} A| = \frac{24}{15\delta} < \frac{2}{\delta}$$

在(6)中调整常数使(8)能够成立.（在极坐标中计算积分.）因为 A 有紧支集，故(9)很易得到. 为了计算(10)，象在引理20.3的证明一样，用极坐标表示 $\bar{\partial} A$，并且注意到 $\partial A/\partial\theta = 0$，$|\partial A/\partial r| = -a'(r)$.

现在定义

(11) $$\Phi(z) = \iint_{R^2} f(z-\zeta) A(\zeta)\, d\xi d\eta = \iint_{R^2} A(z-\zeta) f(\zeta) d\xi d\eta$$

因为 f 和 A 有紧支集，于是 Φ 也有. 因为

(12) $$\Phi(z) - f(z) = \iint_{R^2} [f(z-\zeta) - f(z)] A(\zeta) d\xi d\eta$$

又如果 $|\zeta| > \delta$，则 $A(\zeta) = 0$，所以由(8)得出(3). 因为 $A \in \mathcal{C}_c^\infty(R^2)$，$A$ 的差商有界收敛于对应的 A 的偏导数. 因此，(11)的最后一个表达式可以在积分号下求导. 于是我们得到

(13) $$\begin{aligned}(\bar{\partial}\Phi)(z) &= \iint_{R^2} (\bar{\partial} A)(z-\zeta) f(\zeta) d\xi d\eta \\ &= \iint_{R^2} f(z-\zeta)(\bar{\partial} A)(\zeta)\, d\xi d\eta \\ &= \iint_{R^2} [f(z-\zeta) - f(z)](\bar{\partial} A)(\zeta) d\xi d\eta\end{aligned}$$

最后一个等式依赖于(9). 现在由(10)和(13)给出(4). 如果我们用 Φ_x 和 Φ_y 代替 $\bar{\partial}\Phi$ 写出(13)式，则看出 Φ 有连续偏导数. 因此，应用引理 20.3 于 Φ，并且，如果我们能证明在 G 内 $\bar{\partial}\Phi = 0$，其

中 G 是所有与 K 的余集的距离超过 δ 的 $z\in K$ 的集，则(5)便成立. 我们将通过证明

$$\Phi(z)=f(z) \qquad (z\in G) \tag{14}$$

来做到这一点. 注意，因为 f 在 G 内全纯，故在 G 内 $\bar{\partial}f=0$. (我们记住 $\bar{\partial}$ 是Cauchy-Riemann 算子，已在 11.1 节定义.)现在，如果 $z\in G$，则对于所有满足 $|\zeta|<\delta$ 的 ζ，$z-\zeta$ 在 K 的内部. 由(11)的第一个等式，调和函数的平均值性质因此给出：对所有 $z\in G$，

$$\begin{aligned}\Phi(z)&=\int_0^{\delta}a(r)r\,dr\int_0^{2\pi}f(z-re^{i\theta})\,d\theta\\ &=2\pi f(z)\int_0^{\delta}a(r)r\,dr=f(z)\iint_{R^2}A=f(z)\end{aligned} \tag{15}$$

我们现在已经证明了(3)，(4)和(5)式.

X 的定义表明，X 是紧集并且可以被有限个半径为 2δ，圆心不在 K 内的开圆盘 $D_1,\cdots,D_n$ 所覆盖. 因为 S^2-K 是连通的，每一个 D_j 的圆心可以被一条在 S^2-K 内的多边形路径连到 ∞. 结果是每一个 D_j 包含一个紧连通集 E_j，其直径至少是 2δ，使得 S^2-E_j 是连通的，且 $K\cap E_j=\varnothing$.

现在我们对 $r=2\delta$ 应用引理 20.2. 存在函数 $g_j\in H(S^2-E_j)$ 和常数 b_j，如果

$$Q_j(\zeta,z)=g_j(z)+(\zeta-b_j)g_j^2(z) \tag{16}$$

则不等式

$$|Q_j(\zeta,z)|<\frac{50}{\delta} \tag{17}$$

$$\left|Q_j(\zeta,z)-\frac{1}{z-\zeta}\right|<\frac{4000\delta^2}{|z-\zeta|^3} \tag{18}$$

对所有 $z\in E_j$ 和 $\zeta\in D_j$ 成立.

设 Ω 是 $E_1\cup\cdots\cup E_n$ 的余集. 则 Ω 是包含 K 的开集.

令 $X_1=X\cap D_1$ 和对于 $2\leqslant j\leqslant n$，$X_j=(X\cap D_j)-(X_1\cup\cdots\cup$

X_{j-1})。

定义

(19) $$R(\zeta, z)=Q_j(\zeta, z) \qquad (\zeta\in X_j, z\in\Omega)$$

并且

(20) $$F(z)=\frac{1}{\pi}\iint_X(\bar{\partial}\Phi)(\zeta)R(\zeta, z)d\xi d\eta \qquad (z\in\Omega)$$

因为

(21) $$F(z)=\sum_{j=1}^{n}\frac{1}{\pi}\iint_{X_j}(\bar{\partial}\Phi)(\zeta)Q_j(\zeta, z)d\xi d\eta$$

(16)式表明 F 是函数 g_j 和 g_j^2 的有限线性组合. 因此 $F\in H(\Omega)$.

由(20), (4)和(5), 我们有

(22) $$|F(z)-\Phi(z)|<\frac{2\omega(\delta)}{\pi\delta}\iint_X\left|R(\zeta, z)-\frac{1}{z-\zeta}\right|d\xi d\eta$$

$$(z\in\Omega)$$

注意到, 如果 $\zeta\in X$ 和 $z\in\Omega$, 不等式(17)和(18)用 R 代替 Q_j 仍然真确. 因为, 若 $\zeta\in X$, 则对于某个 j, $\zeta\in X_j$, 并且, $R(\zeta,z)=Q_j(\zeta,z)$ 对于所有 $z\in\Omega$ 成立.

现在固定 $z\in\Omega$, 令 $\zeta=z+\rho e^{i\theta}$, 并且当 $\rho<4\delta$ 时用(17)式, 当 $4\delta\leqslant\rho$ 时用(18)式来估计(22)式中的被积函数. 因此看出(22)式的积分小于

(23) $$2\pi\int_0^{4\delta}\left(\frac{50}{\delta}+\frac{1}{\rho}\right)\rho d\rho=808\pi\delta$$

与

(24) $$2\pi\int_{4\delta}^{\infty}\frac{4000\delta^2}{\rho^3}\rho d\rho=2000\pi\delta$$

的和. 这样, (22)式导出

(25) $$|F(z)-\Phi(z)|<6000\omega(\delta) \qquad (z\in\Omega)$$

因为 $F\in H(\Omega)$, $K\subset\Omega$, 且S^2-K 是连通的，所以 Runge 定理指出F可以在 K 上用多项式一致逼近. 因此(3)和(25)证明(2)式可以成立.

这就完成了证明.

应该指出这个证明的一个特色. 我们曾经不得不证明给定函数 f 是在 $C(K)$的闭子空间 $P(K)$中的. (我们用 20.1 节的术语.)我们的第一个步骤在于用Φ逼近 f. 但是这一步使我们超出了 $P(K)$的范围，因为Φ是这样构造的，使得一般说来，Φ 在 K 的整个内部不是全纯的. 因此Φ与$P(K)$ 有正距离. 然而，(25)指出这个距离小于一个常数乘 $\omega(\delta)$. [事实上，已证明这定理之后，由(3)知道这个距离至多是 $\omega(\delta)$，而不是 $6000\omega(\delta)$.] (25)的证明依赖于不等式 (4)和在G内$\bar{\partial}\Phi=0$ 这两个事实. 因为全纯函数 φ 被 $\bar{\partial}\varphi=0$ 所刻划，(4)可以看成是Φ与全纯函数离得并不太远，并且这个解释已被(25)所证实.

习　题

1　推广 Mergelyan 定理到 S^2-K 有有限个分支的情况：证明每一个在K内部全纯的 $f\in C(K)$，可以在K上用有理函数一致逼近.

2　证明习题 1 的结果不能推广到平面内任意紧集 K. 这可由详细验证下面的例子证实. 对于 $n=1,2,3,\cdots$，设 $D_n=D(a_n;r_n)$ 是U内不相交的开圆盘，它们的并V在U内稠密，并满足 $\Sigma r_n<\infty$. 令 $K=\overline{U}-V$. 设Γ和 γ_n 是路径

$$\Gamma(t)=e^{it},\ \gamma_n(t)=a_n+\gamma_n e^{it},\qquad 0\leqslant t\leqslant 2\pi$$

并定义

$$L(f)=\int_\Gamma f(z)dz-\sum_{n=1}^{\infty}\int_{\gamma_n}f(z)dz\qquad (f\in C(K))$$

证明 L 是 $C(K)$ 上有界线性泛函. 证明对每一个极点在K外面的有理函数 R, $L(R)=0$. 并证明存在 $f\in C(K)$，使 $L(f)\neq 0$.

3　证明在证明引理 20.2 时所构造的函数 g 在满足当 $z\to\infty$ 时，$zf(z)\to 1$ 的全体 $f\in H(\Omega)$ 中，具有最小的上确界范数．（这就给出了引理证明的线索．）

同时证明在那个证明中 $b=c_0$，并且不等式 $|b|<4r$，因此可用 $|b|<r$ 替换．事实上，b 在集 E 的凸包内．

附　录

Hausdorff 极大性定理

我们首先证明一个引理，当它与选择公理配合时，导出定理 4.21 的一个几乎自明的证明.

如果$\mathscr{F}$是一个集族，并且$\Phi\subset\mathscr{F}$. 我们称Φ是$\mathscr{F}$的一个子链，如果Φ按集的包含关系是全序的. 显然，这意味着如果 $A\in\Phi$ 及 $B\in\Phi$，则不是 $A\subset B$ 就是 $B\subset A$. Φ的并将被认为是Φ的全体成员之并.

引理　假设$\mathscr{F}$是集X的一个非空的子集族，使得$\mathscr{F}$的每一个子链的并属于$\mathscr{F}$. 假设 g 是一个函数，使每一个 $A\in\mathscr{F}$ 对应于一个集 $g(A)\in\mathscr{F}$，满足 $A\subset g(A)$，且 $g(A)-A$ 至多由一个元素组成. 则存在 $A\in\mathscr{F}$，使 $g(A)=A$.

证明　固定$A_0\in\mathscr{F}$. 称$\mathscr{F}$的子族$\mathscr{F}'$为塔，如果$\mathscr{F}'$具有下述三个性质：

(a)　$A_0\in\mathscr{F}'$，

(b)　$\mathscr{F}'$ 的每一个子链的并属于$\mathscr{F}'$，

(c)　如果 $A\in\mathscr{F}'$，则亦有 $g(A)\in\mathscr{F}'$.

所有塔的族是非空的. 因为，如果$\mathscr{F}_1$是所有满足 $A_0\subset A$ 的 $A\in\mathscr{F}$ 的族，则$\mathscr{F}_1$是一个塔. 设$\mathscr{F}_0$是所有塔的交，则(容易证明)$\mathscr{F}_0$是塔，但$\mathscr{F}_0$的真子族没有一个是塔. 如果 $A\in\mathscr{F}_0$，亦有 $A_0\subset A$. 证明的思想是指出$\mathscr{F}_0$是$\mathscr{F}$的一个子链.

设Γ是所有这样一些 $C\in\mathscr{F}_0$ 的族，即对每一个 $A\in\mathscr{F}_0$，或者 $A\subset C$ 或者 $C\subset A$.

对于每个 $C\in\Gamma$，设 $\Phi(C)$ 是所有满足 $A\subset C$ 或者 $g(C)\subset A$ 的 $A\in\mathscr{F}_0$ 的族.

Γ以及每个$\Phi(C)$都明显满足性质(a)和(b)，固定$C\in\Gamma$，并假设$A\in\Phi(C)$我们需要证明 $g(A)\in\Phi(C)$. 如果 $A\in\Phi(C)$，就有三种可能：或者 $A\subset C$ 及 $A\neq C$，或者 $A=C$，或者$g(C)\subset A$. 如果 A 是C的真子集，则C不能是 $g(A)$ 的真子集，否则 $g(A)-A$ 至少包含两个元素；因为 $C\in\Gamma$，所以 $g(A)\subset C$. 如果 $A=C$，则 $g(A)=g(C)$. 如果 $g(C)\subset A$，因为 $A\subset g(A)$，则亦有 $g(C)\subset$

$g(A)$. 于是 $g(A)\in\Phi(C)$, 并且我们证明了 $\Phi(C)$ 是塔. 现在 $\mathscr{F}_0$ 的极小性蕴涵着对于每一个 $C\in\Gamma$, $\Phi(C)=\mathscr{F}_0$.

换言之, 如果 $A\in\mathscr{F}_0$ 及 $C\in\Gamma$, 则不是 $A\subset C$ 就是 $g(C)\subset A$. 但是, 这也就是说 $g(C)\in\Gamma$. 因此 Γ 是塔, 并且 $\mathscr{F}_0$ 的极小性表明 $\Gamma=\mathscr{F}_0$. 由 Γ 的定义立即知道 $\mathscr{F}_0$ 是全序的.

设 A 是 $\mathscr{F}_0$ 的并. 因为 $\mathscr{F}_0$ 满足性质 (b), $A\in\mathscr{F}_0$. 由性质 (c), $g(A)\in\mathscr{F}_0$. 因为 A 是 $\mathscr{F}_0$ 的最大元及 $A\subset g(A)$, 由此得出 $A=g(A)$.

定义 关于集 X 的一个选择函数是指一个函数 f, 它对 X 的每一个非空子集 E 都对应于 E 的一个元素: $f(E)\in E$.

用非正式语言说, f 对 X 的每一个非空子集"选出"一个元素.

选择公理 对于每一个集都存在一个选择函数.

Hausdorff 极大性定理 每一个非空半序集 P 有一个极大全序子集.

证明 设 $\mathscr{F}$ 是 P 的所有全序子集的族. 因为 P 的每一个只由单一元素组成的子集是全序集, 所以 $\mathscr{F}$ 是非空的. 注意, 任何全序集的链的并是全序的.

设 f 是关于 P 的一个选择函数. 如果 $A\in\mathscr{F}$, 设 A^* 是在 A 的余集中满足 $A\cup\{x\}\in\mathscr{F}$ 的所有 x 的集. 如果 $A^*\neq\varnothing$, 令

$$g(A)=A\cup\{f(A^*)\}$$

如果 $A^*=\varnothing$, 令 $g(A)=A$.

由引理, 由于至少有一 $A\in\mathscr{F}$, $A^*=\varnothing$, 并且任何这样的 A 都是 $\mathscr{F}$ 的极大元素.

注　释

第　一　章

关于微分和积分的现代理论的第一本著作是 Lebesgue 于 1904 年出版的"Leçons sur l'intégration".

要想知道针对着想作出一种合适的积分理论所进行的早期尝试的光辉历史，可以参看[37]†，其中包含有在关于简单的集论概念被正式定义和被人们充分了解之前，甚至是第一流的数学家们所遇到的困难的有趣讨论.

本书所提出的抽象积分处理方式，是受 Saks 的[28]所启发的. 若用 σ-环(公理：如果对所有 $i=1,2,3,\cdots$, $A_i\in\mathscr{R}$，则 $\bigcup A_i\in\mathscr{R}$，且 $A_1-A_2\in\mathscr{R}$；并不要求 $X\in\mathscr{R}$)取代 σ-代数，可以得到更大的普遍性. 但其代价必然是使可测性定义变得更为烦琐. 这可以参看[7]的 18 节. 在全部经典性的应用中，X 的可测性或多或少是自动满足的. 这就是我们选择以 σ-代数为基础的比较简单的测度论的理由.

1.11 节. $\mathscr{B}$ 的定义同[12]. 在[7]中，Borel 集族被定义为由紧集生成的 σ-环. 当空间不是 σ-紧时，这个集族比我们这里定义的要小一些.

第　二　章

2.12 节. Urysohn 引理通常的说法是：若 K_0 和 K_1 是正规Hausdorff空间 X 的两个互不相交的闭集，则存在 X 上的一个连续函数，它在 K_0 上等于 0 而在 K_1 上等于1. 其证明恰如本书所述.

2.14 节. 这个定理当 $X=[0,1]$时的原始形式是 F. Riesz (1909) 作出的. 关于其进一步的历史，可参看[5], pp. 373, 380—381 和[12], pp. 134—135. 此处定理的陈述形式与[12]中的具有同样的普遍性. 定理 2.14 的证明中，对 X 的全体子集所定义的集函数 μ，由于它的可数次可加性(第一步)，而称为外测度. 这一概念的系统探讨(由Carathéodory 开创的)可参看[25]

† 方括号内的数字参看参考文献目录. ——译者注.

[28].

2.20节. 关于沿着更为古典的途径直接构造 Lebesgue 测度的情况可参看[31], [35]和[26].

2.21节. 任意可数生成 σ-代数的势$\leqslant c$ 的证明可以在[38]的pp. 133—134 找到. 它的势或者有限或者$\geqslant c$ 这一点, 在做过第一章习题 1 之后应当是很清楚的.

2.22节. 关于不可测集与在一个群的作用之下测度保持不变二者之间的关系这个论题, J. Von Neumann 有一篇非常富有启发性的文章: Zur allgemeinen Theorie des Masses, *Fundamenta Math.*, Vol. 13, pp. 73—116, 1929. 也可以参看 Halmos 发表在 *Bull. Am. Math. Soc.* (1958 年 5 月) 专门介绍 Von Neumann 的工作的特辑中的文章.

2.23节. [28], p. 72.

2.24节. [28], p. 75. 对于 Lebesgue 积分理论, 还有另外一种由Daniell(*Ann. Math.*, Vol. 19, pp. 279—294, 1917—1918)作出的、以正线性泛函的开拓为基础的处理方式. 当应用于 $C_c(X)$时它就导致这样一种结构, 使得Vitali-Carathéodory 定理上实质上转化为可测性的定义. 参看[17], 关于完整的讨论可参看[18].

习题 8. R. B. Kirk (*Am. Math. Monthly*, Vol. 79, pp. 884—886, 1972.)在 Baire 类定理的基础上找到了一个很有趣的推广.

习题 17. 这个例子出现在 *Acta Math.*, Vol. 89, p. 160, 1953, R. E. Edwards 的"A Theory of Radon Measures on Locally Compact Spaces"中. 遗憾的是[27]中忽略了它的存在性.

习题 18. [7], p. 231; 由 Dieudonné 最先作出.

第 三 章

最普通的参考书是[9]. 也可以参看[36]的第一章.

习题 3. *Fundamenta Math.*, Vol. 1(1920)包含有与附带性注解有关的三篇文章.

习题 16. [28], p. 18.

习题 17. (*b*) 这一部分的第二种提示性证明由 W. P. Novinger 发表于 *Proc. Am. Math. Soc.*, Vol. 34, pp. 627—628, 1972. 它也为 David Hall 所发现.

习题 18. 在概率论中，依测度收敛性是一个很自然的概念. 参看[7]的第九章.

第 四 章

论及 Hilbert 空间理论的书很多，我们列出 [6] 和 [24] 作为主要参考书，也可参看[5]、[17]和[19].

Fourier 级数的标准著作是[36]. 作为比较简单的引论，可参看 [10]，[31]和[39].

第 五 章

这方面的经典著作是[2]. 更为近代的文献是[5]，[14]和[24]. 也可参看[17]和[19].

5.22 节. 关于表示测度的比较深入的讨论可参看 Arens 和 Singer 的 *Proc. Am. Math. Soc.*, Vol. 5, pp. 735—745, 1954.

第 六 章

6.3 节. 常数 1/6 可用 $1/\pi$ 代替，而且这也是最好的可能值. 参看 R. P. Kaufman 和 N. W. Rickert, *Bull. Am. Math. Soc.*, Vol. 72, pp. 672—676, 1966, 和 W. W. Bledsoe, *Am. Math. Monthly*, Vol. 77, pp. 180—182, 1970(一种较简单的处理).

6.9 节. Von Neumann 的证明见于他的论文: On Rings of Operators, III, *Ann. Math.*, Vol. 41, pp. 94—161, 1940 的关于测度论一节. 参看该文的 pp. 124—130.

6.15 节. $L^\infty \neq (L_1)^*$ 的现象由 J. T. Schwartz 在 *Proc. Am. Math. Soc.*, Vol. 2, pp. 270—275, 1951, 以及 H. W. Ellis 与 D. O. Snow 在 *Can. Math. Bull.*, Vol. 6, pp. 211—229, 1963 中讨论过. 也可参看[7], p. 131 和[28], p. 36.

6.19 节. 定理 2.14 的参考文献此处同样适用.

习题 6. 参看[17], p. 43.

习题10(g). 参看[36], Vol. I. p. 167.

第 七 章

Fubini 定理在本书讨论的与[7] 和 [28]中的一样. 不同的处理可参看[25].

7.9 节的(c). 见于 *Fundamenta Math.*, Vol. 1, p. 145, 1920.

习题 2. 与积分是一条曲线所包围的面积这一观念相对应, Lebesgue 积分可以用坐标集的测度来开展讨论. 这就是[16]中所做的.

习题 8. (b) 这一部分, 甚至在更加严格的形式之下, 是 Lebesgue 在 *J. Mathématiques, Ser.* 6, Vol. 1 p. 201, 1905 中证明的, 并且似乎已被人们遗忘. 鉴于 Sierpinski 的另一个例子 (*Fundamenta Math.* Vol. 1, p. 114, 1920), 值得注意的是, 存在一个不是 Lebesgue 可测的平面点集 E, 它和每一条直线至多有两个交点. 若 $f=\chi_E$, 则 f 不是 Lebesgue 可测的. 尽管其所有截口 f_x 和 f^y 上半连续; 事实上, 它们之中每一个至多有两个不连续点. (这个例子依赖选择公理, 但不依赖连续统假设.)

第 八 章

基本定理 8.6 的证明与本书第一版给出的颇为类似. 但现在的这个定理比较强, 因为它所讨论的商 $\mu(E)/m(E)$ 中, E 取遍任意的 Borel 集的序列而仅仅要求它服从正则性条件 8.2 (1). 早期的处理则有赖于一致正则性条件, 同时也限于 E 是开集. 这些更改是 Jim Serrin 1967 年提醒我的. 当然在多数应用中, E 取的都是一些诸如球、立方体之类很规则的集, 而定理 8.6 的全部威力通常是毋需发挥出来的.

微分法定理的经典性证明要用到 Vitali([28], p. 109)作出的一个相当难的覆盖引理. 此处它的位置为一个几乎是显而易见的引理 8.4 所取代. 这个引理看来首次出现在 Wiener 关于遍历定理的文章中 (*Duke Math. J.*, Vol. 5, pp. 1—18, 1939). 注意, 该引理的证明用到 R^k 的一个几何性质. (但是定理 8.6 证明中的余下部分却并不用到它!)

近年来, M. de Guzman 和 B. Rubio (*Am. Math. Monthly*, Vol. 79, pp. 341—348, 1972.) 也采用了另外一些覆盖引理.

E. M. Stein 的书[41]讨论了微分理论其它许多方面的课题.

8.10, 8.11 节 A. S. Besicovitch (*Proc. Cambridge Philos. Soc.*, Vol. 41, pp. 103—110, 1945)证明了. 若 μ 是 R^k (对任何 $k>1$) 上的正的

奇异 Borel 测度，则 μ 的对称导数是 $+\infty$ a. e. $[\mu]$. （为了获得这个结论，在他的定理 3 中取了倒数.）他也使用了另外一个覆盖引理.

8.19 节. 每一个单调函数（从而每一个 $f\in BV$）几乎处处可微的初等证明参看[24]的 pp. 5—9. 在那本书中这个定理作为 Lebesgue 理论的出发点. 由 D. Austin 在 *Proc. Am. Math. Soc.* Vol. 16, pp. 201—221, 1965 中最近所作的证明甚至更为简单.

8.21 节. 参看[16], 定理 260—264, 在那里同一结论已在比较弱的假定之下得到. 注意, 定理8.21的证明仅仅用到 f 的右导数的存在性、可积性及 f 的连续性.

8.26 节. 关于在比较弱的假定之下的证明，可参看 D. E. Varberg 的论文. (*Am. Math. Monthly*, Vol. 78, pp. 42—45, 1971.)

习题 12. 在 *Proc. Am. Math. Soc.* Vol. 36, p. 308, 1972 中, K. Stromberg 给出了一个非常简单的证明.

第 九 章

关于另外一种简短的引论，可参看[36]第十六章. [33]中有 Plancherel 定理的不同证法. [17]、[19]和[27]中讨论了群论方面的问题及与 Banach 代数的联系. 关于不变子空间(9.16 节)的进一步的知识可参看[11]. L^1 中相应问题的现状在[27]第七章有所叙述.

第 十 章

一般参考书: [1], [4], [13], [20], [29]和[31].

10.8 节. 积分也可以在任意的可求长曲线上定义. 参看[13], Vol. I的附录 *C*.

10.10 节. 指数这一拓扑概念在[29]中采用了，并且在[1] 中很完满地使用了它. 定理 10.10 的计算性证明见于[1]的 p. 93.

10.13 节. Cauchy 在附带假设 f' 连续的情况下于 1825 年发表了他的定理. Gousart 指出了这个假设是多余的，并把定理叙述成了现在这种形式. 进一步的历史评注可参看[13], p. 163.

10.16 节. 幂级数表示以及 $f\in H(\Omega)$ 蕴涵 $f'\in H(\Omega)$ 这一事实的标准证法和本书一样都是通过 Cauchy 积分公式进行的. 最近已经构造出一种利

用环绕数（winding number）而不必借助于积分的证明．其细节可参考[34]．

10.25 节．[26]，p. 170 有一个关于复数域的代数完备性的很初等的证明．

10.30 节．(*b*)这部分的证明见于[40]．

10.32 节．开映射定理及 $Z(f)$ 的离散性是全体非常数全纯函数类的拓扑性质．这些性质刻画这个函数类直到同胚的程度．这就是 Stoilov 定理．参看[34]．

10.35 节．Cauchy 定理整体形式这一引人注目的简单，其初等的证明是由 John D. Dixon 发现的．见 *Proc. Am. Math. Soc.*, Vol. 29, pp. 625—626, 1971．[1]中的证明基于恰当微分的理论．本书第一版中则是从 Runge 定理推出的．这种处理更早一些在[29]，p. 177 中就已采用了，然而在那里仅仅是就单连通区域来讨论的．

第十一章

一般参考书：[1]，第五章；[20]，第一章．

11.12 节．“非切向”极限在圆上确实是几乎处处存在的．参看[15]，pp. 34—37 和[36]，Vol. I, pp. 96—106，特别是其定理(7.8)．

11.17 节．反射原理曾被 H. A. Schwarz 用来解决关于多边形区域的保形映射问题．参看[13]的 17.6 节．这方面的进一步结果由 Carathéodory 得到；参看[4]，Vol. II, pp. 82—92，以及 *Commentarii Mathematici Helvetici*, Vol. 19, pp. 263—278, 1946—1947．

11.19 节．这是 Herglotz 作出的，*Leipziger Berichte*, Vol. 63, pp. 501—511, 1911．

11.21 节．参看 Fatou 的论文 Séries trigonométriques et Séries de Taylor．*Acta Math.*, Vol. 30, pp. 335—400, 1906．这是将 Lebesgue 积分理论应用于全纯函数方面的问题的第一部主要著作．

第十二章

12.7 节．进一步的例子参看[31]，pp. 176—186．

12.11 节．这个定理就三角级数来说，首先由 W. H. Young (1912; $q=2,4,6,\cdots$)和 F. Hausdorff (1923; $2\leqslant q<\infty$)予以证明．F. Riesz (1923)将

它推广到一致有界的正交集. M. Riesz(1926)从一个一般的内插定理导出了这一推广. 而 G. O. Thorin(1939) 则找出了 M. Riesz 定理的复变量时的证明. 本书的证明是 Carderón-Zygmund(1950) 就 Thorin 的思想作的改写. 其它内插定理的完整的参考文献和讨论可在[36]的第十二章找到.

12.12 节. 一种稍微不同的形式见于 *Duke Math. J.*, Vol. 20, pp. 449—458, 1953.

12.13 节. 这一证明实质上是属于 R. Kaufman 的(*Math. Ann.*, Vol. 169, p. 282, 1963). E. L. Stout(*Math. Ann.*, Vol. 177, pp. 339—340, 1968)得到过一个更强的结果.

第十三章

13.9 节. Runge 定理发表于 *Acta Math.*, Vol. 6, 1885. (附带说一句, 关于区间上的多项式一致逼近的 Weierstrass 定理也正是在同一年发表的. 见 *Mathematische Werke*, Vol. 3, pp. 1—37.)与原证明很接近的证明可参看[29], pp. 171—177. 本书的泛函分析证法是分析学者们所熟悉的, 而且近年来可能已多次独立地被发现. 这是 L. A. Rubel 写信告诉我的. 在[13], Vol. II, pp. 299—308 中, 注意的重点是放在当多项式的次数固定时逼近的密切程度上.

习题 5、6. 另外一种方法可参看 D. G. Cantor 的 *Proc. Am. Math. Soc.*, Vol. 15, pp. 335—336, 1964.

第十四章

一般参考书: [20]. 很多特殊的映射函数在那里叙述得很详尽.

14.3 节. 关于线性分式变换的更多的细节可在下列文献中找到: [1], pp. 22—35; [13], pp. 46—57; [4]; 特别是 L. R. Ford的书 "Automorphic Functions", McGraw-Hill Book Company, New York, 1929, 第一章.

14.5 节. 正规族是由 Montel 引出的. 参看[13]的第十五章.

14.8 节. Riemann 定理的历史探讨见[13], pp. 320—321, 和[29], p. 230. Koebe 的证明 (习题 26) 见 *J. für Math.*, Vol. 145, pp. 177—223, 1915, 那里也讨论双连通区域.

14.10 节. 至今尚未获证的 Bieberbach 猜测是说当 $f\in\mathcal{S}$ 时, 对任意 n,

$|a_n| \leqslant n$. 参看[13], pp. 346—358.

14.18 节之(*b*). 这一推证见于[20], p. 179.

14.19 节. 保形映射的边界性态曾由 Carathéodory 在 *Math. Ann.*, Vol. 73, 1913, pp. 323—370 中研究过. 定理 14.19 在那里就 Jordan 曲线围成的区域作了证明，并且引出了素端点的概念. 也可参看 [4], Vol. II, pp. 88—107.

习题 24. 证明是 Y. N. Moschovakis 作出的.

第十五章

15.9 节. 典范积与有限阶整函数之间的关系的讨论可参看 [3] 的第二章；[29] 的第七章和[31]的第八章.

15.25 节. 这个方向的进一步结果可参看 Szasz 的 *Math. Ann.*, Vol. 77, pp. 482—496, 1916 以及[21]的第二章.

第十六章

关于 Riemann 面的经典著作是[32](初版于 1913 年). 其它参考书有：[1]的第六章；[13]的第十章；[29]的第六章和[30].

16.5 节. Ostrowski 定理见于 *J. London Math. Soc.*, Vol. 1, pp. 251—263, 1926. 关于空隙级数方面的近代论述，可参看 J. P. Kahane的Lacunary Taylor and Fourier Series. *Bull. Am. Math. Soc.*, Vol. 70, pp. 199—213, 1964.

16.16 节. 单值性定理的处理比本书第一版稍微简单一些. 它用到了这样一个事实，每个单连通平面区域都同胚于一个凸的单连通区域，即U. 现在的证明是这样安排的，使它不用修改就能应用于多复变全纯函数. (要注意，当 $k>2$ 时，R^k 中总存在一个不能同胚于任何凸集的单连通开集；球壳就提供了这样的实例.)

16.17 节. 参考[13]的第十三章，[29]的第八章和[4]的第七部分.

16.21 节. 在[4]的第七部分中，借助于模函数证明了 Picard 大定理. "初等"的证明可在[31], pp. 277—284 和[29]的第七章找到.

习题 10. 在 Conformal Invariants and Function-theoretic Null-Sets, *Acta Math.*, Vol. 83, pp. 101—129, 1950 一文中，Ahlfors 和 Beurling 讨论了各种各样可去集的类.

第十七章

最普通的参考书是 [15]. 也可参看[36]的第七章. 虽然[15]主要是讨论单位圆盘, 但大多数证明都安排得使之能应用于本书所叙述的更广泛一些的场合. 这些推广的一部分在[27]的第八章也提出来了.

17.1 节. 关于次调和函数的完整论述可参看[22].

17.13 节. 在[15]或者 Helson 和 Lowdenslager 在 *Acta Math.*, Vol. 99, pp. 165—202, 1958 发表的文章中, 有不同的证明.

17.14 节. "内函数"和"外函数"这两个名词是 Beurling 在一篇论文中新造出来的. 在该文中也证明了定理 17.21. 这篇文章是: On Two Problems Concerning Linear Transforma—tions in Hilbert Space. *Acta Math.*, Vol. 81, pp. 239—255, 1949. 进一步的发展见[11].

17.25, 17.26 节. M. Riesz 定理的这一证明由 A. P. Calderón 给出, 参看 *Proc. Am. Math. Soc.*, Vol. 1, pp. 533—535, 1950. 也可参看[36], Vol. I, pp. 252—262.

习题 3. 这形成了在其它区域内 H^p 空间定义的基础. 参看 *Trans. Am. Math. Soc.*, Vol. 78, pp. 46—66, 1955.

第十八章

一般参考书: [17], [19]和[23]; 也可参看[14]. 这一理论是由 Gelfand 于 1941 年开创的.

18.18 节. 这是由 P. J. Cohen 在 *Proc. Am. Math. Soc.*, Vol. 12, 1961, pp. 159—163 中以初等方式予以证明的.

18.20 节. 这个定理属于 Wermer, 见 *Proc. Am. Math. Soc.*, Vol. 4, pp. 866—869, 1953. 本书的证明由 Hoffman和 Singer 给出. 参看[15], pp. 93—94. 在那里, 也给出了 P. J. Cohen 作出的极简单的证明. (参看 18.18 节所列的文献.)

18.21 节. 这是 Wiener 关于他的 Tauber 定理的原始证明中的主要步骤之一. 参看[33], p. 91. 本书中给出的不费力的证明乃是 Gelfand 理论的第一个辉煌成果.

习题 14. 在集 Δ 上可以给出一个紧的 Hausdorff 拓扑, 使得函数 $\hat{x}$ 关于这个拓扑是连续的. 这样 $x\to\hat{x}$ 是 A 到 $C(\Delta)$ 内的一个同态. A 作为连续

函数代数的这一表示在研究交换 Banach 代数时是一个最重要的工具.

第十九章

19.2, 19.3 节. [21], pp. 1—13. 也可参看[3]. 在那里指数型函数是讨论的主要对象.

19.5 节. 关于 $C\{M_n\}$ 类的更为详尽的介绍可参看 S. Mandelbrojt, "Séries de Fourier et classes quasi-analytiques," Gauthier-Villars, Paris, 1935.

19.11 节. 在[21]中这一定理的证明是基于定理 19.2而不是定理19.3.

习题 4. 函数Φ称为 f 的 Borel 变式. 见[3], 第五章.

习题 12. 提示性证明是 H. Mirkil 给出的. 见 *Proc. Am. Math. Soc.*, Vol. 7, pp. 650—652, 1956. 定理由 Borel 于 1895 年证明.

第二十章

参看 S. N. Mergelyan, Uniform Approximations to Functions of a Complex Variable, *Uspehi Mat. Nauk.* (N. S.) 7, no. 2(48), 31—122, 1952; *Amer. Math. Soc. Translation* No. 101, 1954. 本书定理 20.5 即 Mergelyan 论文中的定理 1.4.

近年来, L. Carleson 在 *Math. Scandinavica*, Vol. 15, 1964, pp. 167—175 中发表了一个以测度论研究为基础的泛函分析的证明.

附 录

Hausdorff 1914 年在他的书"Grundzüge der Mengenlehre," p. 140 中首次陈述了极大性原理. 本书的证明仿自 Halmos 的书[8] 的第 16 节. 在作出"塔"这一名词的同时, 选择 g 使 $g(A)-A$ 至多有一个元素的想法在那里也出现了. 其证明与良序定理的 Zermelo 证法相类似; 见 *Math. Ann.*, Vol. 65, pp. 107—128, 1908.

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专用和缩写符号一览表†

$\exp(z)$	1	Z	104
τ	8	$\hat{f}(n)$	107
$\mathfrak{M}$	9	c_0	122
χ_E	13	$\|f\|_E$	128
lim sup	15	U	130
lim inf	15	$P_r(\theta-t)$	132
f^+, f^-	17	$\|\mu\|(E)$	138
$L^1(\mu)$	28	μ^+, μ^-	142
a. e.	32	$\lambda \ll \mu$	143
$\bar{E}$	42	$\lambda_1 \perp \lambda_2$	143
$C_c(X)$	44	$A \Delta B$	159
$K \prec f \prec V$	45	E_x, E^y	162
$\mathfrak{M}_F$	49	f_x, f^y	164
m, $\mathfrak{M}_k$	59, 60	$\mu \times \lambda$	167
$L^1(R^k)$, $L^1(E)$	61	$f*g$	175
$\|f\|_p$, $\|f\|_\infty$	75	$\mu*\lambda$	177
$L^p(\mu)$, $L^p(R^k)$, l^p	75	$D\mu$	182
$L^\infty(\mu)$, $L^\infty(R^k)$, l^∞	75	$\mathscr{D}^Q$, $\mathscr{D}_Q$	188
$C_0(X)$, $C(X)$	81	$T_f, V(f)$	190
(x, y)	88	BV, NBV	191
$\|x\|$	88	$f(x-), f(x+)$	191
$x \perp y$, $M^\perp$	92	$T'(x)$	202
$\hat{x}(\alpha)$	96	$\Delta(A)$	202
T	103	$\hat{f}(t)$	214
$L^p(T)$, $C(T)$	104	C^∞	233

† 标准的集论符号叙述在第 6 页和第 7 页，此处没有列入。

C_*^-	233	$f^*(e^{it})$	293
$D(a;r)$, $D'(a;r)$, $\bar{D}(a;r)$	236	H^∞	293
Ω	237	$\varphi_a(z)$	300
$H(\Omega)$	237	S^2	314
γ, γ^*	241	$\mathcal{S}$	338
$\partial\Delta$	244	$E_p(z)$	358
$\mathrm{Ind}_\gamma(z)$	244	$\log^+ t$	369
$Z(f)$	250	N	369
$\dot{+}$	261	$(f_0, D_0) \sim (f_1, D_1)$	384
$I^2 = I \times I$	266	$M_p(f;r)$	402
∂, $\bar{\partial}$	277	H^p	402
Δ	278	M_f, Q_f	411
$P[f]$	280	$C\{M_n\}$	452
$P[d\mu]$	280	$P(K)$	463
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说　明

这本书是由六位同志合译的．李世余译序言和第二、九章，戴牧民译注释和第四、五、十八章，李奕华译第一、十、十六、十九章，陈庆祺译第三、六、七、八、十七章，张耀勋译第十一、十二、十三章，仇焕章译第十四、十五、二十章．最后全书由戴牧民、李奕华校核．

因我们的水平不高，译文会出现不少的缺点和错误，希望读者批评指正。

译　者